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Scheduling Bipartite Tournaments to Minimize Total Travel Distance

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Abstract

In many professional sports leagues, teams from opposing leagues/conferences compete against one another, playing *inter-league* games. This is an example of a *bipartite* tournament. In this paper, we consider the problem of reducing the total travel distance of bipartite tournaments, by analyzing inter-league scheduling from the perspective of discrete optimization. This research has natural applications to sports scheduling, especially for leagues such as the National Basketball Association (NBA) where teams must travel long distances across North America to play all their games, thus consuming much time, money, and greenhouse gas emissions.

We introduce the *Bipartite Traveling Tournament Problem (BTTP)*, the inter-league variant of the well-studied Traveling Tournament Problem. We prove that the $2n$-team *BTTP* is NP-complete, but for small values of n, a distance-optimal inter-league schedule can be generated from an algorithm based on minimum-weight 4-cycle-covers. We apply our theoretical results to the 12-team Nippon Professional Baseball (NPB) league in Japan, producing a provably-optimal schedule requiring 42950 kilometres of total team travel, a 16% reduction compared to the actual distance traveled by these teams during the 2010 NPB season. We also develop a nearly-optimal inter-league tournament for the 30-team NBA league, just 3.8% higher than the trivial theoretical lower bound.

1. Introduction

Consider a tournament involving two teams X and Y, each with n players. In a *bipartite tournament*, players from team X compete against players from team Y, with the goal of determining the superior team. Labeling the players $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$, we represent each match by the ordered pair (x_i, y_j), with indices $i, j \in \{1, 2, \ldots, n\}$.

The Davis Cup is an example of a bipartite tournament, where each country fields a tennis squad consisting of two singles players and a doubles team. Their are five matches played between the two countries, with the doubles teams squaring off on Day 2, sandwiched between the singles matches $(x_1, y_1), (x_2, y_2)$ on Day 1, and $(x_1, y_2), (x_2, y_1)$ on Day 3. Another example is the biennial Ryder Cup championship, where the United States and Europe field teams consisting of the top twelve male golfers. The competition culminates with twelve head-to-head matches on the last day, with the i^{th} ranked golfer from the United States facing off against the i^{th} ranked golfer from Europe.

In a single round-robin (SRR) bipartite tournament, each player from X competes against every player from Y once, with everyone playing one match in each time slot. This produces a tournament with n^2 matches spread out over n time slots. In a double round-

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robin (DRR) bipartite tournament, each pair plays twice, thus producing a tournament with $2n^2$ matches spread out over $2n$ time slots. SRR bipartite tournaments are common in tennis and ping-pong, while DRR bipartite tournaments are used in chess, so that x_i plays against each y_j twice, with one game as white and one game as black. The aforementioned Ryder Cup is an example of a *partial* bipartite tournament, where each player from X plays against some proper subset of players from Y.

While there has been much research conducted on the theory of bipartite tournaments (Kendall, Knust, Ribeiro, & Urrutia, 2010), all previous papers have dealt with feasibility and fairness, specifically in constructing balanced tournament designs and minimizing carry-over effects (Easton, Nemhauser, & Trick, 2004) to ensure competitive balance for all the players on each team.

By replacing the words "team" and "player" by "league" and "team", respectively, we can view X and Y as two n-team sports leagues, where a bipartite tournament between X and Y represents *inter-league play.* For example, Major League Baseball (MLB) holds four weeks of inter-league games each season, with every American League team playing 18 games against a half-dozen teams from the National League. MLB inter-league play is an example of a partial bipartite tournament, where some/many of the scheduled games are based on historical rivalry or geographic proximity.

In this light, we consider the problem of minimizing the *total travel distance* of bipartite tournaments. For chess and tennis, the issue of travel is irrelevant as all tournament matches take place in the same venue. However, in the case of inter-league play in professional baseball, teams must travel long distances to play their games all across North America, and so finding a schedule that reduces total travel distance is important, for both economic and environmental reasons.

To answer this question of creating a distance-optimal inter-league schedule, we introduce a variant of the *Traveling Tournament Problem* (TTP), in which every pair of teams plays twice, with one game at each team's home stadium. The output is an optimal schedule that minimizes the sum total of distances traveled by the teams as they move from city to city, subject to several natural constraints that ensure balance and fairness. Unlike the TTP which models a double round-robin *intra*-league tournament, our variant, the *Bipartite Traveling Tournament Problem (BTTP)*, seeks the best possible double round-robin *inter*-league tournament.

Since its introduction (Easton, Nemhauser, & Trick, 2001), the TTP has emerged as a popular area of study within the operations research community (Kendall et al., 2010) due to its incredible complexity, where challenging benchmark problems remain unsolved. Research on the TTP has led to the development of powerful techniques in integer programming, constraint programming, as well as advanced heuristics such as simulated annealing (Anagnostopoulos, Michel, Hentenryck, & Vergados, 2006) and hill-climbing (Lim, Rodrigues, & Zhang, 2006). More importantly, the TTP has direct applications to scheduling optimization, and can aid professional sports leagues as they make their regular season schedules more efficient, saving time and money, as well as reducing greenhouse gas emissions.

The purpose of this paper is to consider the problem of creating distance-optimal inter-league tournaments, thus connecting the techniques and methods of sports scheduling to the theory of bipartite tournaments, producing new directions for research in scheduling optimization. Optimizing inter-league tournaments is a natural next step in the field of sports scheduling, especially since the introduction of inter-league play to professional sports. For example, in Major League Baseball, inter-league play began only in 1997, six decades after it was first proposed. In Japan, the Nippon Professional Baseball (NPB) league was formed in 1950, yet NPB inter-league play did not commence until 2005.

The authors were motivated to analyze the Japanese NPB schedule, due to puzzling inefficiencies in the regular season schedule that we believed could be improved. We developed a multi-round generalization of the TTP (Hoshino & Kawarabayashi, 2011c) based on Dijkstra's shortest path algorithm to create a distance-optimal intra-league schedule that reduced the total travel distance by over 60000 kilometres as compared to the 2010 NPB schedule. We elaborated further on the intricacies of intra-league scheduling in a journal paper (Hoshino & Kawarabayashi, 2011d). Inspired by the success of analyzing intra-league scheduling, we asked whether our techniques and methods could be extended to inter-league play, wondering whether the 2010 NPB schedule requiring 51134 kilometres of total team travel could be minimized to optimality. We answered that question by presenting the *Bipartite Traveling Tournament Problem* (Hoshino & Kawarabayashi, 2011b), and providing a rigorous analysis of *BTTP* for the NPB distance matrix, producing a provably-optimal inter-league schedule requiring 42950 kilometres of total team travel (Hoshino & Kawarabayashi, 2011a).

The purpose of this paper is to expand upon our two inter-league conference papers and provide a more thorough discussion of *BTTP* and its properties. We present a rigorous proof to a lemma we omitted due to space constraints (Hoshino & Kawarabayashi, 2011b), that is key to proving the NP-completeness of *BTTP*. We also present an application of *BTTP* beyond Japanese baseball, by considering the problem of optimizing inter-league scheduling for the 30-team National Basketball Association (NBA) in North America. While we briefly alluded to the NBA inter-league problem (Hoshino & Kawarabayashi, 2011b), we are able to provide a full analysis in this paper.

In Section 2, we formally define *BTTP* and discuss uniform and non-uniform schedules. In Section 3, we prove that *BTTP* on $2n$ teams is NP-complete by obtaining a reduction from 3-SAT, the well-known NP-complete problem on boolean satisfiability (Garey & Johnson, 1979). Despite its computational intractability for general n, we present a simple yet powerful heuristic involving minimum-weight 4-cycle-covers and apply it to the 12-team NPB league in Japan, as well as the 30-team NBA.

In Section 4, we solve *BTTP* for the NPB, producing an optimal schedule whose total travel distance of 42950 kilometres is 16% less than the 51134 kilometres traveled by these teams during the five weeks of inter-league play in the 2010 season. In Section 5, we produce a nearly-optimal solution to *BTTP* for the NBA, developing a bipartite tournament schedule whose total travel distance of 537791 miles is just 3.8% higher than the trivial theoretical lower bound. In Section 6, we conclude the paper with several open problems and present directions for future research.

2. Definitions

Let there be $2n$ teams, with n teams in each league. Let X and Y be the two leagues, with $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Let D be the $2n \times 2n$ *distance matrix*, where entry $D_{p,q}$ is the distance between the home stadiums of teams p and q. By definition, $D_{p,q} = D_{q,p}$ for all $p, q \in X \cup Y$, and all diagonal entries $D_{p,p}$ are zero. Similar to the original TTP, we require a compact double round-robin bipartite tournament schedule satisfying the following conditions:

(a) *at-most-three*: No team may have a home stand or road trip lasting more than three games.

(b) *no-repeat*: A team cannot play against the same opponent in two consecutive games.

(c) *each-venue*: For all $1 \leq i, j \leq n$, teams x_i and y_j play twice, once in each other's home venue.

	1	2	3	4	5	6
x_1	y_1	y_2	y_3	$\mathbf{y_1}$	$\mathbf{y_2}$	$\mathbf{y_3}$
x_2	y_2	y_3	y_1	$\mathbf{y_2}$	$\mathbf{y_3}$	$\mathbf{y_1}$
x_3	y_3	y_1	y_2	$\mathbf{y_3}$	$\mathbf{y_1}$	$\mathbf{y_2}$
y_1	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_2}$	x_1	x_3	x_2
y_2	$\mathbf{x_2}$	$\mathbf{x_1}$	$\mathbf{x_3}$	x_2	x_1	x_3
y_3	$\mathbf{x_3}$	$\mathbf{x_2}$	$\mathbf{x_1}$	x_3	x_2	x_1

	1	2	3	4	5	6
x_1	y_3	y_2	$\mathbf{y_1}$	$\mathbf{y_3}$	y_1	$\mathbf{y_2}$
x_2	$\mathbf{y_1}$	y_3	$\mathbf{y_2}$	y_1	y_2	$\mathbf{y_3}$
x_3	$\mathbf{y_2}$	y_1	$\mathbf{y_3}$	y_2	y_3	$\mathbf{y_1}$
y_1	x_2	$\mathbf{x_3}$	x_1	$\mathbf{x_2}$	$\mathbf{x_1}$	x_3
y_2	x_3	$\mathbf{x_1}$	x_2	$\mathbf{x_3}$	$\mathbf{x_2}$	x_1
y_3	$\mathbf{x_1}$	$\mathbf{x_2}$	x_3	x_1	$\mathbf{x_3}$	x_2

Table 1: Two feasible inter-league tournaments for $n = 3$.

To illustrate, Table 1 provides two feasible tournaments satisfying all of the above conditions for the case $n = 3$. In this table, as in all other schedules that will be subsequently presented, home games are marked in bold.

Following the convention of the TTP, whenever a team is scheduled for a road trip consisting of multiple away games, the team doesn't return to their home city but rather proceeds directly to their next away venue. Furthermore, we assume that every team begins the tournament at home, and returns home after playing their last away game. For example, in Table 1, team x_1 would travel a distance of $D_{x_1,y_1} + D_{y_1,y_2} + D_{y_2,y_3} + D_{y_3,x_1}$ when playing the schedule on the left and a distance of $D_{x_1,y_3} + D_{y_3,y_2} + D_{y_2,x_1} + D_{x_1,y_1} + D_{y_1,x_1}$ when playing the schedule on the right. The desired solution to $BTTP$ is the tournament schedule that minimizes the total distance traveled by all $2n$ teams subject to the given conditions.

Define a *trip* to be a pair of consecutive games not occurring in the same city, i.e., any situation where that team doesn't play at the same location in time slots s and $s+1$, and therefore has to travel from one venue to another. In Table 1, the schedule on the left has 24 total trips, while the schedule on the right has 32 trips. One may conjecture that the total distance of schedule S_1 is lower than the total distance of schedule S_2 iff S_1 has fewer trips than S_2.

To see that this is actually not the case, let the teams x_1, x_3, y_1, and y_2 be located at $(0, 0)$ and let x_2 and y_3 be located at $(1, 0)$. Then the schedule on the left has total distance 16 and the schedule on the right has total distance 12. So minimizing trips does not correlate to minimizing total travel distance; while the former is a trivial problem, the latter is extremely difficult, even for the case $n = 3$.

The six teams $x_1, x_2, x_3, y_1, y_2, y_3$ can be located in the Cartesian plane so that the distance-optimal solution occurs via a schedule with 27 trips, although in the majority of cases, the distance-optimal schedule consists of 24 trips, the fewest number possible. This inspires several interesting open problems which we will present at the conclusion of this paper. To provide an example with 27 trips, locate the six teams at $x_1 = (8, 0)$, $x_2 = (9, 0)$, $x_3 = (0, 4)$, $y_1 = (6, 1)$, $y_2 = (0, 7)$, and $y_3 = (3, 5)$. Then a computer search proves that the minimal distance is

$$18 + 16\sqrt{5} + 16\sqrt{2} + 3\sqrt{13} + 5\sqrt{10} + 2\sqrt{130} + \sqrt{61} \sim 133.646,$$

with equality iff the inter-league schedule is one of the two appearing in Table 2. Note that each of these 27-trip distance-optimal schedules is a mirror image of the other.

	1	2	3	4	5	6
x_1	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$
x_2	y_2	y_3	y_1	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	$\boldsymbol{y_1}$
x_3	$\boldsymbol{y_3}$	$\boldsymbol{y_1}$	y_2	y_3	y_1	$\boldsymbol{y_2}$
y_1	$\boldsymbol{x_1}$	x_3	$\boldsymbol{x_2}$	x_1	$\boldsymbol{x_3}$	x_2
y_2	$\boldsymbol{x_2}$	$\boldsymbol{x_1}$	$\boldsymbol{x_3}$	x_2	x_1	x_3
y_3	x_3	$\boldsymbol{x_2}$	$\boldsymbol{x_1}$	$\boldsymbol{x_3}$	x_2	x_1

	1	2	3	4	5	6
x_1	$\boldsymbol{y_3}$	$\boldsymbol{y_2}$	$\boldsymbol{y_1}$	y_3	y_2	y_1
x_2	$\boldsymbol{y_1}$	$\boldsymbol{y_3}$	$\boldsymbol{y_2}$	y_1	y_3	y_2
x_3	$\boldsymbol{y_2}$	y_1	y_3	y_2	$\boldsymbol{y_1}$	$\boldsymbol{y_3}$
y_1	x_2	$\boldsymbol{x_3}$	x_1	$\boldsymbol{x_2}$	x_3	$\boldsymbol{x_1}$
y_2	x_3	x_1	x_2	$\boldsymbol{x_3}$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$
y_3	x_1	x_2	$\boldsymbol{x_3}$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	x_3

Table 2: The 27-trip distance-optimal schedules for a special selection of 6 points.

Let $BTTP*$ be the restriction of $BTTP$ to the set of tournament schedules where in any given time slot, the teams in each league either all play at home, or all play on the road. For example, the left schedule in Table 1 is a feasible solution of both $BTTP$ and $BTTP*$. We say that such schedules are *uniform*. While this uniformity constraint significantly reduces the number of potential tournaments, it allows us to quickly generate an approximate solution to $BTTP$ from an algorithm based on minimum-weight 4-cycle-covers.

We now prove that both $BTTP$ and $BTTP*$ are NP-complete by obtaining a reduction from 3-SAT, the well-known NP-complete problem of deciding whether a boolean formula in conjunctive normal form with three literals per clause admits a satisfying assignment (Garey & Johnson, 1979).

3. Theoretical Results

To establish our reduction, we first express $BTTP$ in its decision form:

INSTANCE:

(a) $2n$ teams, in which n teams belong to league X and n teams belong to league Y.

(b) A $2n \times 2n$ distance matrix D whose entries are the distances between each pair of teams in $X \cup Y$.

(c) An integer $T \geq 0$.

QUESTION: does there exist a double round-robin bipartite tournament for which:

(a) The *at-most-three*, *no-repeat*, and *each-venue* conditions are all satisfied.

(b) The sum of the distances traveled by the $2n$ teams is at most T.

Similarly, we can express *BTTP** in its decision form, by adding the uniformity constraint (i.e., for any given time slot, a team plays at home iff every other team in that league also plays at home). We now reduce these two problems to 3-SAT.

Let $S = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be the conjunction of m clauses with three literals on the variables $\{u_1, u_2, \ldots, u_l\}$. From S, we will define the sets X_S and Y_S representing the teams in leagues X and Y. From this set of $|X_S| + |Y_S|$ vertices, we will describe a polynomial-time algorithm that constructs a complete graph and assigns edge weights to produce the distance matrix D_S. We then prove the existence of an integer $T = T(m)$ for which the solutions to *BTTP* and *BTTP** have total travel distance $\leq T$ iff S is satisfiable. This will establish the desired polynomial-time reductions.

We can assume that literals u_i and $\overline{u}_i$ occur equally often in S for each $1 \leq i \leq l$. To see why, assume without loss that u_i occurs less frequently than $\overline{u}_i$. By repeated addition of the tautological clause $(u_i \vee u_{i+1} \vee \overline{u}_{i+1})$, which does not affect the satisfiability of S, we can ensure that the number of occurrences of u_i in S matches that of $\overline{u}_i$.

Let $r(i)$ denote the number of occurrences of u_i in S. In Figure 1, we present a "gadget" for each variable u_i, where the vertices $u_{i,r}$ and $\overline{u}_{i,r}$ correspond respectively to the r^{th} occurrence of u_i and $\overline{u}_i$ in S, vertex $a_{i,r}$ is adjacent to $\overline{u}_{i,r-1}$ and $u_{i,r}$, and vertex $b_{i,r}$ is adjacent to $u_{i,r}$ and $\overline{u}_{i,r}$. (Note: $\overline{u}_{i,0} := \overline{u}_{i,r(i)}$ for all i.)

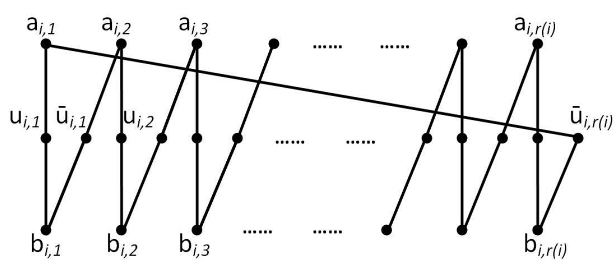

Figure 1: Gadget for 3-SAT reduction.

This gadget was used to establish the NP-completeness of deciding whether an undirected graph contains a given number of vertex-disjoint s-t paths of a specified length (Itai, Perl, & Shiloach, 1982) and to prove that the original TTP is NP-complete (Thielen & Westphal, 2010).

There are l gadgets, one for each u_i, $i = 1, 2, \ldots, l$. Now we define the *gadget graph* G_S. We create vertices c_j and d_j for $1 \leq j \leq m$, one pair for each clause in S. Join c_j to d_j. Now connect c_j to vertex $u_{i,r}$ iff clause C_j contains the r^{th} occurrence of u_i in S. Similarly, connect c_j to vertex $\overline{u}_{i,r}$ iff clause C_j contains the r^{th} occurrence of $\overline{u}_i$ in S.

To illustrate, let $S = C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5 \wedge C_6 \wedge C_7 \wedge C_8$, where $C_1 = (u_1 \vee u_2 \vee u_3)$, $C_2 = (\overline{u}_1 \vee \overline{u}_2 \vee \overline{u}_3)$, $C_3 = (u_1 \vee \overline{u}_2 \vee u_4)$, $C_4 = (u_2 \vee \overline{u}_3 \vee u_4)$, $C_5 = (\overline{u}_1 \vee u_3 \vee u_4)$, $C_6 = (u_1 \vee \overline{u}_2 \vee \overline{u}_4)$, $C_7 = (u_2 \vee \overline{u}_3 \vee \overline{u}_4)$, and $C_8 = (\overline{u}_1 \vee u_3 \vee \overline{u}_4)$. By definition, S is an instance of 3-SAT. The gadget graph G_S is given in Figure 2.

Since each literal occurs as often as its negation, and each clause has three literals, the number of clauses in S must be even. Hence, $m = 2k$ for some integer $k \geq 1$. From the instance S, we will define a set X_S with $18k$ vertices corresponding to the teams in league X. We will then define another set Y_S, with just 3 vertices (labelled p, q, and r), and place

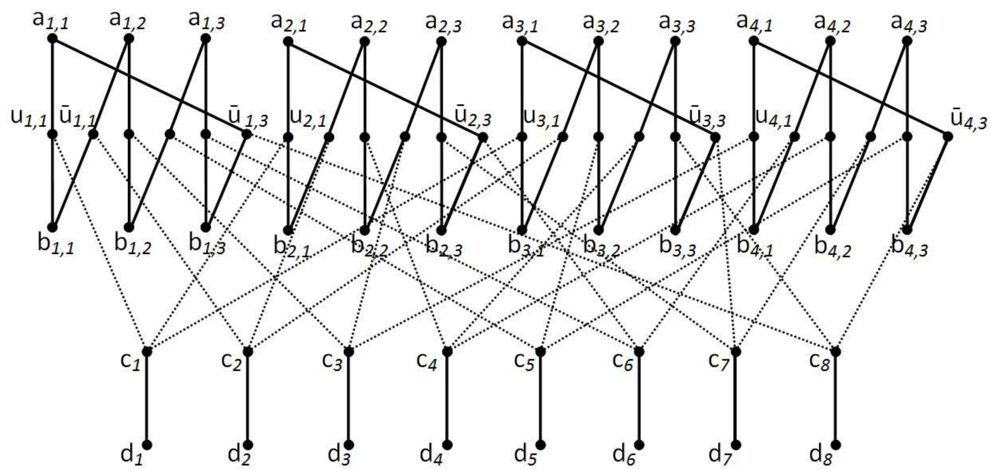

Figure 2: The gadget graph G_S for the instance S.

$6k$ teams at each of these three vertices. This will create a $36k$-team league, with $18k$ teams in both X and Y. The weight of each edge will just correspond to the distance between the teams located at those vertices. Using the gadget graph G_S, we will define the edge weights in such a way that S is satisfiable iff the solutions to $BTTP$ and $BTTP*$ have total distance at most $T = T(k) = 96k^2(2900k^2 + 375k + 11)$. This will establish the desired polynomial-time reductions from 3-SAT.

We first define X_S. Let $C = \{c_1, c_2, \ldots, c_{2k}\}$ and $D = \{d_1, d_2, \ldots, d_{2k}\}$, which are the same set of vertices from the corresponding gadget graph G_S. Let U be the set of $6k$ vertices of the form $u_{i,r}$ or $\overline{u}_{i,r}$ that appear in G_S, and let A and B be respectively the set of vertices of the form $a_{i,r}$ and $b_{i,r}$ that appear in G_S. Finally, we present two additional sets, $E = \{e_1, e_2, \ldots, e_k\}$ and $F = \{f_1, f_2, \ldots, f_k\}$, which will be matched up to the vertices of U in our cycle cover.

We define $X_S = A \cup B \cup C \cup D \cup E \cup F \cup U$. Hence, $|X_S| = |A|+|B|+|C|+|D|+|E|+|F|+|U| = 3k+3k+2k+2k+k+k+6k = 18k$.

Having defined X_S, we now define the edge weights connecting each pair of vertices in X_S, thus producing a complete graph on $18k$ vertices. The weight of each edge will be a function of k. For readability, we will express each weight as a function of z, where $z := 20k + 1$. To each edge in this complete graph, we assign a weight from the set $\{z^2, z^2 + z, 2z^2 - 1\}$ as follows:

(1) A weight of z^2 is given to every edge that appears in the gadget graph G_S, the $6k^2$ edges from U to E, and the k edges connecting e_i to f_i (for each $1 \le i \le k$).

(2) A weight of $z^2 + z$ is given to the $6k^2$ edges from U to F, the $6k$ edges connecting A to B through a common neighbour in U, and the $6k$ edges connecting D to U through a common neighbour in C.

(3) A weight of $2z^2 - 1$ is given to every other edge.

We now create an inter-league tournament with $36k$ total teams. First, we assign the $18k$ teams in league X to the $18k$ vertices of graph X_S, where the distance between the home venues of two teams is the edge weight between the corresponding two vertices in the complete graph.

Let $Y_S = \{p, q, r\}$. Now define the $18k$ teams in league Y as follows: place $6k$ teams at point p, $6k$ teams at point q, and $6k$ teams at point r.

Therefore, $|X_S \cup Y_S| = 18k + 3$. We now extend our complete graph on $18k$ vertices to include these three additional vertices. To assign an edge weight connecting each pair of "inter-league" vertices, we read off the matrix given in Table 3.

	$p \in Y_S$	$q \in Y_S$	$r \in Y_S$
$a \in A$	z^2	$z^2 + z$	$2z^2 - 1$
$b \in B$	z^2	$2z^2 - 1$	$z^2 + z$
$c \in C$	$2z^2 - 1$	z^2	$z^2 + z$
$d \in D$	z^2	$2z^2 - 1$	z^2
$e \in E$	$2z^2 - 1$	$z^2 + z$	z^2
$f \in F$	z^2	z^2	$2z^2 - 1$
$u \in U$	$z^2 + z$	$z^2 + 2z$	$z^2 + 2z$

Table 3: Weights of all edges connecting X_S to Y_S.

For example, the edge from c_i to p is given a weight of $2z^2 - 1$, for all $i = 1, 2, \ldots, 2k$. We repeat the same process for each of the $7 \times 3 = 21$ pairs connecting a vertex in $X_S = A \cup B \cup C \cup D \cup E \cup F \cup U$ to a vertex in $Y_S = \{p, q, r\}$.

Finally, let the weights of edges pq, pr, and qr all be $2z^2 - 1$. As a result, we have now created a complete graph on the vertex set $X_S \cup Y_S$, and assigned a weight to each edge. Moreover, the weight of each edge appears in the set $\{z^2, z^2 + z, z^2 + 2z, 2z^2 - 1\}$, where $z = 20k + 1$. As most versions of the TTP require the teams to be located at points satisfying the Triangle Inequality, we have chosen the weights in our inter-league variant $BTTP$ to ensure that the Triangle Inequality holds for any triplet of points in $X_S \cup Y_S$.

We now partition the $18k$ vertices of X_S into groups of cardinality at most three and attach them to each $y \in \{p, q, r\} = Y_S$ to produce a union of cycles of length at most 4. More formally, we define the following:

Definition 1. *For each $y \in Y_S$, a y-*rooted 4-cycle-cover *is a union of cycles of length at most 4, where every cycle contains y, no cycle contains a vertex from $Y_S \backslash \{y\}$, and every vertex of X_S appears in exactly one cycle.*

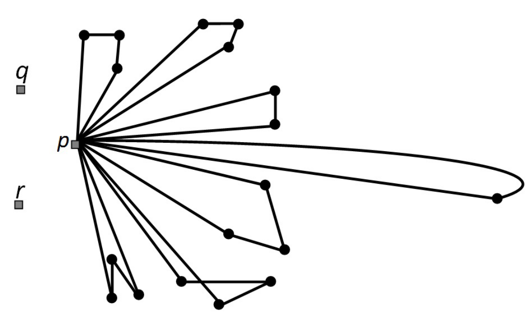

Figure 3: A p-rooted 4-cycle-cover with 18 vertices in set X_S.

To illustrate, Figure 3 gives a p-rooted 4-cycle-cover with $|X_S| = 18$. This definition is motivated by our tournament construction, where we will show that the total travel distance is minimized by creating a uniform schedule where each team takes the maximum number of three-game road trips to play their $18k$ away games. In the case of the $6k$ teams of Y_S located at vertex p, their $6k$ three-game road trips will correspond to the $6k$ 4-cycles in the minimum weight p-rooted 4-cycle-cover. For example, if p-$u_{1,1}$-c_5-d_5-p appears as one of the $6k$ cycles, then each team in Y_S located at vertex p will play three consecutive road games during the tournament against the teams of X_S located at $u_{1,1}$, c_5, and d_5.

So the total distance traveled by each team at $y \in Y_S$ is bounded below by the sum of the edge weights of the minimum weight y-rooted 4-cycle-cover.

Definition 2. *We define three special types of cycles that may appear in a p-rooted 4-cycle-cover.*

(1) A (p, a, u, b, p)-cycle is a 4-cycle with vertices p, a, u, b in that order, where $p \in Y_S$, $a \in A$, $u \in U$, $b \in B$, where au and ub are both edges in the gadget graph G_S.

(2) A (p, u, c, d, p)-cycle is a 4-cycle with vertices p, u, c, d in that order, where $p \in Y_S$, $u \in U$, $c \in C$, $d \in D$, where uc and cd are both edges in the gadget graph G_S.

(3) A (p, u, e, f, p)-cycle is a 4-cycle with vertices p, u, e, f in that order, where $p \in Y_S$, $u \in U$, $e \in E$, $f \in F$, where e and f have the same index (i.e., e_i and f_i for some $1 \leq i \leq k$.)

For example, for our instance S whose gadget graph was illustrated in Figure 2, p-$a_{1,2}$-$\overline{u}_{1,1}$-$b_{1,1}$-p is a (p, a, u, b, p)-cycle, but p-$a_{1,2}$-$\overline{u}_{1,1}$-$b_{4,2}$-p is not. Similarly, p-$\overline{u}_{4,3}$-c_8-d_8-p is a (p, u, c, d, p)-cycle, but p-$\overline{u}_{4,3}$-c_7-d_7-p is not.

Following the convention of the TTP (Easton, Nemhauser, & Trick, 2002), we define ILB_t to be the *individual lower bound* for team t. This value represents the minimum possible distance that can be traveled by team t in order to complete all of their games under the constraints of $BTTP$, independent of the other teams' schedules. By definition, for each team t located at $y \in Y_S$, the value of ILB_t is the minimum weight of a y-rooted 4-cycle-cover.

Similarly, we define the *league lower bound* LLB_T to be the minimum possible distance traveled by all of the teams t in league T, and the *tournament lower bound* TLB to be the minimum possible distance traveled by all the teams in both leagues. We note the following trivial inequalities:

$$TLB \geq LLB_X + LLB_Y$$
$$LLB_X \geq \sum_{t \in X} ILB_t \ , \quad LLB_Y \geq \sum_{t \in Y} ILB_t \ .$$

By definition, the solution to $BTTP$ is a tournament schedule whose total travel distance is TLB.

We now have all of the definitions we need to complete the proof of the NP-completeness of $BTTP$ and $BTTP^*$. We will create an inter-league tournament between the $18k$ teams of X_S and the $18k$ teams of Y_S (with one-third of the teams at each vertex of Y_S), and show that there exists a distance-optimal uniform tournament with total distance at most $T(k) = 96k^2(2900k^2 + 375k + 11)$ iff S is satisfiable. This will establish our polynomial-time reduction from 3-SAT, since all of the transformations in our construction are clearly polynomial.

The desired result will follow from the next four lemmas. In each lemma, we let K_S be the complete graph on the $18k + 3$ vertices of $X_S \cup Y_S$, with edge weights as described in our construction. For the interested reader, the proofs to these three lemmas appear in Appendix A.

Lemma 1. *The following statements are equivalent:*

(i) $S = C_1 \wedge C_2 \wedge \ldots \wedge C_{2k}$ *is satisfiable.*

(ii) There exists a p*-rooted* 4*-cycle-cover of* K_S *with exactly* $3k$ (p, a, u, b, p)*-cycles,* $2k$ (p, u, c, d, p)*-cycles, and* k (p, u, e, f, p)*-cycles.*

Lemma 2. *The following statements are equivalent:*

(i) A p*-rooted* 4*-cycle-cover of* K_S *has exactly* $3k$ (p, a, u, b, p)*-cycles,* $2k$ (p, u, c, d, p)*-cycles, and* k (p, u, e, f, p)*-cycles.*

(ii) A p*-rooted* 4*-cycle-cover of* K_S *has total edge weight* $k(24z^2 + 3z)$*.*

Lemma 3. *Let* ILB_y *be the minimum total edge weight of a* y*-rooted* 4*-cycle-cover of* K_S*. Then*

$$ILB_y = \begin{cases} k(24z^2 + 3z) & \text{if } y = p \\ k(24z^2 + 20z) & \text{if } y = q \\ k(24z^2 + 19z) & \text{if } y = r \end{cases}$$

Let us illustrate these three lemmas with a specific example. Let $S = C_1 \wedge C_2 \wedge C_3 \wedge C_4 \wedge C_5 \wedge C_6 \wedge C_7 \wedge C_8$ be the instance of 3-SAT whose gadget graph G_S was presented in Figure 2. Recall that we defined $C_1 = (u_1 \vee u_2 \vee u_3)$, $C_2 = (\overline{u}_1 \vee \overline{u}_2 \vee \overline{u}_3)$, $C_3 = (u_1 \vee \overline{u}_2 \vee u_4)$, $C_4 = (u_2 \vee \overline{u}_3 \vee u_4)$, $C_5 = (\overline{u}_1 \vee u_3 \vee u_4)$, $C_6 = (u_1 \vee \overline{u}_2 \vee \overline{u}_4)$, $C_7 = (u_2 \vee \overline{u}_3 \vee \overline{u}_4)$, and $C_8 = (\overline{u}_1 \vee u_3 \vee \overline{u}_4)$.

Suppose S is satisfiable, i.e., there is a function $\phi : \{u_1, u_2, u_3, u_4\} \to \{\text{TRUE}, \text{FALSE}\}$ so that each clause C_i evaluates to TRUE for all $1 \leq i \leq 8$. By symmetry, we may assume without loss that $\phi(u_4)$ is TRUE. Then from clauses C_6, C_7, and C_8, we see that for $1 \leq i \leq 3$, $\phi(u_i)$ must be all TRUE or all FALSE. In the former, clause C_2 is FALSE, and in the latter, clause C_1 is FALSE. Therefore, S is not satisfiable.

Since S is not satisfiable, by Lemma 1, there does not exist a p-rooted 4-cycle-cover of K_S with 12 (p, a, u, b, p)-cycles, 8 (p, u, c, d, p)-cycles, and 4 (p, u, e, f, p)-cycles. And by Lemma 2 and Lemma 3, the minimum weight of a p-rooted 4-cycle-cover of K_S is strictly larger than $4(24z^2 + 3z)$.

We now show that such a (non-satisfiable) instance S cannot yield a graph K_S forming a distance-optimal inter-league tournament, but that a satisfiable instance S indeed does.

Just as we defined special 4-cycles rooted at p (e.g. (p, a, u, b, p)-cycles), we can similarly define 4-cycles rooted at q and r. In Lemma 3, the lower bound ILB_q occurs when the q-rooted 4-cycle-cover consists of $3k$ (q, u, b, a, q)-cycles, $2k$ (q, c, d, u, q)-cycles, and k

(q, f, u, e, q)-cycles, with total edge weight $3k(4z^2 + 4z) + 2k(4z^2 + 3z) + k(4z^2 + 2z) = k(24z^2 + 20z)$. The lower bound ILB_r occurs when the r-rooted 4-cycle-cover consists of $3k$ (r, b, a, u, r)-cycles, $2k$ (r, d, u, c, r)-cycles, and k (r, e, f, u, r)-cycles, with total edge weight $3k(4z^2 + 4z) + 2k(4z^2 + 2z) + k(4z^2 + 3z) = k(24z^2 + 19z)$. We apply this information in the following lemma in constructing our distance-optimal bipartite tournament.

Lemma 4. *If S is satisfiable, then there exists a uniform schedule (i.e., a solution to BTTP as well as BTTP*) whose total travel distance is $\sum ILB_t = k^2(696z^2 + 408z - 48) = 96k^2(2900k^2 + 375k + 11)$.*

Proof. From Lemma 1, if S is satisfiable, there exists a p-rooted 4-cycle-cover of K_S with exactly $3k$ (p, a, u, b, p)-cycles, $2k$ (p, u, c, d, p)-cycles, and k (p, u, e, f, p)-cycles. Consider such a p-rooted 4-cycle-cover. We now relabel the teams in X_S as follows:

First let $\{x_0, x_1, x_2\}, \{x_3, x_4, x_5\}, \ldots, \{x_{9k-3}, x_{9k-2}, x_{9k-1}\}$ be the vertices of the $3k$ (p, a, u, b, p)-cycles, where $x_{3i} \in A$, $x_{3i+1} \in U$, and $x_{3i+2} \in B$ for all $0 \le i \le 3k - 1$.

Then let $\{x_{9k}, x_{9k+1}, x_{9k+2}\}, \{x_{9k+3}, x_{9k+4}, x_{9k+5}\}, \ldots, \{x_{15k-3}, x_{15k-2}, x_{15k-1}\}$ be the vertices of the $2k$ (p, u, c, d, p)-cycles, where $x_{3i} \in U$, $x_{3i+1} \in C$, and $x_{3i+2} \in D$ for all $3k \le i \le 5k - 1$.

Finally, let $\{x_{15k}, x_{15k+1}, x_{15k+2}\}, \{x_{15k+3}, x_{15k+4}, x_{15k+5}\}, \ldots, \{x_{18k-3}, x_{18k-2}, x_{18k-1}\}$ be the vertices of the k (p, u, e, f, p)-cycles, where $x_{3i} \in U$, $x_{3i+1} \in E$, and $x_{3i+2} \in F$ for all $5k \le i \le 6k - 1$.

To explain our proof more clearly, we use this relabeling of the teams in X_S, letting each vertex be x_i for some $0 \le i \le 18k - 1$. We also relabel the teams in Y_S, so that $\{p_0, p_1, \ldots, p_{6k-1}\}$ are the teams at p, $\{q_0, q_1, \ldots, q_{6k-1}\}$ are the teams at q, and $\{r_0, r_1, \ldots, r_{6k-1}\}$ are the teams at r.

Since every team plays two games against each of the $18k$ teams in the other league, the tournament has $36k$ time slots. We now build a double round-robin bipartite tournament where the teams in each league play their home games in the same slots (i.e., the schedule is *uniform*.) Specifically, each team in X_S will play three consecutive home games followed by three consecutive road games and repeat that pattern $6k$ times. Similarly, each team in Y_S will play three consecutive road games followed by three consecutive home games and repeat that pattern until the end of the tournament. Given the way we constructed the edge weights, this is the natural way to construct a distance-optimal tournament, where each team takes as few trips as possible.

In Lemma 3, we determined the value of ILB_v for each $v \in Y_S = P \cup Q \cup R$. we have $ILB_{p_i} = k(24z^2 + 3z)$, $ILB_{q_i} = k(24z^2 + 20z)$, and $ILB_{r_i} = k(24z^2 + 19z)$, for all $0 \le i \le 6k - 1$. Therefore, $LLB_{Y_S} \ge 6k^2(24z^2 + 3z) + 6k^2(24z^2 + 20z) + 6k^2(24z^2 + 19z) = 6k^2(72z^2 + 42z)$.

We now determine the value of ILB_t for each $t \in X_S = A \cup B \cup C \cup D \cup E \cup F \cup U$. Every team $t \in X_S$ plays a road game against each of the $18k$ teams in Y_S, with $6k$ teams located at points p, q, and r. Team t must make at least $\frac{6k}{3} = 2k$ trips to each of p, q, and r, since the maximum length of a road trip is three games. Therefore, $ILB_t \ge 2k(D_{t,p} + D_{t,q} + D_{t,r})$, where $D_{t,v}$ is the distance from $t \in X_S$ to $y \in Y_S$ for all choices of t and y. Note that equality can occur, specifically when the road trips of team t are scheduled in the most efficient way, with each trip consisting of three consecutive games against three teams located at the same point.

From Table 3, we determine that $ILB_t = 2k(D_{t,p} + D_{t,q} + D_{t,r}) = 4k(4z^2 + z - 1)$ for all $t \in A \cup B \cup C \cup E$. Similarly, $ILB_t = 4k(4z^2 - 1)$ for all $t \in D \cup F$, and $ILB_t = 4k(3z^2 + 5z)$ for all $t \in U$. Thus, we have

$$
\begin{aligned}
& LLB_{X_S} \\
\geq\ & 4k(4z^2 + z - 1)(|A| + |B| + |C| + |E|) + 4k(4z^2 - 1)(|D| + |F|) + 4k(3z^2 + 5z)(|U|) \\
=\ & 4k(4z^2 + z - 1)(3k + 3k + 2k + k) + 4k(4z^2 - 1)(2k + k) + 4k(3z^2 + 5z)(6k) \\
=\ & 36k^2(4z^2 + z - 1) + 12k^2(4z^2 - 1) + 24k^2(3z^2 + 5z) \\
=\ & k^2(264z^2 + 156z - 48).
\end{aligned}
$$

Therefore, $TLB \geq LLB_{X_S} + LLB_{Y_S} \geq \sum ILB_t = k^2(264z^2 + 156z - 48) + 6k^2(72z^2 + 42z) = k^2(696z^2 + 408z - 48)$. To complete the proof, it suffices to construct a tournament for which each team's total travel distance matches its individual lower bound. This will prove that $TLB = \sum ILB_t = k^2(696z^2 + 408z - 48)$.

For each $0 \leq i \leq 6k - 1$ and $0 \leq j \leq 6k - 1$, we determine the opponent of teams p_i, q_i, r_i in time slots $6j + 1$, $6j + 2$, $6j + 3$, $6j + 4$, $6j + 5$, and $6j + 6$. In Table 4, we provide the schedule of games in slots $6j + 1$, $6j + 2$, and $6j + 3$, where the teams in X_S play at home and the teams in Y_S play on the road. In this table, the function $f(i, j)$ is always reduced modulo $18k$, so that $x_{18k+z} := x_z$ for all $0 \leq z \leq 18k - 1$.

Game	$6j+1$	$6j+2$	$6j+3$
p_i	$x_{3(i+j)+0}$	$x_{3(i+j)+1}$	$x_{3(i+j)+2}$
q_i	$x_{3(i+j)+1}$	$x_{3(i+j)+2}$	$x_{3(i+j)+0}$
r_i	$x_{3(i+j)+2}$	$x_{3(i+j)+0}$	$x_{3(i+j)+1}$

Game	$6j+1$	$6j+2$	$6j+3$
p_i	$x_{3(i+j)+0}$	$x_{3(i+j)+1}$	$x_{3(i+j)+2}$
q_i	$x_{3(i+j)+2}$	$x_{3(i+j)+0}$	$x_{3(i+j)+1}$
r_i	$x_{3(i+j)+1}$	$x_{3(i+j)+2}$	$x_{3(i+j)+0}$

Table 4: The left table lists the schedule of matches when i and j satisfy $i + j \in \{0, 1, \ldots, 5k - 1\} \pmod{6k}$, while the right table lists the schedule when i and j satisfy $i + j \in \{5k, 5k + 1, \ldots, 6k - 1\} \pmod{6k}$.

Fix i. By this construction, each team p_i, q_i, r_i will play each of $\{x_0, x_1, \ldots, x_{18k-1}\}$ on the road exactly once. Now fix j. In time slot $6j + k$ (with $1 \leq k \leq 3$), each team in X_S appears exactly once, playing a unique opponent from Y_S. Each team's schedule corresponds to a rooted 4-cycle-cover. By our labeling scheme, the 4-cycle-cover of each team p_i consists of $3k$ (p, a, u, b, p)-cycles, $2k$ (p, u, c, d, p)-cycles and k (p, u, e, f, p)-cycles. Similarly, the 4-cycle-cover of each team q_i consists of $3k$ (q, u, b, a, q)-cycles, $2k$ (q, c, d, u, q)-cycles, and k (q, f, u, e, q)-cycles. Finally, the 4-cycle-cover of each team r_i consists of $3k$ (r, b, a, u, r)-cycles, $2k$ (r, d, u, c, r)-cycles, and k (r, e, f, u, r)-cycles. Therefore, each team in Y_S plays their $6k$ road trips so that its total travel distance is equal to the minimum weight of a 4-cycle-cover rooted at that vertex, which by definition is equal to that team's individual lower bound. Thus, we have constructed a schedule with $LLB_{Y_S} = 6k^2(72z^2 + 42z)$.

Now we construct the other half of our schedule, where the teams in Y_S play at home and the teams in X_S play on the road. This is a much simpler construction. For example, one way to build this half of the schedule is to match each triplet of teams in X_S (e.g. $\{x_0, x_1, x_2\}$) with a triplet of teams from the same vertex in Y_S (e.g. $\{p_0, p_1, p_2\}$), and have three consecutive slots of games between the two triplets all at the home venues of the teams in league Y_S. Repeating this process, we can ensure that each of the $6k$ triplets in X_S play all $6k$ triplets of Y_S via three-game road trips. Thus, this schedule satisfies $LLB_{X_S} = k^2(264z^2 + 156z - 48)$.

All that is required when putting the schedules together is to ensure the *no-repeat* rule, which is a simple matter given all of the flexibility we have in constructing this half of the tournament schedule.

Therefore, we have completed our proof. If S is satisfiable, then the bipartite tournament with teams $X_S \cup Y_S$ has $TLB = \sum ILB_t = k^2(696z^2 + 408z - 48)$. Recalling that $z = 20k+1$, we conclude that $TLB = 96k^2(2900k^2 + 375k + 11)$. □

To illustrate the preceding proof, Table 5 gives a distance-optimal schedule for the case $k = 1$, with 18 teams in each league. We just present the schedule for the teams in Y_S since we can immediately derive the schedule for the teams in X_S from this table. As always, home games are marked in bold.

Game	1	2	3	4	5	6	7	8	9	10	11	12	…	…	31	32	33	34	35	36
p_0	x_0	x_1	x_2	$\boldsymbol{x_0}$	$\boldsymbol{x_2}$	$\boldsymbol{x_1}$	x_3	x_4	x_5	$\boldsymbol{x_3}$	$\boldsymbol{x_5}$	$\boldsymbol{x_4}$	…	…	x_{15}	x_{16}	x_{17}	$\boldsymbol{x_9}$	$\boldsymbol{x_{11}}$	$\boldsymbol{x_{10}}$
q_0	x_1	x_2	x_0	$\boldsymbol{x_6}$	$\boldsymbol{x_8}$	$\boldsymbol{x_7}$	x_4	x_5	x_3	$\boldsymbol{x_9}$	$\boldsymbol{x_{11}}$	$\boldsymbol{x_{10}}$	…	…	x_{17}	x_{15}	x_{16}	$\boldsymbol{x_{15}}$	$\boldsymbol{x_{17}}$	$\boldsymbol{x_{16}}$
r_0	x_2	x_0	x_1	$\boldsymbol{x_{12}}$	$\boldsymbol{x_{14}}$	$\boldsymbol{x_{13}}$	x_5	x_3	x_4	$\boldsymbol{x_{15}}$	$\boldsymbol{x_{17}}$	$\boldsymbol{x_{16}}$	…	…	x_{16}	x_{17}	x_{15}	$\boldsymbol{x_3}$	$\boldsymbol{x_5}$	$\boldsymbol{x_4}$
p_1	x_3	x_4	x_5	$\boldsymbol{x_1}$	$\boldsymbol{x_0}$	$\boldsymbol{x_2}$	x_6	x_7	x_8	$\boldsymbol{x_4}$	$\boldsymbol{x_3}$	$\boldsymbol{x_5}$	…	…	x_0	x_1	x_{12}	$\boldsymbol{x_{10}}$	$\boldsymbol{x_9}$	$\boldsymbol{x_{11}}$
q_1	x_4	x_5	x_3	$\boldsymbol{x_7}$	$\boldsymbol{x_6}$	$\boldsymbol{x_8}$	x_7	x_8	x_6	$\boldsymbol{x_{10}}$	$\boldsymbol{x_9}$	$\boldsymbol{x_{11}}$	…	…	x_1	x_2	x_0	$\boldsymbol{x_{16}}$	$\boldsymbol{x_{15}}$	$\boldsymbol{x_{17}}$
r_1	x_5	x_3	x_4	$\boldsymbol{x_{13}}$	$\boldsymbol{x_{12}}$	$\boldsymbol{x_{14}}$	x_8	x_6	x_7	$\boldsymbol{x_{16}}$	$\boldsymbol{x_{15}}$	$\boldsymbol{x_{17}}$	…	…	x_2	x_0	x_1	$\boldsymbol{x_4}$	$\boldsymbol{x_3}$	$\boldsymbol{x_5}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
p_5	x_{15}	x_{16}	x_{17}	$\boldsymbol{x_5}$	$\boldsymbol{x_4}$	$\boldsymbol{x_3}$	x_0	x_1	x_2	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_0}$	…	…	x_{12}	x_{13}	x_{14}	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_6}$
q_5	x_{17}	x_{15}	x_{16}	$\boldsymbol{x_{11}}$	$\boldsymbol{x_{10}}$	$\boldsymbol{x_9}$	x_1	x_2	x_0	$\boldsymbol{x_7}$	$\boldsymbol{x_8}$	$\boldsymbol{x_6}$	…	…	x_{13}	x_{14}	x_{12}	$\boldsymbol{x_{13}}$	$\boldsymbol{x_{14}}$	$\boldsymbol{x_{12}}$
r_5	x_{16}	x_{17}	x_{15}	$\boldsymbol{x_{17}}$	$\boldsymbol{x_{16}}$	$\boldsymbol{x_{15}}$	x_2	x_0	x_1	$\boldsymbol{x_{13}}$	$\boldsymbol{x_{14}}$	$\boldsymbol{x_{12}}$	…	…	x_{14}	x_{12}	x_{13}	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_0}$

Table 5: A distance-optimal inter-league tournament with 18 teams in each league.

Having provided all of the lemmas, we can now prove the main theorem of this paper.

Theorem 1. *BTTP and BTTP* are NP-complete.*

Proof. Let S be an instance of 3-SAT with $2k$ clauses, and create sets X_S and Y_S, with edge weights as described in our construction. Consider an inter-league tournament between the $18k$ teams at X_S and the $18k$ teams at Y_S (with one-third of the teams at each vertex of Y_S).

By Lemma 4, if S is satisfiable, then there exists a uniform double round-robin bipartite tournament with total distance at most $96k^2(2900k^2 + 375k + 11)$. By definition, this tournament is a feasible solution to $BTTP$ and $BTTP^*$. We now prove the converse.

Let $T(k) = 96k^2(2900k^2 + 375k + 11)$. Consider an inter-league tournament between these $36k$ teams with total travel distance at most $T(k)$. By Lemma 4, $T(k) = \sum ILB_t$. Hence, *every* team $t \in X_S \cup Y_S$ must travel the shortest possible distance of ILB_t to play all of their games. By Lemma 3, this implies that every team located at $p \in Y_S$ must travel a distance of $ILB_p = k(24z^2 + 3z)$.

By Lemma 2, if each team $p \in Y_S$ travels a distance of $k(24z^2 + 3z)$, then the graph K_S must contain exactly $3k$ (p, a, u, b, p)-cycles, $2k$ (p, u, c, d, p)-cycles, and k (p, u, e, f, p)-cycles. And by Lemma 1, this occurs iff S is satisfiable.

Therefore, we have constructed a double round-robin bipartite tournament K_S on $36k$ teams with distance matrix D_S for which the solutions to $BTTP$ and $BTTP^*$ have total distance $\leq T(k)$ iff the instance S with $2k$ clauses is satisfiable. This establishes the desired polynomial-time reduction from 3-SAT, proving the NP-hardness of *BTTP* and *BTTP**. Finally, we note that both problems are clearly in NP, since the distance traveled by the teams can be calculated in polynomial time. Therefore, we conclude that *BTTP* and *BTTP** are NP-complete. $\square$

To illustrate the difference between *BTTP* and *BTTP**, we provide a concrete illustration for the case $n = 3$. Let the teams be $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. In Figure 4, the teams are located on the Cartesian plane, where x_1 and x_2 represent the same point, y_1 and y_2 represent the same point, and the non-negative distances a, b, c satisfy the Pythagorean equation $a^2 + b^2 = c^2$.

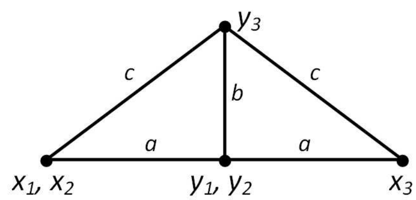

Figure 4: Illustration of *BTTP* for the case $n = 3$.

It is straightforward to show that $ILB_{x_1} = ILB_{x_2} = ILB_{x_3} = a + b + c$, $ILB_{y_1} = ILB_{y_2} = 4a$ and $ILB_{y_3} = 2a + 2c$. Hence, $TLB \geq LLB_X + LLB_Y \geq (3a + 3b + 3c) + (10a + 2c) = 13a + 3b + 5c$.

In order for $TLB = 13a + 3b + 5c$, each of the teams in X must play a three-game road stand with consecutive games against y_1 and y_2, and each of the teams in Y must play a three-game road stand with consecutive games against x_1 and x_2. One can quickly show that such a scenario is impossible, but that a "nearly-best" schedule can be achieved by making either y_1 or y_2 take an extra trip, adding $2a$ to the total distance. Hence, the solution to *BTTP* must have distance at least $15a + 3b + 5c$.

For *BTTP**, we have $TLB \geq 16a + 4b + 4c$ since $LLB_X = 4a + 4b + 2c$ and $LLB_Y = 12a + 2c$ in a uniform schedule. For both problems, we justify optimality by presenting a feasible tournament satisfying the stated tournament lower bounds. This is presented in Table 6.

Team	1	2	3	4	5	6
x_1	y_1	y_2	y_3	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$
x_2	y_2	y_3	y_1	$\boldsymbol{y_2}$	$\boldsymbol{y_3}$	$\boldsymbol{y_1}$
x_3	y_3	y_1	y_2	$\boldsymbol{y_3}$	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$
y_1	$\boldsymbol{x_1}$	$\boldsymbol{x_3}$	$\boldsymbol{x_2}$	x_1	x_3	x_2
y_2	$\boldsymbol{x_2}$	$\boldsymbol{x_1}$	$\boldsymbol{x_3}$	x_2	x_1	x_3
y_3	$\boldsymbol{x_3}$	$\boldsymbol{x_2}$	$\boldsymbol{x_1}$	x_3	x_2	x_1

Team	1	2	3	4	5	6
x_1	$\boldsymbol{y_1}$	$\boldsymbol{y_3}$	y_2	y_1	y_3	$\boldsymbol{y_2}$
x_2	$\boldsymbol{y_2}$	$\boldsymbol{y_1}$	$\boldsymbol{y_3}$	y_2	y_1	y_3
x_3	$\boldsymbol{y_3}$	$\boldsymbol{y_2}$	$\boldsymbol{y_1}$	y_3	y_2	y_1
y_1	x_1	x_2	x_3	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$
y_2	x_2	x_3	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$	x_1
y_3	x_3	x_1	x_2	$\boldsymbol{x_3}$	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$

Table 6: Solutions to *BTTP** and *BTTP*, with total distance $16a+4b+4c$ and $15a+3b+5c$, respectively.

For this example, the solution to $BTTP$ requires 25 trips, one more trip than the solution to $BTTP$*, yet the tournament lower bound is reduced by $a + b - c > 0$. As we will see in the concluding section, there are many examples where the solution to the $n = 3$ $BTTP$ requires more than 24 trips.

To illustrate with an example for the case $n = 6$, consider a 12-team league with six teams in each of X and Y. Place three points A, B, C equally spaced around a unit circle, so that $\triangle ABC$ is equilateral. Place $\{x_1, x_2\}$ at A, $\{x_3, x_4\}$ at B, and $\{x_5, x_6\}$ at C. Now place $\{y_1, y_2, \ldots, y_6\}$ at the centre of the circle. Then the best lower bound of $ILB_{y_j} = 6$ occurs when y_j plays two-game road trips against $\{x_1, x_2\}$, $\{x_3, x_4\}$, $\{x_5, x_6\}$ in pairs rather than in three-game road trips such as $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$, which has total distance $4+2\sqrt{3} > 6$. And clearly the best lower bound $ILB_{x_i} = 4$ occurs when x_i plays three-game road trips against the teams in Y, making just two trips to the centre of the circle.

Table 7 provides an distance-optimal schedule which is uniform, thus proving that for this simple example, the solution to $BTTP$ is the same as $BTTP$*. However, note that unlike our proof of Theorem 1, in this 12-team scenario, the best schedule requires 102 total trips, six more than the fewest possible number of total trips.

	1	2	3	4	5	6	7	8	9	10	11	12
x_1	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	y_1	y_2	y_3	$\boldsymbol{y_4}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$
x_2	$\boldsymbol{y_2}$	$\boldsymbol{y_1}$	y_2	y_3	y_1	$\boldsymbol{y_3}$	$\boldsymbol{y_4}$	y_5	y_6	y_4	$\boldsymbol{y_6}$	$\boldsymbol{y_5}$
x_3	$\boldsymbol{y_3}$	$\boldsymbol{y_4}$	y_3	y_1	y_2	$\boldsymbol{y_6}$	$\boldsymbol{y_5}$	y_6	y_4	y_5	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$
x_4	$\boldsymbol{y_4}$	$\boldsymbol{y_3}$	y_4	y_5	y_6	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_1	y_2	y_3	$\boldsymbol{y_2}$	$\boldsymbol{y_1}$
x_5	$\boldsymbol{y_5}$	$\boldsymbol{y_6}$	y_5	y_6	y_4	$\boldsymbol{y_2}$	$\boldsymbol{y_1}$	y_2	y_3	y_1	$\boldsymbol{y_3}$	$\boldsymbol{y_4}$
x_6	$\boldsymbol{y_6}$	$\boldsymbol{y_5}$	y_6	y_4	y_5	$\boldsymbol{y_1}$	$\boldsymbol{y_2}$	y_3	y_1	y_2	$\boldsymbol{y_4}$	$\boldsymbol{y_3}$
y_1	x_1	x_2	$\boldsymbol{x_1}$	$\boldsymbol{x_3}$	$\boldsymbol{x_2}$	x_6	x_5	$\boldsymbol{x_4}$	$\boldsymbol{x_6}$	$\boldsymbol{x_5}$	x_3	x_4
y_2	x_2	x_1	$\boldsymbol{x_2}$	$\boldsymbol{x_1}$	$\boldsymbol{x_3}$	x_5	x_6	$\boldsymbol{x_5}$	$\boldsymbol{x_4}$	$\boldsymbol{x_6}$	x_4	x_3
y_3	x_3	x_4	$\boldsymbol{x_3}$	$\boldsymbol{x_2}$	$\boldsymbol{x_1}$	x_2	x_1	$\boldsymbol{x_6}$	$\boldsymbol{x_5}$	$\boldsymbol{x_4}$	x_5	x_6
y_4	x_4	x_3	$\boldsymbol{x_4}$	$\boldsymbol{x_6}$	$\boldsymbol{x_5}$	x_1	x_2	$\boldsymbol{x_1}$	$\boldsymbol{x_3}$	$\boldsymbol{x_2}$	x_6	x_5
y_5	x_5	x_6	$\boldsymbol{x_5}$	$\boldsymbol{x_4}$	$\boldsymbol{x_6}$	x_4	x_3	$\boldsymbol{x_2}$	$\boldsymbol{x_1}$	$\boldsymbol{x_3}$	x_1	x_2
y_6	x_6	x_5	$\boldsymbol{x_6}$	$\boldsymbol{x_5}$	$\boldsymbol{x_4}$	x_3	x_4	$\boldsymbol{x_3}$	$\boldsymbol{x_2}$	$\boldsymbol{x_1}$	x_2	x_1

Table 7: Solution to $BTTP$ and $BTTP$* for a scenario with $n = 6$.

Having provided simple illustrations for $n = 3$ and $n = 6$, we now analyze $BTTP$ for two professional sports leagues, namely the Nippon Professional Baseball league (with $n = 6$) and the National Basketball Association (with $n = 15$).

4. Japanese Baseball

Nippon Professional Baseball (NPB) is Japan's largest professional sports league. In the NPB, the teams are split into two leagues of six teams, with each team playing 120 intra-league and 24 inter-league games during the regular season. The intra-league problem was analyzed recently by the authors (Hoshino & Kawarabayashi, 2011c), where we developed a multi-round generalization of the TTP based on Dijkstra's shortest path algorithm and applied it to produce a distance-optimal schedule reducing the total travel distance by over 60000 kilometres (a 25% reduction) as compared to the 2010 NPB intra-league schedule (Hoshino & Kawarabayashi, 2011d). Given that Japan is a small island country, a 60000 kilometre reduction represents a significant amount.

We now consider the inter-league problem, where the six teams in the NPB Pacific League each play four games against all six teams in the NPB Central League, with one two-game *set* played at the home of the Pacific League team, and the other two-game set played at the home of the Central League team. All inter-league games take place during a five-week stretch between mid-May and mid-June, with no intra-league games occurring during that period. Thus, the NPB inter-league scheduling problem is precisely $BTTP$, for the case $n = 6$.

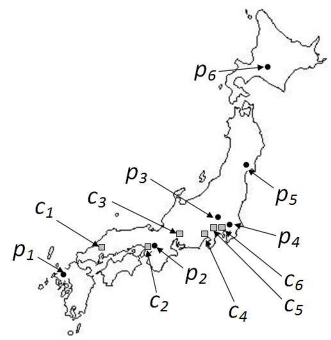

Figure 5: Location of the 12 teams in the NPB.

The locations of each team's home stadium is marked in Figure 5. For readability, we label each team as follows: the Pacific League teams are p_1 (Fukuoka), p_2 (Orix), p_3 (Saitama), p_4 (Chiba), p_5 (Tohoku), p_6 (Hokkaido), and the Central League teams are c_1 (Hiroshima), c_2 (Hanshin), c_3 (Chunichi), c_4 (Yokohama), c_5 (Yomiuri), and c_6 (Yakult). The actual 12×12 NPB distance matrix is provided in Appendix B.

We now solve $BTTP$ for the NPB, producing an inter-league schedule requiring 42950 kilometres of total travel, representing a 16% reduction compared to the 51134 kilometres traveled by these teams during the 2010 inter-league schedule (Hoshino & Kawarabayashi, 2011a). To accomplish this, we present two powerful reduction heuristics. To motivate these heuristics, we first require several key definitions.

For each $t \in X \cup Y$, let S_t be the set of possible schedules that can be played by team t satisfying the *at-most-three* and *each-venue* constraints. Let $\pi_t \in S_t$ be a possible schedule for team t. For each π_t, we just list the opponents of the six *road* sets, and ignore the home sets, since we can determine the total distance traveled by team t just from the road sets. To give an example, below is a feasible schedule $\pi_{x_1} \in S_{x_1}$ for the case $n = 6$:

	1	2	3	4	5	6	7	8	9	10	11	12
x_1	y_1	y_6	$\boldsymbol{y}$	$\boldsymbol{y}$	y_3	y_5	y_4	$\boldsymbol{y}$	$\boldsymbol{y}$	$\boldsymbol{y}$	y_2	$\boldsymbol{y}$

In the following team schedule π_{x_1}, each $\boldsymbol{y}$ represents a home set played by x_1 against a unique opponent in Y. Note that π_{x_1} satisfies the *at-most-three* and *each-venue* constraints.

Let $\Phi = (\pi_{x_1}, \pi_{x_2}, \ldots, \pi_{x_n}, \pi_{y_1}, \pi_{y_2}, \ldots, \pi_{y_n})$, where $\pi_t \in S_t$ for each $t \in X \cup Y$. Since road sets of X correspond to home sets of Y and vice-versa, it suffices to list just the time slots and opponents of the n road sets in each π_t, since we can then uniquely determine the full schedule of $2n$ sets for every team $t \in X \cup Y$, thus producing an inter-league tournament schedule Φ. We note that Φ is a feasible solution to $BTTP$ iff each team plays a unique opponent in every time slot, and no team schedule π_t violates the *no-repeat* constraint. In this section, we will frequently refer to *team* schedules π_t and *tournament* schedules Φ. From the context it will be clear whether the schedule is for an individual team $t \in X \cup Y$, or for all $2n$ teams in $X \cup Y$.

As before, define ILB_t to be the individual lower bound of team t, the minimum possible distance that can be traveled by team t in order to complete its $2n$ sets.

For each $\pi_t \in S_t$, let $d(\pi_t)$ be the integer for which $d(\pi_t)+ILB_t$ equals the total distance traveled by team t when playing the schedule π_t. By definition, $d(\pi_t) \geq 0$.

For each $\Phi = (\pi_{x_1}, \ldots, \pi_{x_n}, \pi_{y_1}, \ldots, \pi_{y_n})$, define

$$d(\Phi) = \sum_{t \in X \cup Y} d(\pi_t).$$

Since $\sum ILB_t$ is fixed, the optimal solution to $BTTP$ is the schedule Φ for which $d(\Phi)$ is minimized. This is the motivation for the function $d(\Phi)$.

For each subset $S_t^* \subseteq S_t$, define the *lower bound* function

$$B(S_t^*) = \min_{\pi_t \in S_t^*} d(\pi_t).$$

If $S_t^* = S_t$, then $B(S_t^*) = 0$ by the definition of ILB_t. For each subset S_t^*, we define $|S_t^*|$ to be its cardinality.

For example, consider the $n = 6$ instance in Table 7, where we located the six teams in league X so that two teams were assigned to each vertex of an equilateral triangle. As mentioned at the end of Section 3, $ILB_{y_1} = 6$, with equality occurring iff $y_1 \in Y$ plays two-set road trips against $\{x_1, x_2\}$, $\{x_3, x_4\}$, and $\{x_5, x_6\}$. Now let $S_{y_1}^*$ be the restriction of S_{y_1} to the subset of schedules where y_1 starts with three consecutive road sets against teams x_1, x_2, and x_3. Then any such schedule must have total distance $\geq 4+2\sqrt{3}$, implying that $B(S_{y_1}^*) = 4 + 2\sqrt{3} - ILB_{y_1} = 2\sqrt{3} - 2 > 0$.

If n is a multiple of 3, we define for each team the set R_3^t as the subset of schedules in S_t for which the n road sets occur in $\frac{n}{3}$ blocks of three (i.e., team t takes $\frac{n}{3}$ three-set road trips). For example, in Table 5 (which has $n = 18$), every team t plays a schedule $\pi_t \in R_3^t$.

Finally, we define Γ to be a *global constraint* that fixes some subset of matches, and S_t^Γ to be the subset of schedules in S_t which are consistent with that global constraint. For example, if Γ is the simple constraint that forces y_2 to play against x_1 at home in time slot 3, then $S_{x_1}^\Gamma$ would only consist of the team schedules where slot 3 is a road set against y_2. If Γ is a much more complex global constraint (e.g. where the number of fixed matches is large), then each $|S_t^\Gamma|$ will be significantly less than $|S_t|$.

To illustrate this concept, consider the above $n = 6$ instance and the global constraint Γ that y_1 starts with three consecutive road sets against teams x_1, x_2, and x_3 (in that order). One can show that there are only 34 valid home-road patterns in S_{y_1}, including RRR-HHH-RRR-HHH and RRR-H-R-HH-R-HH-R-H. For each home-road pattern, there are $3! = 6$ ways to assign $\{x_4, x_5, x_6\}$ to the last three road sets. Thus, $|S_{y_1}^\Gamma| = 34 \times 3! = 204$, which is significantly less than $|S_{y_1}|$ which can be shown to equal $616 \times 6! = 443520$.

This simple notion of global constraints inspires our first result, a powerful reduction heuristic that drastically cuts down the computation time.

Proposition 1. *Let M be a fixed positive integer. For any global constraint Γ, define for each $t \in X \cup Y$,*

$$Z_t = \left\{ \pi_t \in S_t^\Gamma : d(\pi_t) \leq M + B(S_t^\Gamma) - \sum_{u \in X \cup Y} B(S_u^\Gamma) \right\}.$$

If $\Phi = (\pi_{x_1}, \ldots, \pi_{x_n}, \pi_{y_1}, \ldots, \pi_{y_n})$ is a feasible tournament schedule consistent with Γ so that $d(\Phi) \leq M$, then for each $t \in X \cup Y$, team t's schedule π_t appears in Z_t.

Proof. Consider all tournament schedules consistent with Γ. If there is no Φ with $d(\Phi) \leq M$, then there is nothing to prove. So assume some schedule Φ satisfies $d(\Phi) \leq M$. Letting $Q = \sum_{u \in X \cup Y} B(S_u^\Gamma)$, we have $M \geq d(\Phi) = \sum_{u \in X \cup Y} d(\pi_u) \geq \sum_{u \in X \cup Y} B(S_u^\Gamma)$, so that $M \geq Q$.

If $\pi_t \in Z_t$, then $Z_t \subseteq S_t^\Gamma$ implying that $d(\pi_t) \geq B(S_t^\Gamma)$. Now suppose there exists some $v \in X \cup Y$ with $\pi_v \notin Z_v$. Since π_v is consistent with Γ, $\pi_v \in S_v^\Gamma$ and $d(\pi_v) > M + B(S_v^\Gamma) - Q \geq B(S_v^\Gamma)$. This is a contradiction, as

$$\begin{aligned} d(\Phi) &= d(\pi_v) + \sum_{u \in X \cup Y, u \neq v} d(\pi_u) \\ &> (M + B(S_v^\Gamma) - Q) + \sum_{u \in X \cup Y, u \neq v} B(S_u^\Gamma) \\ &= (M + B(S_v^\Gamma) - Q) + (Q - B(S_v^\Gamma)) = M. \end{aligned}$$

Hence, if $\Phi = (\pi_{x_1}, \ldots, \pi_{x_n}, \pi_{y_1}, \ldots, \pi_{y_n})$ is a feasible tournament schedule consistent with Γ so that $d(\Phi) \leq M$, then $\pi_t \in Z_t$ for all $t \in X \cup Y$. ☐

Proposition 1 shows how to perform some reduction *prior* to propagation, and may be applicable to other problems. To apply this proposition, we will reduce $BTTP$ to k scenarios where in each scenario the six home sets for four of the Pacific League teams are pre-determined. Expressing these scenarios as the global constraints $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, each Γ_i fixes 24 of the 72 total matches.

For every Γ_i, we determine Z_{c_j} for the Central League teams and by setting a low threshold M, we show that each $|Z_{c_j}|$ is considerably smaller than $|S_{c_j}^\Gamma|$, thus reducing the search space to an amount that can be quickly analyzed. From there, we run a simple six-loop that generates all 6-tuples $(\pi_{c_1}, \pi_{c_2}, \pi_{c_3}, \pi_{c_4}, \pi_{c_5}, \pi_{c_6})$ that can appear in a feasible schedule Φ with $d(\Phi) \leq M$. By Proposition 1, each $\pi_{c_j} \in Z_{c_j}$ for $1 \leq j \leq 6$. From this list of possible 6-tuples, we can quickly find the optimal schedule Φ which corresponds to the solution to $BTTP$.

We now present a result that works only for the case $n = 6$, when two teams from one league are located quite far from the other 10 teams, forcing the distance-optimal schedule Φ to have a particular structure.

Proposition 2. *Let M be a fixed positive integer, and define $S_t^* = \{\pi_t \in S_t : d(\pi_t) \leq M\}$. Suppose there exist two teams $x_i, x_j \in X = \{x_1, x_2, \ldots, x_6\}$ for which $S_{x_i}^* \subseteq R_3^{x_i}$, $S_{x_j}^* \subseteq R_3^{x_j}$, and for each team $y_k \in Y$, every schedule in $S_{y_k}^*$ has the property that y_k plays their road sets against x_i and x_j in two consecutive time slots. If $\Phi = (\pi_{x_1}, \ldots, \pi_{x_6}, \pi_{y_1}, \ldots, \pi_{y_6})$ is a feasible tournament schedule with $d(\Phi) \leq M$ where each $\pi_t \in S_t^*$, then the team schedules*

π_{x_i} and π_{x_j} both have the home-road pattern HH-RRR-HH-RRR-HH; moreover, each team's six home slots must have the following structure for some permutation (a, b, c, d, e, f) of $\{1, 2, 3, 4, 5, 6\}$:

	1	2	3	4	5	6	7	8	9	10	11	12
x_i	$\boldsymbol{y_a}$	$\boldsymbol{y_b}$	y	y	y	$\boldsymbol{y_c}$	$\boldsymbol{y_d}$	y	y	y	$\boldsymbol{y_e}$	$\boldsymbol{y_f}$
x_j	$\boldsymbol{y_b}$	$\boldsymbol{y_a}$	y	y	y	$\boldsymbol{y_d}$	$\boldsymbol{y_c}$	y	y	y	$\boldsymbol{y_f}$	$\boldsymbol{y_e}$

Proof. We first note that if π_{x_i} and π_{x_j} have the above structure, they satisfy all the given conditions since $\pi_{x_i} \in R_3^{x_i}$, $\pi_{x_j} \in R_3^{x_j}$, and every team $y_k \in Y$ plays road sets against x_i and x_j in two consecutive time slots. For example, y_d plays road sets against x_j in slot 6 and against x_i in slot 7. We now prove that π_{x_i} and π_{x_j} must have this structure.

For each team $x_t \in X$ and time slot $s \in [1, 12]$, define $O(x_t, s)$ to be the *opponent* of team x_t in set s. We define $O(x_t, s)$ only when x_t is playing at *home*; for the sets when x_t plays on the road, $O(x_t, s)$ is undefined.

Since $\pi_{x_i} \in S^*_{x_i}$ and $S^*_{x_i} \subseteq R_3^{x_i}$, there are four possible cases to consider:

(1) x_i plays set 1 at home, and sets 2 to 4 on the road.

(2) x_i plays sets 1 and 2 at home, and sets 3 to 5 on the road.

(3) x_i plays sets 1 to 3 at home, and sets 4 to 6 on the road.

(4) x_i plays sets 1 to 3 on the road, and set 4 at home.

We examine the cases one by one. In each, suppose there exists a feasible schedule Φ satisfying all the given conditions. We finish with case (2).

In (1), let $O(x_i, 1) = y_a$. Then $O(x_j, 2) = y_a$, since y_a must play road sets against x_i and x_j in consecutive time slots. Since $\pi_{x_j} \in R_3^{x_j}$ and x_j plays at home in set 2, x_j must also play at home in set 1. Thus, $O(x_j, 1) = y_b$ for some y_b, which forces $O(x_i, 2) = y_b$. This is a contradiction as x_i plays set 2 on the road.

In (3), let $O(x_i, 1) = y_a$, $O(x_i, 2) = y_b$, and $O(x_i, 3) = y_c$. Then $O(x_j, 2) = y_a$ and $O(x_j, 4) = y_c$. Either $O(x_j, 1) = y_b$ or $O(x_j, 3) = y_b$. In either case, we violate the *at-most-three* constraint or the condition that $\pi_{x_j} \in R_3^{x_j}$.

In (4), team x_i starts with a three-set road trip. In order to satisfy the *at-most-three* constraint, π_{x_i} must have the pattern RRR-HHH-RRR-HHH. Then this reduces to case (3), as we can read the schedule Φ backwards, letting $O(x_i, 12) = y_a$, $O(x_i, 11) = y_b$, $O(x_i, 10) = y_c$, and applying the argument in the previous paragraph.

In (2), let $O(x_i, 1) = y_a$ and $O(x_i, 2) = y_b$. Then $O(x_j, 2) = y_a$ and $O(x_j, 1) = y_b$. If $O(x_j, 3) = y_c$ for some y_c, then $O(x_i, 4) = y_c$, forcing x_i to play a single road set in slot 3. Thus, x_j must play on the road in set 3, and therefore also in sets 4 and 5. Hence, both x_i and x_j start with two home sets followed by three road sets. Since this is the only case remaining, by symmetry x_i and x_j must end with two home sets preceded by three road sets. Thus, these two teams must have the pattern HH-RRR-HH-RRR-HH.

In order for each y_k to play their road sets against x_i and x_j in two consecutive time slots, we must have $O(x_i, 6) = O(x_j, 7)$, $O(x_i, 7) = O(x_j, 6)$, $O(x_i, 11) = O(x_j, 12)$, and $O(x_i, 12) = O(x_j, 11)$. This completes the proof. ◻

We will use Proposition 2 to solve $BTTP$, since teams p_5 and p_6 are located quite far from the other ten teams (see Figure 5). This heuristic of isolating two teams and finding its common structure significantly reduces the search space and enables us to solve $BTTP$ for the 12-team NPB in hours rather than weeks.

By applying these results, we do not require weeks of computation time on multiple processors. With these two heuristics, $BTTP$ can be solved in less than ten hours on a single laptop. All of the code was written in Maple and compiled using Maplesoft 13 using a single Toshiba laptop under Windows with a single 2.10 GHz processor and 2.75 GB RAM.

Table 8 presents an inter-league tournament schedule Φ that is a solution to $BTTP$ with $d(\Phi) = (0+4+0+0+1+1) + (51+9+31+58+19+13) = 187$.

	1	2	3	4	5	6	7	8	9	10	11	12
p_1	$\boldsymbol{c_3}$	$\boldsymbol{c_5}$	$\boldsymbol{c_1}$	c_3	c_2	c_1	$\boldsymbol{c_6}$	$\boldsymbol{c_2}$	c_4	c_5	c_6	$\boldsymbol{c_4}$
p_2	$\boldsymbol{c_5}$	$\boldsymbol{c_3}$	c_2	c_1	c_3	$\boldsymbol{c_6}$	$\boldsymbol{c_1}$	c_4	c_5	c_6	$\boldsymbol{c_4}$	$\boldsymbol{c_2}$
p_3	$\boldsymbol{c_4}$	$\boldsymbol{c_2}$	c_6	c_5	c_4	$\boldsymbol{c_3}$	$\boldsymbol{c_5}$	$\boldsymbol{c_1}$	c_3	c_2	c_1	$\boldsymbol{c_6}$
p_4	$\boldsymbol{c_2}$	$\boldsymbol{c_4}$	$\boldsymbol{c_5}$	c_4	c_6	c_5	$\boldsymbol{c_3}$	$\boldsymbol{c_6}$	$\boldsymbol{c_1}$	c_3	c_2	c_1
p_5	$\boldsymbol{c_1}$	$\boldsymbol{c_6}$	c_4	c_6	c_5	$\boldsymbol{c_2}$	$\boldsymbol{c_4}$	c_3	c_2	c_1	$\boldsymbol{c_5}$	$\boldsymbol{c_3}$
p_6	$\boldsymbol{c_6}$	$\boldsymbol{c_1}$	c_3	c_2	c_1	$\boldsymbol{c_4}$	$\boldsymbol{c_2}$	c_5	c_6	c_4	$\boldsymbol{c_3}$	$\boldsymbol{c_5}$
c_1	p_5	p_6	p_1	$\boldsymbol{p_2}$	$\boldsymbol{p_6}$	$\boldsymbol{p_1}$	p_2	p_3	p_4	$\boldsymbol{p_5}$	$\boldsymbol{p_3}$	$\boldsymbol{p_4}$
c_2	p_4	p_3	$\boldsymbol{p_2}$	$\boldsymbol{p_6}$	$\boldsymbol{p_1}$	p_5	p_6	p_1	$\boldsymbol{p_5}$	$\boldsymbol{p_3}$	$\boldsymbol{p_4}$	p_2
c_3	p_1	p_2	$\boldsymbol{p_6}$	$\boldsymbol{p_1}$	$\boldsymbol{p_2}$	p_3	p_4	$\boldsymbol{p_5}$	$\boldsymbol{p_3}$	$\boldsymbol{p_4}$	p_6	p_5
c_4	p_3	p_4	$\boldsymbol{p_5}$	$\boldsymbol{p_4}$	$\boldsymbol{p_3}$	p_6	p_5	$\boldsymbol{p_2}$	$\boldsymbol{p_1}$	$\boldsymbol{p_6}$	p_2	p_1
c_5	p_2	p_1	p_4	$\boldsymbol{p_3}$	$\boldsymbol{p_5}$	$\boldsymbol{p_4}$	p_3	$\boldsymbol{p_6}$	$\boldsymbol{p_2}$	$\boldsymbol{p_1}$	p_5	p_6
c_6	p_6	p_5	$\boldsymbol{p_3}$	$\boldsymbol{p_5}$	$\boldsymbol{p_4}$	p_2	p_1	p_4	$\boldsymbol{p_6}$	$\boldsymbol{p_2}$	$\boldsymbol{p_1}$	p_3

Table 8: Solution to $BTTP$ with total distance 42950 km.

In Table 8, we see that only seven of the twelve teams satisfy $\pi_t \in R_3^t$, namely c_1 and all six of the Pacific League teams. However, every Central League team in this schedule plays road sets against p_5 and p_6 in consecutive time slots. This explains why each $d(c_j)$ in Φ is small.

We claim that Φ is an optimal solution, with total distance $d(\Phi) + \sum ILB_t = 187 + 42763 = 42950$. To prove this, we set $M = 187$. Define $S_t^* = \{\pi_t \in S_t : d(\pi_t) \leq M\}$, from which we determine that $S_{p_5}^* \subseteq R_3^{p_5}$ and $S_{p_6}^* \subseteq R_3^{p_6}$.

Define $T_{c_i} \subseteq S_{c_i}^{\Gamma}$ to be the subset of schedules for which c_i does *not* play their road sets against p_5 and p_6 in two consecutive time slots. From this, we can show that $B(T_{c_3}) = 153$, and that $B(T_{c_j}) > M = 187$ for $j \in \{1,2,4,5,6\}$. We claim that if Φ satisfies $d(\Phi) \leq 187$, then $\pi_{c_j} \notin T_{c_j}$ for all $1 \leq j \leq 6$.

It suffices to prove the claim for $j = 3$. There are 144 schedules in T_{c_3}, all of which belong to the set $R_3^{c_3}$. For example, one such schedule π_{c_3} is

	1	2	3	4	5	6	7	8	9	10	11	12
c_3	$\boldsymbol{p}$	p_2	p_1	p_6	$\boldsymbol{p}$	$\boldsymbol{p}$	$\boldsymbol{p}$	p_3	p_4	p_5	$\boldsymbol{p}$	$\boldsymbol{p}$

Suppose there exists a tournament schedule Φ with $d(\Phi) \leq 187$ and $\pi_{c_3} \in T_{c_3}$. There are nine possible home-road patterns for $\pi_{p_5} \in R_3^{p_5}$ (e.g. HHH-RRR-H-RRR-HH and H-RRR-HHH-RRR-HH), each of which gives rise to $6! = 720$ possible orderings for the six home sets. Thus, there are $9 \times 720 = 6480$ ways we can select the time slots and opponents for the six home sets in π_{p_5}. Similarly, there are 6480 ways to do this for π_{p_6}. A simple Maplesoft procedure shows that only 140 of the 6480^2 possible pairs (π_{p_5}, π_{p_6}) are consistent with at least one $\pi_{c_3} \in T_{c_3}$.

For each of these 140 cases, define the global constraints $\Gamma_1, \Gamma_2, \ldots, \Gamma_{140}$, obtained from fixing the twelve home sets in $\{\pi_{p_5}, \pi_{p_6}\}$. For each Γ_k, define for each $j \in \{1, 2, 4, 5, 6\}$ the set $Z_{c_j} = \{\pi_{c_j} \in S_{c_j}^{\Gamma_k} : d(\pi_{c_j}) \leq M - B(T_{c_3}) = 34\}$. Then we run our six-loop to compute all possible 6-tuples $(\pi_{c_1}, \pi_{c_2}, \ldots, \pi_{c_6})$ satisfying the given conditions with $\pi_{c_3} \in T_{c_3}$ and $\pi_{c_j} \in Z_{c_j}$ for $j \neq 3$. Within twenty minutes, Maplesoft solves all 140 cases and returns no feasible 6-tuples that can appear in a schedule Φ with $d(\Phi) \leq 187$.

Therefore, in Φ, each c_j must play road sets against p_5 and p_6 in consecutive time slots. Thus, teams p_5 and p_6 satisfy the conditions of Proposition 2. Hence, the home-road pattern of π_{p_5} and π_{p_6} in Φ must be HH-RRR-HH-RRR-HH.

Without loss, assume that p_5 plays a home set against c_1 within the first *six* time slots; otherwise we can read the schedule backwards by symmetry. Thus, there are $\frac{6!}{2} = 360$ ways to assign opponents to the six home sets in π_{p_5}. By Proposition 2, each of these 360 arrangements uniquely determines the six home sets in π_{p_6}.

A short calculation shows that in order for $d(\Phi) \leq M = 187$, teams p_1 and p_3 must also play their six road sets in two blocks of three. In other words, $\pi_{p_1} \in R_3^{p_1}$ and $\pi_{p_3} \in R_3^{p_3}$. As mentioned earlier, there are $9 \times 6!$ possible ways to select the six home sets for each of π_{p_1} and π_{p_3}.

Thus, there are $360 \times (9 \cdot 6!) \times (9 \cdot 6!)$ ways we can select the 24 home sets played by the teams in $\{p_1, p_3, p_5, p_6\}$. We eliminate all scenarios in which some p_i and p_j play against some c_k in the same time slot. For the possibilities that remain, we create a global constraint to apply Proposition 1.

Let $\{\Gamma_1, \Gamma_2, \ldots, \Gamma_k\}$ be the complete set of global constraints derived from the above process, where each Γ_i fixes 24 of the 72 matches, corresponding to the home sets of $\{p_1, p_3, p_5, p_6\}$. The reduction heuristic of Proposition 1 allows us to quickly verify the existence of feasible tournament schedules Φ consistent with Γ_i for which $d(\Phi) \leq M$.

To explain this procedure, let us illustrate with the inter-league schedule in Table 8. Let Γ be the constraint that fixes the 24 home sets of teams p_1, p_3, p_5, and p_6 in that table. Then $S_{c_5}^{\Gamma}$, defined as the subset of schedules in S_{c_5} consistent with Γ, consists only of team schedules π_{c_5} for which c_5 plays road sets against p_1 in slot 2, p_3 in slot 7, p_5 in slot 11, and p_6 in slot 12.

We find that there are only 11 schedules $\pi_{c_5} \in S_{c_5}^{\Gamma}$ with $d(\pi_{c_5}) \leq M$ that are consistent with Γ. Furthermore, each $d(\pi_{c_5}) \in \{19, 41, 46, 48\}$, implying that $B(S_{c_5}^{\Gamma}) = 19$. Similarly, we can calculate the other values of $B(S_{c_j}^{\Gamma})$.

We find that $\sum_{j=1}^{6} B(S_{p_j}^{\Gamma}) = 0$ and $\sum_{j=1}^{6} B(S_{c_j}^{\Gamma}) = 51 + 9 + 31 + 58 + 19 + 13 = 181$, implying that $Z_{c_5} = \{\pi_{c_5} \in S_{c_5}^{\Gamma} : d(\pi_{c_5}) \leq 187 + 19 - 181 = 25\}$. Hence, Z_{c_5} reduces to just the *two* schedules with $d(\pi_{c_5}) = 19$, including the team schedule π_{c_5} in Table 8.

By Proposition 1, any schedule Φ consistent with Γ satisfying $d(\Phi) \leq M$ must have the property that $\pi_t \in Z_t$ for each team t. Since each $|Z_{c_j}|$ is small, the calculation is extremely fast. Of course, if any $|Z_{c_j}| = 0$, then no schedule Φ can exist.

This algorithm, based on Propositions 1 and 2, runs in 34716 seconds in Maplesoft (just under 10 hours). Maplesoft generates zero inter-league schedules with $d(\Phi) < 187$ and 14 inter-league schedules with $d(\Phi) = 187$, including the schedule given in Table 8. Since we made the assumption that p_5 plays a home set against c_1 within the first six time slots, there are actually twice as many distance-optimal schedules by reading each schedule Φ backwards.

In each of the 28 distance-optimal schedules Φ, we find that $(d(\pi_{p_1}), d(\pi_{p_2}), \ldots, d(\pi_{p_6})) = (0, 4, 0, 0, 1, 1)$ and $(d(\pi_{c_1}), d(\pi_{c_2}), \ldots, d(\pi_{c_6})) = (51, 9, 31, 58, 19, 13)$.

Therefore, we have proven that Table 8 is an optimal inter-league schedule for the NPB, reducing the total travel distance by 8184 kilometres, or 16.0%, compared to the 2010 NPB schedule.

5. American Basketball

The National Basketball Association (NBA) is one of the world's most lucrative sports leagues, with over four billion dollars in annual revenue, and an average franchise value of 400 million dollars. There are 15 teams in the Western Conference and 15 teams in the Eastern Conference. Every NBA team plays 82 regular-season games, of which 30 are inter-league (with one home game and one away game against each of the 15 teams from the other conference.) The geographic location of each team is provided in Figure 6.

Figure 6: Map of the NBA's 15 Western Conference teams and 15 Eastern Conference teams.

Given that NBA teams play inter-league games, we consider *BTTP* for this league, where we attempt to find a distance-optimal inter-league tournament. In this theoretical problem, we will assume that all inter-league games take place during a consecutive stretch in the regular season, as is done currently in the Japanese NPB. We will also enforce all the constraints of *BTTP*, including no team having a home stand or road trip lasting more than 3 games. We note that these strict conditions are not part of the NBA scheduling requirement, as evidenced by the San Antonio Spurs playing 6 consecutive home games followed immediately by 8 consecutive road games during the 2009-10 regular season. Furthermore, we will require that our inter-league schedule be *compact*, i.e., having each team play one game in each time slot. Of course, this compactness condition is not part of a typical NBA schedule, as one team might play five games by the time another team has played just two.

We determine the 30×30 NBA distance matrix from an online website[1] that lists the flight distance (in statute miles) between each pair of major cities in North America. This matrix is found in Appendix B.

Unlike the 12-team NPB where we could solve *BTTP*, it appears highly unlikely that we can solve this problem for the 30-team NBA. Nonetheless, we can generate an inter-league tournament whose total distance is close to the trivial lower bound of $\sum ILB_t$, by grouping each league's fifteen teams into five triplets so that the travel distance of each team t is extremely close to ILB_t, the minimum weight of a t-rooted 4-cycle-cover. From this, we construct a uniform tournament, i.e., a feasible solution to *BTTP**, where each Western Conference team alternates by playing three away games followed by three home games.

Given the geographic location of the 30 teams, it is easy to show that each team's ILB_t occurs when playing the fifteen away games in five groups of three. We note that this is not always the case; to give a concrete example, consider a variant of the scenario we presented at the end of Section 3. Let points X, Y, Z be equally spaced around a unit circle, so that $\triangle XYZ$ is equilateral. Place $\{e_1, e_2\}$ at X, $\{e_3, e_4\}$ at Y, and $\{e_5, e_6\}$ at Z. Now place $\{w, e_7, e_8, \ldots, e_{15}\}$ at the centre of the circle. Then the best lower bound of $ILB_w = 6$ occurs when w plays two-game road trips against $\{e_1, \ldots, e_6\}$ in pairs rather than three-game road trips like $\{e_1, e_2, e_3\}$ and $\{e_4, e_5, e_6\}$, which has total distance $4 + 2\sqrt{3} > 6$. However, for the NBA distance matrix, each team's ILB_t occurs when that team has five road trips, where in each trip that team plays three opponents located close to each other.

Thus, for each team w_i, there exists some permutation π for which the lower bound ILB_{w_i} is attained by playing away games against the fifteen Eastern Conference teams in the order $e_{\pi(1)}, e_{\pi(2)}, \ldots, e_{\pi(15)}$. Note that for this permutation, the total distance traveled by w_i is

$$ILB_{w_i} = \sum_{j=1}^{5} \{D_{w_i, e_{\pi(3j-2)}} + D_{e_{\pi(3j-2)}, e_{\pi(3j-1)}} + D_{e_{\pi(3j-1)}, e_{\pi(3j)}} + D_{e_{\pi(3j)}, w_i}\}.$$

The five triplets $\{\{e_{\pi(3j-2)}, e_{\pi(3j-1)}, e_{\pi(3j)}\} : j = 1, 2, \ldots, 5\}$ can be permuted in 5! ways without changing the total distance. Also, within each triplet, we can change the order of the first and third element while retaining the same total. Thus, we can compute ILB_{w_i} from a simple enumeration of $\frac{15!}{5! \cdot 2^5}$ cases, which can be done in minutes using Maplesoft. From this, we calculate ILB_t for each team t, giving $LLB_W \geq \sum_{t \in W} ILB_t = 251795$. Similarly, we have $LLB_E \geq \sum_{t \in E} ILB_t = 266137$, and so $TLB \geq LLB_W + LLB_E \geq 517932$.

In nearly every case, the bounds ILB_{w_i} and ILB_{e_i} are attained by selecting the road trips as indicated in Figure 7, corresponding to the minimum-weight *triangle packing* for each league. For example, in this minimum-weight triangle packing, every Eastern Conference team makes just one trip to the northwest, to play Portland, Golden State, and Sacramento in some order. Similarly, every Western Conference team makes just one trip to the southeast, to play Atlanta, Orlando, and Miami in some order. We note the natural connection between minimum-weight triangle packings and minimum-weight 4-cycle-covers, remarking that the former generates an approximation for the latter.

Re-label the fifteen Western Conference teams so that five triplets occur side-by-side (i.e., w_1 is Portland, w_2 is Golden State, w_3 is Sacramento), and similarly re-label the Eastern

1. http://www.savvy-discounts.com/discount-travel/JavaAirportCalc.html

Figure 7: The minimum-weight triangle packing for the 30 NBA teams.

Conference teams. Similar to our construction in Table 5, we build a tournament from the $5 \times 5 = 25$ pairs of inter-league triplets, where each team from one triplet plays the three teams from the other triplet in three consecutive time slots (e.g., e_1 plays $\{w_1, w_2, w_3\}$, e_2 plays $\{w_2, w_3, w_1\}$ and e_3 plays $\{w_3, w_1, w_2\}$.) This construction produces a schedule where the Eastern Conference teams travel 286683 miles and the Western Conference teams travel 258443 miles, for a total of 545126 miles.

We can improve this bound slightly by noting that the *away* teams are not forced to travel according to the triplets given in Figure 7. Specifically, suppose we are considering the road trips for the Eastern Conference. Let the triplets be $\{\{e_{\pi(3j-2)}, e_{\pi(3j-1)}, e_{\pi(3j)}\} : j = 1, 2, \ldots, 5\}$, for some permutation π. Then each triplet $\{e_{\pi(3j-2)}, e_{\pi(3j-1)}, e_{\pi(3j)}\}$ travels west for three-game road trips against each of $\{w_1, w_2, w_3\}, \{w_4, w_5, w_6\}, \ldots, \{w_{13}, w_{14}, w_{15}\}$. Examining all $\frac{15!}{5! \cdot 2^5}$ non-equivalent possibilities for π, we show that the best permutation is $\pi = (1, 6, 12, 2, 8, 13, 3, 7, 11, 4, 10, 14, 5, 9, 15)$, so that the teams in $\{e_1, e_6, e_{12}\}$ play their first three games on the road against each of $\{w_1, w_2, w_3\}$, the teams in $\{e_2, e_8, e_{13}\}$ play their first three games on the road against each of $\{w_4, w_5, w_6\}$, and so on. In this optimal schedule, the Eastern Conference teams travel a total of 280294 miles. Similarly, in the best possible case, the Western Conference teams travel a total of 257497 miles.

From this, we produce Table 9, a uniform inter-league tournament with total distance $280294 + 257497 = 537791$ miles, just 3.8% more than the trivial lower bound of $\sum ILB_t$. The labeling of the 30 teams (e.g. PT = Portland Trailblazers, MB = Milwaukee Bucks) is given in Appendix B.

While we are certain that the trivial lower bound of $\sum ILB_t$ cannot be achieved for either the $BTTP$ or $BTTP^*$, we conjecture that the 3.8% figure can be reduced using more sophisticated techniques. But how close can we get? We leave this as a challenge for the interested reader.

Problem 1. *Determine better (best?) bounds for $BTTP$ and $BTTP^*$, for the 30×30 NBA distance matrix.*

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
PT	MB	IP	CU	**MB**	**CB**	**CC**	AH	OM	MH	**CU**	**AH**	**NK**	WW	CB	PS
GW	TR	CC	DP	**CC**	**MB**	**CB**	MB	IP	CU	**NK**	**CU**	**AH**	MH	OM	AH
SK	NK	NN	BC	**CB**	**CC**	**MB**	TR	CC	DP	**AH**	**NK**	**CU**	MB	IP	CU
LC	IP	CU	MB	**NN**	**MH**	**TR**	OM	MH	AH	**CC**	**CB**	**MB**	CB	PS	WW
LL	WW	CB	PS	**TR**	**NN**	**MH**	NK	NN	BC	**MB**	**CC**	**CB**	TR	CC	DP
PS	BC	NK	NN	**MH**	**TR**	**NN**	CC	DP	TR	**CB**	**MB**	**CC**	IP	CU	MB
UJ	OM	MH	AH	**DP**	**PS**	**OM**	PS	WW	CB	**TR**	**NN**	**MH**	BC	NK	NN
DN	CC	DP	TR	**OM**	**DP**	**PS**	IP	CU	MB	**MH**	**TR**	**NN**	OM	AH	MH
OT	AH	OM	MH	**IP**	**WW**	**BC**	WW	CB	PS	**DP**	**OM**	**PS**	NK	NN	BC
SS	CU	MB	IP	**BC**	**IP**	**WW**	MH	AH	OM	**PS**	**DP**	**OM**	PS	WW	CB
DM	DP	TR	CC	**WW**	**BC**	**IP**	CU	MB	IP	**OM**	**PS**	**DP**	AH	MH	OM
HR	PS	WW	CB	**AH**	**NK**	**CU**	BC	NK	NN	**WW**	**BC**	**IP**	DP	TR	CC
MT	CB	PS	WW	**PS**	**OM**	**DP**	NN	BC	NK	**NN**	**MH**	**TR**	CC	DP	TR
MG	NN	BC	NK	**CU**	**AH**	**NK**	DP	TR	CC	**IP**	**WW**	**BC**	CU	MB	IP
NH	MH	AH	OM	**NK**	**CU**	**AH**	CB	PS	WW	**BC**	**IP**	**WW**	NN	BC	NK

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
PT	**BC**	**IP**	**WW**	NK	NN	BC	**DP**	**OM**	**PS**	TR	CC	DP	**TR**	**MH**	**NN**
GW	**WW**	**BC**	**IP**	PS	WW	CO	**PS**	**DP**	**OM**	NN	NK	BC	**NN**	**TR**	**MH**
SK	**IP**	**WW**	**BC**	AH	OM	MH	**OM**	**PS**	**DP**	WW	CO	PS	**MH**	**NN**	**TR**
LC	**NK**	**AH**	**CU**	NN	BC	NK	**IP**	**WW**	**BC**	DP	TR	CC	**DP**	**OM**	**PS**
LL	**CU**	**NK**	**AH**	MB	CU	IP	**BC**	**IP**	**WW**	OM	MH	AH	**PS**	**DP**	**OM**
PS	**AH**	**CU**	**NK**	MH	AH	OM	**WW**	**BC**	**IP**	PS	WW	CO	**OM**	**PS**	**DP**
UJ	**MB**	**CC**	**CO**	CC	DP	TR	**CU**	**NK**	**AH**	CU	IP	MB	**BC**	**WW**	**IP**
DN	**CO**	**MB**	**CC**	CO	PS	WW	**AH**	**CU**	**NK**	BC	NN	NK	**IP**	**BC**	**WW**
OT	**TR**	**MH**	**NN**	DP	TR	CC	**MB**	**CO**	**CC**	MB	CU	IP	**CU**	**AH**	**NK**
SS	**NN**	**TR**	**MH**	BC	NK	NN	**CC**	**MB**	**CO**	CC	DP	TR	**NK**	**CU**	**AH**
DM	**MH**	**NN**	**TR**	WW	CO	PS	**CO**	**CC**	**MB**	NK	BC	NN	**AH**	**NK**	**CU**
HR	**OM**	**PS**	**DP**	IP	MB	CU	**MH**	**NN**	**TR**	AH	OM	MH	**CO**	**CC**	**MB**
MT	**CC**	**CO**	**MB**	CU	IP	MB	**NK**	**AH**	**CU**	MH	AH	OM	**WW**	**IP**	**BC**
MG	**DP**	**OM**	**PS**	OM	MH	AH	**TR**	**MH**	**NN**	CO	PS	WW	**MB**	**CO**	**CC**
NH	**PS**	**DP**	**OM**	TR	CC	DP	**NN**	**TR**	**MH**	IP	MB	CU	**CC**	**MB**	**CO**

Table 9: A close-to-optimal solution for the NBA *BTTP**.

6. Conclusion

In this paper, we introduced the Bipartite Traveling Tournament Problem and applied it to two professional sports leagues in Japan and North America, illustrating the richness and complexity of bipartite tournament scheduling.

In Section 4, we introduced two heuristics that enabled us to solve $BTTP$ for the $n = 6$ NPB. While Proposition 2 is only applicable for certain 12-team configurations satisfying a specific geometric property, we note that Proposition 1 is a general technique that can be applied to other scheduling problems. Our method of "reduction prior to propagation" breaks a complex problem into a large number of scenarios, and sets up each scenario as a global constraint to reduce the search space. We are confident that Proposition 1 can be applied to more complicated problems in sports scheduling.

In Section 5, we determined an algorithm that produced an approximate solution to $BTTP$ for the $n = 15$ NBA. By finding minimum-weight rooted 4-cycle-covers, we determined a trivial lower bound to $BTTP$, from which our method of creating a uniform schedule based on the minimum-weight triangle packing generated a close-to-optimal feasible solution. For the NBA inter-league problem, this process produced an optimality gap of just 3.8%. We are hopeful that these ideas can be abstracted and refined further, leading to more powerful tools to tackle even harder problem instances.

Perhaps there are other sports leagues for which *BTTP* is applicable, such as in professional hockey and football. We can also expand our analysis to model *tripartite* and *multipartite* tournament scheduling, where a league is divided into three or more conferences. A specific example of this is the newly-created Super 15 Rugby League, consisting of five teams from South Africa, Australia, and New Zealand. In addition to intra-country games, each team plays four games (two home and two away) against teams from each of the other two countries. It would be interesting to see whether we can determine the distance-optimal tripartite tournament schedule using the methods developed in this paper.

We conclude by motivating several interesting questions, including those dealing with geometric probability and extremal combinatorics, and leave them as open problems for the interested reader.

Our solution to the non-uniform *BTTP* required 10 hours of computations. Furthermore, we were only able to solve *BTTP* by applying Proposition 2, whose requirements would not hold for a randomly-selected 12×12 distance matrix. As a result, we require a more sophisticated technique that improves upon our two heuristics, perhaps using methods in constraint programming and integer programming, such as a hybrid CP/IP. We wonder if there exists a general algorithm that would solve *BTTP* given any distance matrix, for "small" values of n such as $n = 6$, $n = 7$, and $n = 8$. We pose this as an open problem.

Problem 2. *Develop a computational procedure (or algorithm) that can routinely solve BTTP and BTTP* instances with* $n \geq 6$.

At the end of Section 3, we presented a simple example (see Figure 4) to illustrate the difference between *BTTP* and *BTTP** for the case $n = 3$. We located the six points to form two sets of Pythagorean triangles, and showed that the solutions to the two problems had total distance $15a + 3b + 5c$ and $16a + 4b + 4c$, respectively. If $(a, b, c) = (3, 4, 5)$, then the tournament lower bounds are 82 and 84, respectively. In other words, by relaxing the uniformity requirement, we can reduce the optimal travel distance from 84 to 82, an improvement of 2.38%. Using elementary calculus, we can show that for this particular choice of six points, the percentage reduction function is at most 2.39%, with equality iff $\frac{b}{a} = \frac{5+3\sqrt{5}}{8}$. However, if we selected a different set of six points, could we achieve a better percentage reduction? This motivates the following question:

Problem 3. *Consider six points* $X = \{x_1, x_2, x_3\}$ *and* $Y = \{y_1, y_2, y_3\}$ *in the Cartesian plane. Let* D^* *and* D *be the tournament lower bounds of BTTP* and BTTP, respectively. Determine the smallest constant* c *for which* $D^* \leq c \cdot D$ *for all possible selections of the six points in* $X \cup Y$.

One may conjecture that in order to minimize the tournament lower bound, we must minimize the total number of trips taken by the $2n$ teams. But as we saw in Table 6, this conjecture is false for $n = 3$, as we located six points for which the solution to *BTTP** requires 24 trips, while the solution to *BTTP* requires 25 trips. However, there are numerous examples (e.g. our scenario in Table 7) where the $2n$ points can be located so that the solution to *BTTP** matches that of *BTTP*. This motivates the following question: given a random selection of $2n$ points, what is the probability that the solutions to *BTTP** and *BTTP* are identical?

To illustrate, consider the case $n = 3$. We can quickly show that there exist $60 \times 2^9 = 30720$ feasible inter-league tournaments, of which $60 \times 2^3 = 480$ are uniform. We run a simulation on Maplesoft, where in each scenario, we randomly select six points (x, y) in the Cartesian plane, calculate the 15×1 column vector of pairwise distances, and apply it to the set of feasible inter-league tournaments to determine the distance-optimal schedule. We run the simulation 100000 times, where in each scenario, we note the number of trips taken in the optimal solution. The results appear in Table 10.

Trips	24	25	26	27	28	≥ 29
Scenarios	55800	33077	10967	43	0	0

Table 10: Results of simulation: number of trips in the distance-optimal tournament.

We note that the sum total is not 100000, as there were 113 scenarios that ended in a tie (e.g. there were two tournaments, one with 24 trips and another with 26 trips, both with equal total distance after rounding to two decimal places.)

As expected, in the majority of scenarios, the six points $X \cup Y$ had the property that the distance-optimal bipartite tournament involved 24 trips, where each team played three consecutive road games. Without much difficulty, one can show (Hoshino & Kawarabayashi, 2011a) that this forces the home game slots to be *uniform*, i.e., all the teams in each league must play their home games at the same time, and so each team in X plays three consecutive home games followed by three consecutive road games, or vice-versa. Therefore, in 55.8% of our randomly-selected scenarios, the solution to $BTTP^*$ is identical to the solution to $BTTP$.

Our simulation motivates an interesting question in geometric probability. Given that the $2n$ points of $X \cup Y$ are chosen at random, what is the probability that the tournament lower bound is achieved with a schedule consisting of t trips? We formally define the question below and present it as an open problem for the reader.

Problem 4. *Let the $2n$ points $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be randomly selected in the Cartesian plane. Let t be the number of trips taken in a distance-optimal solution of BTTP, with the teams located at X and Y. Determine the value $P_n(t)$ for each t, where $P_n(t)$ represents the probability that the distance-optimal tournament involves the $2n$ teams taking exactly t trips.*

For the case $n = 3$, it appears that $P_n(t) = 0$ for all $t \leq 23$ and $t \geq 28$. While it is trivial to show that we must have at least 24 trips, we do not have a formal proof that there cannot exist a selection of six points $X \cup Y$ in the plane for which the solution to $BTTP$ has more than 27 trips. If we could prove that for each n, the number of total trips in a distance-optimal solution is bounded above by some function $f(n)$, then this would enable us to solve $BTTP$ without having to enumerate all feasible schedules, i.e., a small fraction would suffice. Such a result would certainly aid in solving $BTTP$ for larger n, where a full enumeration of all feasible tournament schedules is too computationally laborious. This motivates our final problem.

Problem 5. *Consider a $2n$-team bipartite tournament, with the teams located at $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. For each n, determine the smallest integer $f(n)$*

for which the solution to BTTP involves the teams taking at most $f(n)$ trips, regardless of where the $2n$ teams are located.

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Appendix A.

We provide the proof of our three lemmas (from Section 3), beginning with Lemma 1.

Proof. First, we prove $(i) \rightarrow (ii)$.

If S is satisfiable, then there exists a function ϕ that is a valid truth assignment, i.e., a function for which $\phi(u_i) \in \{\text{TRUE}, \text{FALSE}\}$ for each $1 \leq i \leq l$ that ensures that each clause C_j evaluates to TRUE for all $1 \leq j \leq 2k$. From ϕ, we build a p-rooted 4-cycle-cover of K_S with exactly $3k$ (p, a, u, b, p)-cycles, $2k$ (p, u, c, d, p)-cycles, and k (p, u, e, f, p)-cycles.

We first identify the $3k$ (p, a, u, b, p)-cycles. For each $1 \leq i \leq l$, if $\phi(u_i)$ is FALSE, then select all 4-cycles of the form p-$a_{i,r}$-$u_{i,r}$-$b_{i,r}$-p, for each $r = 1, 2, \ldots, r(i)$. And if $\phi(u_i)$ is TRUE, then select all 4-cycles of the form p-$a_{i,r+1}$-$\overline{u}_{i,r}$-$b_{i,r}$-p, for each r (where $a_{i,r(i)+1} = a_{i,1}$). Repeating this construction for each i, we produce $3k$ (p, a, u, b, p)-cycles, covering the $6k$ vertices of $A \cup B$, as well as $3k$ vertices of U.

Now consider any clause C_j. Since ϕ is a valid truth assignment, at least one of the three literals in C_j evaluates to TRUE. In other words, there must exist some index i for which $u_i \in C_j$ and $\phi(u_i)$ is TRUE, or $\overline{u}_i \in C_j$ and $\phi(u_i)$ is FALSE.

In the former case, where $u_i \in C_j$ and $\phi(u_i)$ is TRUE, there exists some index r for which $u_{i,r}$-c_j is an edge of the gadget graph G_S. Then p-$u_{i,r}$-c_j-d_j-p is a (p, u, c, d, p)-cycle. Note that $u_{i,r}$ has not been previously selected in a (p, a, u, b, p)-cycle since $\phi(u_i)$ is TRUE (and so only the vertices $\overline{u}_{i,1}, \overline{u}_{i,2}, \ldots, \overline{u}_{i,r(i)}$ were covered earlier.)

In the latter case, where $\overline{u}_i \in C_j$ and $\phi(u_i)$ is FALSE, there exists some index r for which $\overline{u}_{i,r}$-c_j is an edge of the gadget graph G_S. Then p-$\overline{u}_{i,r}$-c_j-d_j-p is a (p, u, c, d, p)-cycle. Note that $\overline{u}_{i,r}$ has not been previously selected in a (p, a, u, b, p)-cycle since $\phi(u_i)$ is FALSE (and so only the vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,r(i)}$ were covered earlier.)

Repeating this construction for each j, we produce $2k$ (p, u, c, d, p)-cycles, covering the $4k$ vertices of $C \cup D$. Note that no $u \in U$ can be chosen twice since each vertex in U is adjacent to only one vertex in C. Thus, these $2k$ cycles cover a set of $6k$ vertices in X_S, completely disjoint from the $9k$ vertices covered by the previously-constructed $3k$ (p, a, u, b, p)-cycles. As a result, we are left with $3k$ vertices in X_S still to be covered, specifically k vertices in each of U, E, and F. These vertices can be trivially partitioned into k (p, u, e, f, p)-cycles by just ensuring that e_j and f_j belong to the same cycle for each $1 \leq j \leq k$. When this process is complete, our p-rooted 4-cycle-cover of K_S will contain exactly $3k$ (p, a, u, b, p)-cycles, $2k$ (p, u, c, d, p)-cycles, and k (p, u, e, f, p)-cycles.

Having established the first direction, we now prove $(ii) \rightarrow (i)$.

Consider a p-rooted 4-cycle-cover of K_S containing exactly $3k$ (p,a,u,b,p)-cycles, $2k$ (p,u,c,d,p)-cycles, and k (p,u,e,f,p)-cycles. We prove there exists a function ϕ that is a satisfying truth assignment for S, where $\phi(u_i) \in \{\text{TRUE}, \text{FALSE}\}$ for each $1 \leq i \leq l$.

Define an *a-b path* to be any path on three vertices whose endpoints are $a_{i,j}$ and $b_{i,k}$, for some indices i,j,k. Consider the problem of maximizing the number of vertex-disjoint *a-b* paths in the i^{th} gadget. One can quickly see that a maximum packing of *a-b* paths occurs iff the $r(i)$ paths are chosen in one of the following "trivial" ways:

(a) Taking all paths of the form $a_{i,r}, u_{i,r}, b_{i,r}$ for each $r = 1, 2, \ldots, r(i)$.

(b) Taking all paths of the form $a_{i,r+1}, \overline{u}_{i,r}, b_{i,r}$ for each $r = 1, 2, \ldots, r(i)$. (Note: $a_{i,r(i)+1} = a_{i,1}$.)

In order for us to cover all of the vertices in $A \cup B$, in each gadget we must select our *a-b* paths either vertically (a) or diagonally (b). Thus, in our p-rooted 4-cycle-cover containing $3k$ (p,a,u,b,p)-cycles, one of the following scenarios must hold true in the i^{th} gadget:

(1) For each $r = 1, 2, \ldots, r(i)$, vertex $u_{i,r}$ appears in some (p,a,u,b,p)-cycle, while no vertex $\overline{u}_{i,r}$ appears in any (p,a,u,b,p)-cycle.

(2) For each $r = 1, 2, \ldots, r(i)$, vertex $\overline{u}_{i,r}$ appears in some (p,a,u,b,p)-cycle, while no vertex $u_{i,r}$ appears in any (p,a,u,b,p)-cycle.

In our given p-rooted 4-cycle-cover of K_S, for each i define $\phi(u_i) = \text{FALSE}$ in scenario (1) and define $\phi(u_i) = \text{TRUE}$ in scenario (2). We claim that this is our desired function ϕ.

To prove this, consider the $2k$ (p,u,c,d,p)-cycles in our 4-cycle-cover. For each $1 \leq j \leq 2k$, the (p,u,c,d,p)-cycle containing c_j also contains some other vertex in U. This vertex is either $u_{i,r}$ or $\overline{u}_{i,r}$, for some indices i and r.

In the former case, $u_{i,r}$ and c_j appear in the same (p,u,c,d,p)-cycle, implying that $u_{i,r}$-c_j is an edge of the gadget graph G_S, and that u_i is a literal in clause C_j. Since $u_{i,r}$ appears in this (p,u,c,d,p)-cycle and therefore not in any (p,a,u,b,p)-cycle, this implies scenario (2) above. Since $\phi(u_i) = \text{TRUE}$ and $u_i \in C_j$, clause C_j evaluates to TRUE.

In the latter case, $\overline{u}_{i,r}$ and c_j appear in the same (p,u,c,d,p)-cycle, implying that $\overline{u}_{i,r}$-c_j is an edge of the gadget graph G_S, and that $\overline{u}_i$ is a literal in clause C_j. Since $\overline{u}_{i,r}$ appears in this (p,u,c,d,p)-cycle and therefore not in any (p,a,u,b,p)-cycle, this implies scenario (1) above. Since $\phi(u_i) = \text{FALSE}$ and $\overline{u}_i \in C_j$, clause C_j evaluates to TRUE.

Since C_j evaluates to TRUE for all $1 \leq j \leq 2k$, this implies that ϕ is a valid truth assignment. We conclude that $S = C_1 \wedge C_2 \wedge \ldots \wedge C_{2k}$ is satisfiable. □

We now prove Lemma 2.

Proof. First, we prove $(i) \to (ii)$.

In a (p,a,u,b,p)-cycle, the edges au and ub appear in the gadget graph G_S. Therefore, the edge weights of au and ub are both z^2. From Table 3, we see that a (p,a,u,b,p)-cycle has edge weight $z^2 + z^2 + z^2 + z^2 = 4z^2$. Similarly, a (p,u,c,d,p)-cycle has edge weight $(z^2+z)+z^2+z^2+z^2 = 4z^2+z$, and a (p,u,e,f,p)-cycle has edge weight $(z^2+z)+z^2+z^2+z^2 = 4z^2 + z$.

So if a p-rooted 4-cycle-cover of K_S has exactly $3k$ (p,a,u,b,p)-cycles, $2k$ (p,u,c,d,p)-cycles, and k (p,u,e,f,p)-cycles, then its total edge weight is exactly $3k(4z^2)+2k(4z^2+z)+k(4z^2+z)=k(24z^2+3z)$.

Having established the first direction, we now prove $(ii) \rightarrow (i)$.

Let R be a p-rooted 4-cycle-cover of K_S which is the union of r cycles, with total edge weight $k(24z^2+3z)$. Since each of the 18k vertices of X_S is covered by exactly one cycle of R, the number of edges in R is $|X_S|+r=18k+r$. Since no cycle has length greater than 4, we have $r \geq \frac{18k}{3}=6k$. Now suppose $r \geq 6k+1$. Then there are at least $24k+1$ edges in R, all of which have weight at least z^2 given the construction of our complete graph K_S. Hence, the total edge weight of R is at least $(24k+1)z^2=24kz^2+z^2=24kz^2+z(20k+1)>24kz^2+3zk=k(24z^2+3z)$, a contradiction.

It follows that $r=6k$, and that R must be the union of $6k$ cycles of length 4. Recall that the weight of each edge appears in the set $\{z^2, z^2+z, z^2+2z, 2z^2-1\}$. Suppose that one of these $24k$ edges has weight $2z^2-1$. Then the total edge weight of R is at least $(24k-1)z^2+(2z^2-1)=24kz^2+z^2-1=24kz^2+z(20k+1)-1>24kz^2+3zk=k(24z^2+3z)$, a contradiction. Hence, all edges of R must have weight z^2, z^2+z, or z^2+2z.

From Table 3, we see that no edges p-c and p-e can appear in our 4-cycle-cover R, since all edges from p to $C \cup E$ have weight $2z^2-1$. It follows that there must exist $2k$ 4-cycles of the form p-?-c_i-?-p and k 4-cycles of the form p-?-e_i-?-p, with each of these $2k+k=3k$ 4-cycles containing a unique element from $C \cup E$. Each blank space (denoted by a question mark) can only be filled with a vertex from D, F, or U, as the weights of edges ca, cb, ea, eb are all $2z^2-1$ for all $a \in A$, $b \in B$, $c \in C$, and $e \in E$.

Since edge p-u has weight z^2+z, if some vertex $u \in U$ is chosen to appear in one of these $3k$ 4-cycles, then this adds edge weight z^2+z, producing a 4-cycle of weight at least $4z^2+z$. But if no vertices $u \in U$ are chosen to replace these blank spaces, then the cycles must be of the form p-d_j-c_i-d_k-p or p-f_j-e_i-f_k-p, both of which lead to the addition of at least one edge of weight $2z^2-1$ (since we cannot simultaneously have $i=j$, $i=k$, and $j \neq k$). It follows that these $2k+k=3k$ 4-cycles containing the vertices of $C \cup E$ must each have weight at least $4z^2+z$, thus contributing at least $k(12z^2+3z)$ to the total distance of the p-rooted 4-cycle-cover R.

Since the given 4-cycle-cover R has weight $k(24z^2+3z)$, this implies that the rest of the $3k$ 4-cycles must each have weight exactly $4z^2$, and that in each of the $2k$ cycles of the form p-?-c_i-?-p and k cycles of the form p-?-e_i-?-p, the total edge weight must be *exactly* $4z^2+z$ to ensure that the total edge weight of R does not exceed $k(24z^2+3z)$. This implies that in these two scenarios, we cannot replace the two blank spaces with two distinct vertices from U, as that would create a cycle of weight $4z^2+2z$. It follows that R *must* have $2k$ (p,u,c,d,p)-cycles and k (p,u,e,f,p)-cycles.

We are now left with $3k$ vertices from each of A, B, and U to form our remaining $3k$ 4-cycles. In order for the total edge weight of R to not exceed $k(24z^2+3z)=3k(4z^2+z)+12kz^2$, each of the remaining $12k$ edges must have weight z^2. Since edge p-u has weight z^2+z for all $u \in U$, the $3k$ remaining vertices in U must each appear in a unique 4-cycle, none adjacent to the root vertex p. It follows that the remaining $3k$ 4-cycles of R must all be (p,a,u,b,p)-cycles. $\square$

We now prove Lemma 3.

Proof. From the proof of Lemma 2, we see that $ILB_p = k(24z^2 + 3z)$, so this handles the case $y = p$. We now consider the case $y = q$.

Let R be a q-rooted 4-cycle-cover of K_S which is the union of r cycles. Suppose on the contrary that there exists an R for which its total edge weight is less than $k(24z^2 + 20z)$. We will derive a contradiction.

Since each of the 18k vertices of X_S is covered by exactly one cycle of R, the number of edges in R is $|X_S|+r = 18k+r$. As in the previous proof, $r \geq 6k$. If $r \geq 6k+1$, then the total edge weight of R is at least $(24k+1)z^2 = 24kz^2 + z^2 = 24kz^2 + z(20k+1) > k(24z^2 + 20z)$, a contradiction.

Hence, $r = 6k$, and so R must be the union of $6k$ cycles of length 4. Now suppose that one of these $24k$ edges has weight $2z^2 - 1$. Then the total edge weight of R is at least $(24k-1)z^2 + (2z^2-1) = 24kz^2 + z^2 - 1 = 24kz^2 + z(20k+1) - 1 > k(24z^2 + 20z)$, another contradiction.

Therefore, no edges q-b and q-d can appear in our 4-cycle-cover R, since all edges from p to $B \cup D$ have weight $2z^2 - 1$. It follows that there must exist $3k$ 4-cycles of the form q-?-b_i-?-q and $2k$ 4-cycles of the form q-?-d_i-?-q. No blank space (denoted by a question mark) can be filled with a vertex from E or F since the weights of edges be, bf, de, df are $2z^2 - 1$ for all $b \in B$, $d \in D$, $e \in E$, and $f \in F$.

It follows that the k remaining 4-cycles must include all of the $k + k = 2k$ vertices in $E \cup F$. If any of these 4-cycles contains three elements of $E \cup F$ (e.g. the cycle q-e_i-f_j-e_k-q or the cycle q-e_i-e_j-f_k-q), then that creates at least one edge with weight $2z^2 - 1$, a contradiction. Thus, there must be exactly two vertices from $E \cup F$ in each of these 4-cycles. Moreover, since the weights of edges ae, af, ce, cf are all $2z^2 - 1$ for all $a \in A, c \in C$, it follows that the final vertex of these remaining k 4-cycles must be an element of U, thus producing a 4-cycle such as q-$u_{i,r}$-e_j-f_k-q or q-f_i-$u_{j,r}$-f_k. From Table 3, we see that every valid cycle has edge weight $\geq 4z^2 + 2z$.

Hence, we must have k 4-cycles in the cycle cover R, containing all $2k$ vertices in $E \cup F$ and k vertices in U, contributing total weight $\geq k(4z^2 + 2z)$. Of the $3k$ 4-cycles of the form q-?-b_i-?-q, no vertex in C can appear, as otherwise there would be an edge with weight $2z^2 - 1$. Similarly, of the $2k$ 4-cycles of the form q-?-d_i-?-q, no vertex in A can appear.

Thus, in each of the $3k$ 4-cycles containing b_i, the other two vertices must be selected from $A \cup U$. From Table 3, we see that every such 4-cycle has edge weight $\geq 4z^2 + 4z$. And in each of the $2k$ 4-cycles containing d_i, the other two vertices must be selected from $C \cup U$. Also from Table 3, we see that every such 4-cycle has edge weight $\geq 4z^2 + 3z$.

Therefore, any q-rooted 4-cycle-cover of K_S has total edge weight $\geq k(4z^2 + 2z) + 3k(4z^2 + 4z) + 2k(4z^2 + 3z) = k(24z^2 + 20z)$, establishing our desired contradiction. We conclude that $ILB_q = k(24z^2 + 20z)$.

The proof for the r-rooted 4-cycle-cover is identical. We just apply the mapping $\{a, b, c, d, e, f, u\} \to \{b, a, e, f, c, d, u\}$ to the vertices in the preceding paragraphs to reach the same conclusion. In this case, we have $ILB_r = k(4z^2+3z)+3k(4z^2+4z)+2k(4z^2+2z) = k(24z^2 + 19z)$. □

Appendix B.

We now provide the 12×12 distance matrix for the NPB league (from Section 4), and the 30×30 distance matrix for the NBA (from Section 5).

As mentioned in Section 4, the Pacific League teams are p_1 (Fukuoka), p_2 (Orix), p_3 (Saitama), p_4 (Chiba), p_5 (Tohoku), p_6 (Hokkaido), and the Central League teams are c_1 (Hiroshima), c_2 (Hanshin), c_3 (Chunichi), c_4 (Yokohama), c_5 (Yomiuri), and c_6 (Yakult). In Table 11, we only provide D_{c_i,c_j} and D_{p_i,p_j} for $i < j$ since the case $i > j$ is equivalent by symmetry.

Team	c_1	c_2	c_3	c_4	c_5	c_6	p_1	p_2	p_3	p_4	p_5	p_6
c_1	0	323	488	808	827	829	258	341	870	857	895	1288
c_2		0	195	515	534	536	577	27	577	564	654	1099
c_3			0	334	353	355	742	213	396	383	511	984
c_4				0	37	35	916	533	63	58	364	886
c_5					0	7	926	552	51	37	331	896
c_6						0	923	554	48	39	333	893
p_1							0	595	958	934	1100	1466
p_2								0	595	582	670	1115
p_3									0	86	374	928
p_4										0	361	904
p_5											0	580
p_6												0

Table 11: Distance Matrix for the Japanese NPB League.

To calculate each entry of this distance matrix, we determined how the teams travel from one stadium to another, taking into account the actual mode(s) of transportation. For example, the distance $D_{c_2,c_5} = 534$ was found by adding the travel distance for each component of the trip from Hanshin's home stadium to Yomiuri's home stadium, namely the 15 km bus ride from Koshien Stadium to Shin-Osaka Station, the 515 km bullet-train ride to Tokyo Station, followed by the 4 km bus ride to the Tokyo Dome. This is a more rigorous approach that simply calculating the flight distance between the airports in Osaka and Tokyo. Noting when teams travel by airplane, bullet train, and bus, we repeat the analysis for each of the $\binom{12}{2} = 66$ pairs of cities to produce the matrix in Table 11.

Finally, we provide the 30×30 distance matrix for the NBA, as well the labeling of the 30 teams in Table 9. There are fifteen teams in the Western Conference, namely the Portland Trailblazers (PT), Golden State Warriors (GW), Sacramento Kings (SK), Los Angeles Clippers (LC), Los Angeles Lakers (LL), Phoenix Suns (PS), Utah Jazz (UJ), Denver Nuggets (DN), Oklahoma Thunder (OT), San Antonio Spurs (SS), Dallas Mavericks (DM), Houston Rockets (HR), Minnesota Timberwolves (MT), Memphis Grizzlies (MG), and New Orleans Hornets (NH).

There are fifteen teams in the Eastern Conference, namely the Milwaukee Bucks (MB), Chicago Bulls (CU), Indiana Pacers (IP), Detroit Pistons (DP), Toronto Raptors (TR), Cleveland Cavaliers (CC), Boston Celtics (BC), New York Knicks (NK), New Jersey Nets (NN), Philadelphia Sixers (PS), Washington Wizards (WW), Charlotte Bobcats (CB), Atlanta Hawks (AH), Orlando Magic (OM), and Miami Heat (MH). Note that two teams (Chicago Bulls and Charlotte Bobcats) have the same initials, and thus we have represented the former as CU and the latter as CB to avoid ambiguity.

Team	PT	GW	SK	LC	LL	PS	UJ	DN	OT	SS	DM	HR	MT	MG	NH
PT	0	536	473	824	824	997	620	969	1462	1691	1602	1799	1403	1826	2020
GW		0	75	333	333	636	580	931	1353	1455	1445	1605	1555	1770	1875
SK			0	368	368	637	524	884	1320	1441	1420	1586	1494	1733	1850
LC				0	0	365	582	837	1169	1192	1227	1358	1514	1594	1645
LL					0	365	582	837	1169	1192	1227	1358	1514	1594	1645
PS						0	501	582	820	831	866	994	1258	1244	1281
UJ							0	375	852	1072	985	1178	976	1242	1408
DN								0	493	785	645	853	683	867	1052
OT									0	402	178	391	686	425	559
SS										0	244	187	1084	617	486
DM											0	215	843	417	430
HR												0	1023	462	300
MT													0	692	1027
MG														0	345
NH															0

Table 12: Distance Matrix for the NBA Western Conference (intra-league).

Team	MB	CU	IP	DP	TR	CC	BC	NK	NN	PS	WW	CB	AH	OM	MH
MB	0	66	235	248	414	323	847	734	714	680	602	642	661	1047	1244
CU		0	175	249	430	310	853	728	708	667	580	591	599	985	1183
IP			0	249	433	257	805	655	634	578	468	422	427	811	1009
DP				0	190	90	605	486	466	434	372	502	602	950	1143
TR					0	191	439	361	343	342	341	582	731	1036	1220
CC						0	553	418	398	357	284	425	548	877	1068
BC							0	184	198	276	407	718	933	1101	1243
NK								0	20	93	225	534	749	926	1077
NN									0	80	209	522	735	919	1074
PS										0	133	442	657	844	1002
WW											0	317	526	742	911
CB												0	224	456	643
AH													0	392	589
OM														0	198
MH															0

Table 13: Distance Matrix for the NBA Eastern Conference (intra-league).

Team	PT	GW	SK	LC	LL	PS	UJ	DN	OT	SS	DM	HR	MT	MG	NH
MB	1690	1806	1750	1730	1730	1439	1227	894	726	1082	840	973	292	550	893
CU	1711	1807	1753	1718	1718	1418	1230	886	683	1028	787	915	330	485	827
IP	1848	1903	1855	1786	1786	1466	1333	973	678	973	745	834	495	376	699
DP	1934	2052	1998	1967	1967	1665	1474	1135	908	1220	990	1083	531	624	935
TR	2064	2214	2157	2143	2143	1848	1634	1307	1098	1406	1177	1264	667	801	1097
CC	2014	2117	2064	2022	2022	1711	1540	1194	934	1224	1001	1077	612	614	906
BC	2497	2651	2594	2572	2572	2265	2072	1739	1482	1739	1532	1575	1106	1123	1349
NK	2415	2535	2482	2437	2437	2120	1958	1612	1324	1564	1362	1397	1012	950	1166
NN	2395	2515	2461	2417	2417	2100	1937	1592	1305	1547	1344	1380	992	932	1151
PS	2368	2472	2419	2365	2365	2403	1895	1544	1242	1474	1275	1306	965	861	1074
WW	2291	2372	2320	2252	2252	1926	1799	1441	1118	1342	1147	1173	894	731	942
CB	2247	2251	2209	2092	2092	1747	1700	1327	926	1080	912	899	917	503	642
AH	2140	2097	2060	1917	1917	1562	1565	1190	749	861	710	680	894	327	419
OM	2491	2397	2367	2181	2181	1817	1897	1526	1049	1023	955	839	1286	668	540
MH	2661	2540	2514	2307	2307	1942	2058	1692	1206	1126	1094	950	1483	849	665

Table 14: Distance Matrix between the two NBA Conferences (inter-league).

For readability, the 30×30 distance matrix is broken into 15×15 matrices, providing both intra-league and inter-league distances. We remark that the Los Angeles Clippers and Los Angeles Lakers play their games in the same arena, which explains why their distance is zero. Each entry in Tables 12 through 14 is expressed in miles, unlike the NPB distance matrix which was expressed in kilometres.

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