

Reasoning in Many-Valued Multi-Modal Logic: A Uniform and General Approach

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Background: Multi-modal logic has found many applications in computer science, also providing a framework to model temporal and spatial information. Reasoning in this logic is crucial in many real-world applications from different fields. At the same time, these applications live in scenarios characterized by uncertainty and vagueness in the data.

Objectives: We aim to introduce a possible many-valued extension for multi-modal logic based on the family of finite FL_{ew} -algebras, as well as a sound and complete reasoning system for such logic.

Methods: In order to give a general framework for many-valued temporal and spatial logic, we introduce the notion of a many-valued linear order, allowing for the definition of a many-valued semantics of modal frames. The reasoning system is based on the analytic tableau technique and implemented as part of an open-source framework for representing, reasoning, and learning from structured and unstructured data.

Results: The performance of the tableau implementation has been addressed through extensive experiments, changing both the underlying logic and algebra and solving both the satisfiability and validity problem for formulas of different complexity.

Conclusions: We provide an open-source framework to work with many-valued temporal and spatial logic and a reasoning tool to check satisfiability and validity of formulas from such logic.

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1 Introduction

Temporal and spatial data are undoubtedly two of the most prominent sources of information for most real-world scenarios, ranging from electrical and audio signals, to medical imaging, strategic planning, and many more. One challenge such data presents is identifying and studying logical languages able to correctly describe them. In

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some cases, modal logics (Goranko, Montanari, and Sciavicco 2004; Lutz and Wolter 2006; Marx and Reynolds 1999; Pnueli 1977) have proven their effectiveness, being able to focus their expressive power on specific aspects, usually characterized by two main parameters: ontological base (i.e., points versus extended regions), and number of dimensions (i.e., temporal versus spatial). More recently, some of these languages have also been leveraged in machine learning applications (Manzella, Pagliarini, Sciavicco, et al. 2023; Pagliarini and Sciavicco 2023), effectively opening new lines of research.

In this work, we focus our attention on multi-modal logic, and then, in particular, to four well-known languages for temporal and spatial reasoning, namely Pnueli's Linear Time Logic with Future and Past (LTL[F,P]) (Pnueli 1977), Venema's Compass Logic (CL) (Marx and Reynolds 1999), Halpern and Shoham's Modal Logic of Time Intervals (HS) (Halpern and Shoham 1991), and Lutz and Wolter's Modal Logic of Topological Relations (L_{RCC8}) (Lutz and Wolter 2006), specifically in the case of rectangular areas aligned with the axes, L_{RCC8}^{Rec} . These four languages share a modal nature, and, in some sense, cover a range of natural ways of thinking and reasoning about time with instantaneous events (LTL[F,P]) and events with duration (HS), and about space with punctual properties (CL) and extended areas (L_{RCC8}^{Rec}). While it is true that, historically, these languages have very different origins and applications, and that in some cases their original definition is not purely modal (e.g., LTL with 'next' and 'until'), they are all based on the notion of a linear order, and nicely complement each other.

Many-Valued Logic. In the real world, temporal and spatial data are often characterized by uncertainty and unclear boundaries, most of the time resulting from inaccuracies in the sensors used to capture the signals, loss of information due to the communication of such signals or during discretizations, and more. In the dimensionless case (i.e., at the pure propositional level), one classic approach to deal with uncertainty is resorting to fuzzy logics, which come in a few well-known flavors. The most common examples include logics based on Gödel algebras (G) (Baaz, Preining, et al. 2007), MV-algebras (MV) (Rose 1978), and product algebras (Π) (Hájek 1998); in their typical formulation, the truth value domains are based on the interval $[0, 1]$ in $\mathbb{R}$, and are, therefore, chains. At the modal level, the most important contribution to date is probably Fitting's (Fitting 1991, 1992, 1995), who provided some interesting insights on how to combine modal logic with Heyting algebras (H) (Esakia et al. 2019), aiming to model many-expert scenarios leveraging a partial order between truth degrees. Delving deeper into the temporal case, besides a few early attempts, the most relevant contribution is probably the extension of Linear Time Logic to the standard fuzzy setting (Frigeri et al. 2014), while in the spatial case we mention (Esterline et al. 1997), a topological fuzzy logic with a single relation, on top of a few other contributions mainly focused on the algebraic, rather than logical, aspects of spatial reasoning. Reasoning systems for fuzzy and many-valued logic exist mainly at the propositional, and sometimes first-order level (Hahnle 1994; Hahnle and Escalada Imaz 1997); among the few attempts at proposing working reasoning systems for fuzzy modal logic, we mention (Vidal 2016).

Several considerations stand out at this point. Firstly, there exists no uniform many-valued generalization of spatial and temporal logic that covers both classical fuzzy choices and Fitting's many-expert scenarios. Secondly, there exists no uniform generalization of spatial and temporal logic that allows both propositions and accessibility relations to be graded, based on a notion of a graded linear order. Finally, in sharp opposition to the Boolean case, reasoning systems for non-Boolean spatial and temporal logic seem to be scarce or unavailable.

Our contributions. To address all the above points, in this work we focus our attention on the class of FL_{ew} -algebras (Cintula et al. 2011), which constitute a generalization of both fuzzy logic algebras such as G, MV, and Π, as well as many-valued algebras, such as H. Such a generalization allows us to consider the idea of a logic based on fuzzy truth values as a particular case of many-valued logic. We provide a general definition of multi-modal logic interpreted in some FL_{ew} -algebra, and, then, we show how to generalize spatial and temporal logic to such a case; this, as we shall see, requires a reasonable definition of a many-valued generalization of the notion of linear order. Subsequently, we study a correct and complete set of tableau rules, inspired by Fitting's original proposal for the Heyting case, that immediately extend to the spatial and temporal case. Finally, we present a

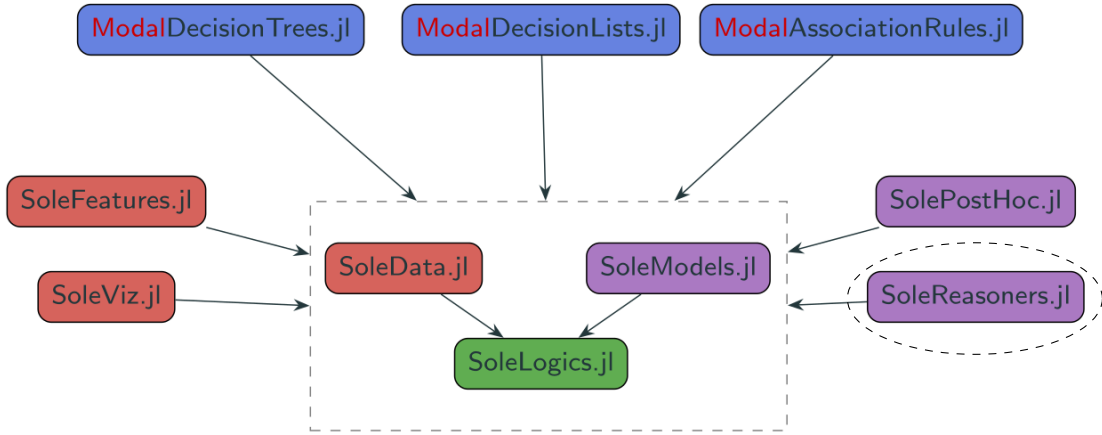


Fig. 1. A high-level overview of the system Sole. Highlighted, the package that contains the reasoning system.

working implementation of such a tableau-based reasoning system that lives in an open-source, comprehensive, end-to-end framework for symbolic learning and reasoning called Sole (Manzella, Pagliarini, Paparella, et al. 2025), depicted in Fig. 1, and an extensive set of experiments meant to address its performance for various many-valued spatial and temporal logics.

Our tableau-based reasoning system can work with any multi-modal propositional logic interpreted in any FL_{ew} -algebra: the four chosen languages can be considered four representative examples of its expressive power. The nature of our tableau rules requires the truth domain to be finite; however, as observed by other authors, this may not be too strong of a limitation (Baaz, Fermüller, et al. 2001). The two-valued Boolean algebra is obviously a particular case of FL_{ew} -algebras; hence, the tableau system covers the classical crisp reasoning tasks as well. While this is not surprising, it helps in a context where there are essentially no available reasoning systems for most temporal and spatial languages. Finally, a not-necessarily-terminating tableau system for logics that have a standard first-order translation may appear easily replaceable by already existing first-order reasoners, such as SMT solvers; however, translating many-valued satisfiability questions into crisp ones is bound to produce formulas with a high degree of quantifier alternation, which notoriously tend to degrade SMT solvers performance. This is an extended and improved version of (Badia, Noguera, et al. 2024; Conradie, Monego, et al. 2023), a tableau system only for many-valued HS (Conradie, Della Monica, et al. 2023) based on Heyting algebras.

Applications. Beside the naturalness of the questions we consider in this paper, studying many-valued temporal and spatial languages is relevant as they can be applied in several different scenarios. To give a few examples, let us consider, first, the problem of representing knowledge that can be extracted from multi-variate time series. Multi-variate time series are a convenient representation of temporal information, that may come from a number of sources. One common case is that of sensors to monitor, among others, machines, processes, behaviors, or even human health indicators. In the context of predictive maintenance, for instance, sensors are usually placed on complex machines in order to monitor their behavior over time. Sometimes, it is the case that relevant information on such behavior may conveniently be expressed in point-based temporal logic, e.g. *there is a moment in which vibration on the sensor S_1 is much higher than vibration on the sensor S_2* . Similarly, prolonged behaviors can be encoded in interval-based temporal logic, e.g.: *there is a period in which the average level of vibration on the sensor S_1 is higher than a certain value, overlapped by a period in which the average level of vibration on sensor S_2 is lower*

than a certain value. In both cases, the rigidity that such logical expressions entails in both the temporal relations and the propositional assertion can be softened by a many-valued interpretation.

The range of potential applications of spatial logic is even wider. Spatial logic can be used to express knowledge that comes from aerial video-photography, medical imagery, autonomous robotic vision, among many other possibilities. In the context of neurophysiology via eye-tracking, for instance, an interesting eye pattern may be expressible in point-based spatial logic, e.g., *there are no three vertically aligned points of interest*. In the same application field, interesting patterns may also involve extended areas, sometimes called areas of interest. In this case, it may be useful to be able to express the relevant topological relations between observed areas of interest, e.g., *there are no points in common between any two areas of interest*. Exactly as in the temporal case, softening both propositional assertions and spatial relations via the many-valued machinery may prove beneficial.

Finally, our reasoner being embedded in the context of a more general learning and reasoning package opens up an even wider range of possibilities, that involves the interplay between learning models and logical rule extraction. Since extracted rules are in fact logical formulas, we can operate several steps that require some form of reasoning, including formula comparison and minimization, as well as expert knowledge integration.

Organization. This paper is organized as follows. In Section 2, after describing the syntax and the semantics of multi-modal logic and the definition of FL_{ew} -algebras, we delve into our proposal for a many-valued generalization of multi-modal logic, and we describe a general tableau system for reasoning with such language. Then, in Section 3, we describe four spatial and temporal specializations of interest, as well as the outcome of a set of experiments to assess the performance of our tableau system when applied to these logics, before concluding.

2 Theory

In this section, we introduce the many-valued multi-modal logic framework through its syntax and semantics. Then, we discuss a tableau system for such framework, proving its soundness and completeness.

2.1 Many-Valued Multi-Modal Logic

In this section, we introduce the syntax and the semantics for multi-modal logic, as well as the notion of an FL_{ew} -algebra. Then, we propose our many-valued generalization of multi-modal logic based on the latter.

Multi-Modal Logic. The language of multi-modal logic K_n (Blackburn et al. 2001) is based on a countable set of propositional letters $\mathcal{P}$, the classical Boolean connectives, and a finite set of existential modalities $\langle X_1 \rangle, \dots, \langle X_n \rangle$.

DEFINITION 1. Let $\mathcal{P}$ be a set of propositional letters. Then, well-formed formulas of the multi-modal logic K_n are obtained by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle X_i \rangle\varphi,$$

for $1 \leq i \leq n$ and $p \in \mathcal{P}$.

The usual (classical) abbreviations apply, so that conjunction ($\wedge$), implication ($\rightarrow$), and one universal modality for each existential one ($[X_i]$) are considered shortcuts; in particular, $[X_i]\varphi \equiv \neg\langle X_i \rangle\neg\varphi$.

Formulas of K_n are interpreted on Kripke structures, as follows.

DEFINITION 2. Given a set of propositional letters $\mathcal{P}$ and a non-empty set of worlds W , a Kripke frame is an object $F = \langle W, R_1, \dots, R_n \rangle$, where each $R_i \subseteq W \times W$ is an accessibility relation. A Kripke structure is a Kripke frame enriched with a valuation function $V: W \rightarrow 2^{\mathcal{P}}$, and it is denoted by $M = \langle F, V \rangle$. Finally, given a well-formed formula φ , we say that φ is satisfied in M at w , for some world w , and we denote it by $M, w \models \varphi$, if and only if

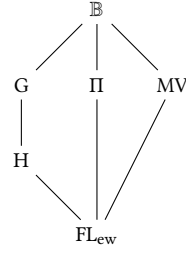


Fig. 2. A taxonomy of well-known many-valued algebras: Boolean algebras ($\mathbb{B}$, corresponding to classical logic) are generalized by algebras in the Gödel variety, the product variety, and the MV variety, whereas the last three are incomparable to each other; a Heyting algebra is a generalization of a Gödel algebra (specifically, the class of Gödel algebras is the variety generated by linearly ordered Heyting algebras), and, finally, the variety of FL_{ew} -algebras is more general than all of the above.

$$\begin{aligned}
 M, w \Vdash p & \quad \text{iff} \quad p \in V(w), \\
 M, w \Vdash \neg\psi & \quad \text{iff} \quad M, w \not\Vdash \psi, \\
 M, w \Vdash \psi \vee \xi & \quad \text{iff} \quad M, w \Vdash \psi \text{ or } M, w \Vdash \xi \\
 M, w \Vdash \langle X_i \rangle \psi & \quad \text{iff} \quad \text{there is an } s \text{ s.t. } wR_i s \text{ and } M, s \Vdash \psi.
 \end{aligned}$$

A formula φ of K_n is satisfied at a world w in a Kripke structure M if and only if

$$M, w \Vdash \varphi.$$

A formula φ is satisfiable if and only if there exists a structure and a world in which it is satisfied, and valid if it is satisfied at every world in every structure.

FL_{ew} -Algebras. Both many-valued and fuzzy logic are based on the idea of having more than two truth values. Truth values are conveniently organized in an algebra based on a partial order.

DEFINITION 3. An algebraic structure

$$\langle A, \cap, \cup, \cdot, \hookrightarrow, 0, 1 \rangle,$$

where we define a binary relation $\preceq$ as $a \preceq b$ if and only if $a \cap b = a$, is called an FL_{ew} -algebra iff: (i) $\langle A, \cap, \cup, 0, 1 \rangle$ is a bounded lattice with upper bound 1 and lower bound 0, (ii) $\langle A, \cdot, 1 \rangle$ is a commutative monoid, (iii) $\cdot$ is monotone w.r.t. $\preceq$, i.e., if $\gamma \preceq \alpha$, $\delta \preceq \beta$, then $\gamma \cdot \delta \preceq \alpha \cdot \beta$, and (iv) for each $a, b, c \in A$, $a \cdot b \preceq c$ if and only if $a \preceq b \hookrightarrow c$.

We refer, as it is customary, to $\cap$ as meet, $\cup$ as join, $\hookrightarrow$ as implication, and $\cdot$ as monoid. If the lattice order is complete (that is, each subset has infimum and supremum), we call the structure a complete FL_{ew} -algebra. In this case, given a subset $A' \subseteq A$, we denote the infimum and the supremum of A' respectively as $\bigcap A'$ and $\bigcup A'$. An FL_{ew} -algebra may be linearly ordered, or a chain, if its lattice order is total ((Badia, Monego, et al. 2025) investigates all possible combinations of FL_{ew} -chains of a given size); a standard algebra is an algebra whose lattice reduct is the real unit interval $[0, 1]$. We are also interested in finite algebras, that is, algebras with a finite number of elements. A many-valued logic, which generalizes Boolean logic, may have its truth values in an FL_{ew} -algebra, interpreting logical conjunction, disjunction, and implication as the monoid, join, and implication operations of the algebra, respectively. FL_{ew} stands for Full Lambek calculus with exchange and weakening.

In the context of many-valued propositional logics, the most common and well-known examples include Gödel algebras (G) (Baaz, Preining, et al. 2007), MV-algebras (MV) (Rose 1978), and product algebras (Π) (Hájek 1998). In their typical formulation, they are standard chains. However, they can all be considered special cases of FL_{ew} -chains. In particular, Gödel algebras interpret conjunction as the minimum; so, for example, if p has truth

value $1/3$ and q has truth value $2/3$, then the sentence $p \wedge q$ evaluates to $1/3$, illustrating Gödel's conservative nature: if we know something about two facts, we know the minimum of them about their conjunction. In the case of MV, the conjunction $\alpha \cdot \beta$ is computed as $\max\{0, \alpha + \beta - 1\}$, implying that, in the same scenario as before, the sentence would evaluate to 0, which highlights MV's pessimistic approach: if we do not know enough about two facts, we do not know anything about their conjunction. Conversely, conjunction is the real product in product algebra, evaluating the same sentence as $2/9$, thus reflecting a probabilistic interpretation of independent events. Similarly, a Heyting algebra (H) (Esakia et al. 2019; Fitting 1995) is an FL_{ew} -algebra in which $\cdot$ is defined as $\cap$, and the underlying lattice is not necessarily a chain. For instance, if p is evaluated as α by some expert and q as β by some other expert, and α and β are not necessarily comparable to each other, then the sentence $p \wedge q$ is evaluated as γ , where γ is the maximum value less than, and comparable with, both α and β . FL_{ew} -algebras thus generalize all four prominent cases. Fig. 2 provides a partial taxonomy of algebra varieties corresponding to various many-valued logics.

Many-Valued Multi-Modal logic. Following Fitting, we proceed to give a general definition of many-valued multi-modal logic parametrized to an FL_{ew} -algebra.

DEFINITION 4. Let $\mathcal{P}$ be a set of propositional letters and $\mathbf{A}$ a complete FL_{ew} -algebra. The well-formed formulas of the multi-modal logic $\text{FL}_{\text{ew}}\text{-K}_n$ are obtained by the following grammar:

$$\varphi ::= \alpha \mid p \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \langle X_i \rangle \varphi \mid [X_i] \varphi,$$

for $1 \leq i \leq n$, $p \in \mathcal{P}$, and $\alpha \in \mathbf{A}$.

As it can be observed, the many-valued nature of $\text{FL}_{\text{ew}}\text{-K}_n$ requires an explicit inclusion of Boolean operators otherwise used as abbreviations, as well as the universal version of every modality. This is a consequence of the fact that while in FL_{ew} -algebras, as in Heyting algebras, it is still possible to define negation as $\neg\varphi \equiv \varphi \rightarrow 0$, such a negation is not involutive in the general case (i.e., negation does not satisfy the law of double negation). Formulas of $\text{FL}_{\text{ew}}\text{-K}_n$ are interpreted on FL_{ew} -Kripke structures, as follows.

DEFINITION 5. Given a set of propositional letters $\mathcal{P}$, a non-empty set of worlds W , and a complete FL_{ew} -algebra $\mathbf{A}$, an FL_{ew} -Kripke frame is an object $\bar{F} = \langle W, \bar{R}_1, \dots, \bar{R}_n \rangle$, where each $\bar{R}_i: (W \times W) \rightarrow \mathbf{A}$ is an accessibility function. An FL_{ew} -Kripke structure is an FL_{ew} -Kripke frame enriched with a valuation function $\tilde{V}: (W \times \mathcal{P}) \rightarrow \mathbf{A}$, and it is denoted by $\tilde{M} = \langle \bar{F}, \tilde{V} \rangle$. Given a well-formed formula φ , we compute its value in $\tilde{M}$ at w , for some $w \in W$, by extending $\tilde{V}$ to formulas as follows:

$$\begin{aligned} \tilde{V}(w, \alpha) &= \alpha, \\ \tilde{V}(w, \varphi \wedge \psi) &= \tilde{V}(w, \varphi) \cdot \tilde{V}(w, \psi), \\ \tilde{V}(w, \varphi \vee \psi) &= \tilde{V}(w, \varphi) \cup \tilde{V}(w, \psi), \\ \tilde{V}(w, \varphi \rightarrow \psi) &= \tilde{V}(w, \varphi) \hookrightarrow \tilde{V}(w, \psi), \\ \tilde{V}(w, \langle X_i \rangle \varphi) &= \bigcup_{s \in W} \{ \bar{R}_i(w, s) \cdot \tilde{V}(s, \varphi) \}, \\ \tilde{V}(w, [X_i] \varphi) &= \bigcap_{s \in W} \{ \bar{R}_i(w, s) \hookrightarrow \tilde{V}(s, \varphi) \}. \end{aligned}$$

A formula φ of $\text{FL}_{\text{ew}}\text{-K}_n$ is α -satisfied at world w in an FL_{ew} -Kripke structure $\tilde{M}$ if and only if

$$\tilde{V}(w, \varphi) \succeq \alpha.$$

A formula φ is α -satisfiable if and only if there exists a structure and a world in which it is α -satisfied, and it is satisfiable if it is α -satisfiable for some $\alpha \in \mathbf{A}$, $\alpha \succ 0$; respectively, a formula is α -valid if it is α -satisfied at every world in every model, and valid if it is 1-valid.

2.2 A Tableau System for Many-Valued Multi-Modal Logic

In this section, we describe a general method that allows us to reason about α -satisfiability and α -validity of many-valued multi-modal logic interpreted over some finite FL_{ew} -algebra.

A Tableau System for Many-Valued Multi-Modal Logic. Our system is directly inspired by Fitting's original proposal for many-valued multi-modal logic over some Heyting algebra. It holds some similarities with tableau systems introduced in (Bresolin, Della Monica, Montanari, and Sciavicco 2013; Goranko, Montanari, Sala, et al. 2006; Goranko, Montanari, and Sciavicco 2003; Muñoz-Velasco et al. 2019) for variants, fragments, and generalizations of crisp HS.

DEFINITION 6. Take an FL_{ew} -algebra $\mathbf{A}$, an FL_{ew} - K_n -formula φ , and a finite FL_{ew} -Kripke frame $\tilde{F} = \langle W, R_1, \dots, R_n \rangle$. A decoration is an object of the type

$$Q(\alpha \preceq \varphi, w, \tilde{F}) \text{ or } Q(\varphi \preceq \alpha, w, \tilde{F}),$$

where $\alpha \in \mathbf{A}$, $w \in W$, $\alpha \preceq \varphi$ and $\varphi \preceq \alpha$ are assertions on w , and $Q \in \{\text{true}, \text{false}\}$ is a judgment. The universe of all possible decorations is denoted by $\mathcal{D}$.

DEFINITION 7. Given a finite FL_{ew} -algebra $\mathbf{A}$ and an FL_{ew} - K_n -formula φ , a tableau τ for φ and $\alpha \in \mathbf{A}$ is defined as a tuple

$$\tau = \langle \mathcal{V}, \mathcal{E}, d, f, c \rangle,$$

where $\langle \mathcal{V}, \mathcal{E} \rangle$ is a tree with nodes in $\mathcal{V}$ and edges in $\mathcal{E}$, and whose set of branches is denoted by $\mathcal{B}$. The vertices are partially ordered by a relation $\triangleleft$, which is induced by the edges. The function

$$d: \mathcal{V} \longrightarrow \mathcal{D}$$

is a node labeling function that assigns to each node $v \in \mathcal{V}$ a decoration of the form $Q(\beta \preceq \psi, w, \tilde{F})$ or $Q(\psi \preceq \beta, w, \tilde{F})$, where $\beta \in \mathbf{A}$ and ψ is a subformula of φ . The function

$$f: \mathcal{V} \longrightarrow \{0, 1\}$$

is a node flag function indicating whether nodes are expanded (1) or not expanded (0). The function

$$c: \mathcal{B} \longrightarrow \mathfrak{F},$$

where $\mathfrak{F}$ is the universe of all possible finite FL_{ew} - K_n frames, is a branch labeling function that associates to every branch the frame that belongs to the decoration of its leaf. The initial tableau τ_0 can be either

$$\langle \{v_0\}, \emptyset, \{v_0 \mapsto \text{true}(\alpha \preceq \varphi, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)\}, \{v_0 \mapsto 0\}, \{\{v_0\} \mapsto \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle\} \rangle$$

if considering α -satisfiability, or

$$\langle \{v_0\}, \emptyset, \{v_0 \mapsto \text{false}(\alpha \preceq \varphi, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)\}, \{v_0 \mapsto 0\}, \{\{v_0\} \mapsto \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle\} \rangle$$

if considering α -validity, and it evolves by iteratively applying the branch expansion rules (see Figs. 3 and 4) or, if not applicable, the reverse rules (see Fig. 5). These rules are applied to the node v closest to the root such that $f(v) = 0$, affecting every descendant leaf v' for which $v \triangleleft v'$. An application of either rule to v where $f(v) = 0$ will result in setting $f(v)$ to 1.

Consider, in particular, the rules *false* $\square$ and *false* $\diamond$ in Fig. 4. If we want to prove that $[X_i]\psi$ (resp., $\langle X_i \rangle \psi$) cannot take value greater than β when evaluated on the world w , we need to make sure that in some world w' which is $\tilde{R}_i$ -reachable from w with value γ , the value of ψ is so that $\beta \cdot \gamma$ (resp., $\gamma \hookrightarrow \beta$) is not greater than β . This must be predicated over all existing worlds in the frame at the leaf of the branch and over a possible new existing world.

$$\begin{array}{l}
\text{(true } \wedge) \frac{\text{true}(\beta \preceq (\psi \wedge \xi), w, \widetilde{F})}{\text{true}(\beta_1 \preceq \psi, w, c(B)) \mid \dots \mid \text{true}(\beta_n \preceq \psi, w, c(B)) \\
\quad \text{true}(\gamma_1 \preceq \xi, w, c(B)) \mid \dots \mid \text{true}(\gamma_n \preceq \xi, w, c(B))} \\
\text{where } \beta \succ 0, (\beta_j, \gamma_j) \in A \times A \text{ so that } \beta \preceq \beta_j \cdot \gamma_j \text{ and there is no} \\
\text{other } (\beta'_j, \gamma'_j) \in A \times A \text{ such that } \beta \preceq \beta'_j \cdot \gamma'_j, \beta'_j \preceq \beta_j \text{ and } \gamma'_j \preceq \gamma_j \\
\\
\text{(false } \wedge) \frac{\text{false}(\beta \preceq (\psi \wedge \xi), w, \widetilde{F})}{\text{true}(\psi \preceq \beta_1, w, c(B)) \mid \dots \mid \text{true}(\psi \preceq \beta_n, w, c(B)) \\
\quad \text{true}(\xi \preceq \gamma_1, w, c(B)) \mid \dots \mid \text{true}(\xi \preceq \gamma_n, w, c(B))} \\
\text{where } \beta \succ 0, (\beta_j, \gamma_j) \in A \times A \text{ so that } \beta \not\preceq \beta_j \cdot \gamma_j \text{ and there is no} \\
\text{other } (\beta'_j, \gamma'_j) \in A \times A \text{ such that } \beta \not\preceq \beta'_j \cdot \gamma'_j, \beta_j \preceq \beta'_j \text{ and } \gamma_j \preceq \gamma'_j \\
\\
\text{(true } \vee) \frac{\text{true}((\psi \vee \xi) \preceq \beta, w, \widetilde{F})}{\text{true}(\psi \preceq \beta_1, w, c(B)) \mid \dots \mid \text{true}(\psi \preceq \beta_n, w, c(B)) \\
\quad \text{true}(\xi \preceq \gamma_1, w, c(B)) \mid \dots \mid \text{true}(\xi \preceq \gamma_n, w, c(B))} \\
\text{where } \beta \prec 1, (\beta_j, \gamma_j) \in A \times A \text{ so that } \beta_j \cup \gamma_j \preceq \beta \text{ and there is no} \\
\text{other } (\beta'_j, \gamma'_j) \in A \times A \text{ such that } \beta'_j \cup \gamma'_j \preceq \beta, \beta_j \preceq \beta'_j \text{ and } \gamma_j \preceq \gamma'_j \\
\\
\text{(false } \vee) \frac{\text{false}((\psi \vee \xi) \preceq \beta, w, \widetilde{F})}{\text{true}(\beta_1 \preceq \psi, w, c(B)) \mid \dots \mid \text{true}(\beta_n \preceq \psi, w, c(B)) \\
\quad \text{true}(\gamma_1 \preceq \xi, w, c(B)) \mid \dots \mid \text{true}(\gamma_n \preceq \xi, w, c(B))} \\
\text{where } \beta \prec 1, (\beta_j, \gamma_j) \in A \times A \text{ so that } \beta_j \cup \gamma_j \not\preceq \beta \text{ and there is no} \\
\text{other } (\beta'_j, \gamma'_j) \in A \times A \text{ such that } \beta'_j \cup \gamma'_j \not\preceq \beta, \beta'_j \preceq \beta_j \text{ and } \gamma'_j \preceq \gamma_j \\
\\
\text{(true } \rightarrow) \frac{\text{true}(\beta \preceq (\psi \rightarrow \xi), w, \widetilde{F})}{\text{true}(\psi \preceq \beta_1, w, c(B)) \mid \dots \mid \text{true}(\psi \preceq \beta_n, w, c(B)) \\
\quad \text{true}(\gamma_1 \preceq \xi, w, c(B)) \mid \dots \mid \text{true}(\gamma_n \preceq \xi, w, c(B))} \\
\text{where } \beta \succ 0, (\beta_j, \gamma_j) \in A \times A \text{ so that } \beta \preceq \beta_j \hookrightarrow \gamma_j \text{ and there is no} \\
\text{other } (\beta'_j, \gamma'_j) \in A \times A \text{ such that } \beta \preceq \beta'_j \hookrightarrow \gamma'_j, \beta_j \preceq \beta'_j \text{ and } \gamma'_j \preceq \gamma_j \\
\\
\text{(false } \rightarrow) \frac{\text{false}(\beta \preceq (\psi \rightarrow \xi), w, \widetilde{F})}{\text{true}(\beta_1 \preceq \psi, w, c(B)) \mid \dots \mid \text{true}(\beta_n \preceq \psi, w, c(B)) \\
\quad \text{true}(\xi \preceq \gamma_1, w, c(B)) \mid \dots \mid \text{true}(\xi \preceq \gamma_n, w, c(B))} \\
\text{where } \beta \succ 0, (\beta_j, \gamma_j) \in A \times A \text{ so that } \beta \not\preceq \beta_j \hookrightarrow \gamma_j \text{ and there is no} \\
\text{other } (\beta'_j, \gamma'_j) \in A \times A \text{ such that } \beta \not\preceq \beta'_j \hookrightarrow \gamma'_j, \beta'_j \preceq \beta_j \text{ and } \gamma_j \preceq \gamma'_j
\end{array}$$

Fig. 3. Propositional rules. Every rule is applied to every branch $B \in \mathcal{B}$ containing the node v which is being expanded.

DEFINITION 8. Given a finite FL_{ew} -algebra A and an $\text{FL}_{\text{ew}}\text{-}K_n$ -formula φ , a tableau τ for φ and $\alpha \in A$ is closed if the branch closing rules (see Fig. 6), can be applied to all of its branches. Conversely, τ is open if there exists at least one branch to which the branch closing rules cannot be applied.

$$\begin{array}{l}
\text{(true } \Box) \frac{\text{true}(\beta \preceq [X_i]\psi, w, \tilde{F})}{\text{true}((\beta \cdot \gamma_1) \preceq \psi, w_1, c(B)) \dots \text{true}((\beta \cdot \gamma_m) \preceq \psi, w_m, c(B))} \\
\text{where } \gamma_j = \tilde{R}_i(w, w_j), w_j \in o(c(B)), \\
\gamma_j \succ 0, \text{ and } \beta \cdot \gamma_j \succ 0 \\
\text{(true } \Diamond) \frac{\text{true}(\langle X_i \rangle \psi \preceq \beta, w, \tilde{F})}{\text{true}(\psi \preceq (\gamma_1 \hookrightarrow \beta), w_1, c(B)) \dots \text{true}(\psi \preceq (\gamma_m \hookrightarrow \beta), w_m, c(B))} \\
\text{where } \gamma_j = \tilde{R}_i(w, w_j), w_j \in o(c(B)), \\
\gamma_j \succ 0, \text{ and } \gamma_j \hookrightarrow \beta \prec 1 \\
\text{(false } \Box) \frac{\text{false}(\beta \preceq [X_i]\psi, w, \tilde{F})}{\text{false}((\beta \cdot \gamma_1) \preceq \psi, w_1, c(B')) \mid \dots \mid \text{false}((\beta \cdot \gamma_m) \preceq \psi, w_m, c(B'))} \\
\text{where } \gamma_j = \tilde{R}_i(w, w_j), w_j \in o(c(B)) \cup n(c(B)), \gamma_j \succ 0, \text{ and } \beta \cdot \gamma_j \succ 0 \\
\text{(false } \Diamond) \frac{\text{false}(\langle X_i \rangle \psi \preceq \beta, w, \tilde{F})}{\text{false}(\psi \preceq (\gamma_1 \hookrightarrow \beta), w_1, c(B')) \mid \dots \mid \text{false}(\psi \preceq (\gamma_m \hookrightarrow \beta), w_m, c(B'))} \\
\text{where } \gamma_j = \tilde{R}_i(w, w_j), w_j \in o(c(B)) \cup n(c(B)), \gamma_j \succ 0, \text{ and } \gamma_j \hookrightarrow \beta \prec 1
\end{array}$$

Fig. 4. Modal rules. Every rule is applied to every branch $B \in \mathcal{B}$ containing the node v which is being expanded and, when expanding a node of the type $\text{false}(\cdot, \cdot, \tilde{F})$, will generate a new branch B' for each possible way a new world can be added to the frame $c(B)$, including the case where no new world is added. $o(c(B))$ is the set of all existing worlds in the frame $c(B)$, while $n(c(B))$ is the new world added to $c(B)$ to obtain $c(B')$.

$$\begin{array}{l}
\text{(true } \succeq) \frac{\text{true}(\beta \preceq \psi, w, \tilde{F})}{\text{false}(\psi \preceq \gamma, w, c(B))} \\
\text{where } \beta \succ 0 \text{ and } \gamma \text{ is any maximal} \\
\text{element not above } \beta, \text{ i.e., } \gamma \not\preceq \beta \\
\text{(false } \succeq) \frac{\text{false}(\beta \preceq \psi, w, \tilde{F})}{\text{true}(\psi \preceq \gamma_1, w, c(B)) \mid \dots \mid \text{true}(\psi \preceq \gamma_n, w, c(B))} \\
\text{where } \beta \succ 0 \text{ and } \gamma_1, \dots, \gamma_n \text{ are all maximal} \\
\text{elements not above } \beta, \text{ i.e., } \gamma_1, \dots, \gamma_n \not\preceq \beta \\
\text{(true } \preceq) \frac{\text{true}(\psi \preceq \beta, w, \tilde{F})}{\text{false}(\gamma \preceq \psi, w, c(B))} \\
\text{where } \beta \prec 1 \text{ and } \gamma \text{ is any minimal} \\
\text{element not below } \beta, \text{ i.e., } \gamma \not\preceq \beta \\
\text{(false } \preceq) \frac{\text{false}(\psi \preceq \beta, w, \tilde{F})}{\text{true}(\gamma_1 \preceq \psi, w, c(B)) \mid \dots \mid \text{true}(\gamma_n \preceq \psi, w, c(B))} \\
\text{where } \beta \prec 1 \text{ and } \gamma_1, \dots, \gamma_n \text{ are all minimal} \\
\text{elements not below } \beta, \text{ i.e., } \gamma_1, \dots, \gamma_n \not\preceq \beta
\end{array}$$

Fig. 5. Reverse rules. Every rule is applied to every branch $B \in \mathcal{B}$ containing the node v which is being expanded.

An open (resp., closed) tableau for φ and α , initiated with the decoration $\text{true}(\alpha \preceq \varphi, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)$ (resp., $\text{false}(\alpha \preceq \varphi, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)$), in which case we are looking for a counter-model), effectively demonstrates that φ is α -satisfiable (resp., α -valid, as closing such a tableau proves that no counter-model exists). Conversely, the reversed decorations $\text{true}(\varphi \preceq \alpha, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)$ and $\text{false}(\varphi \preceq \alpha, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)$ are generally not employed as starting decorations to prove substantial statements. For example, $\text{true}(\varphi \preceq \alpha, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)$ might suggest the possibility of constructing a structure where φ has a value less than α . In classical logic, with a two-element Boolean algebra and $\alpha = 1$, this condition would imply that $\neg\varphi$ is valid, as φ would consistently take the value 0. However, in the many-valued case, such duality is absent, and there is no guarantee of a formula that consistently attains the value 1 when φ is valued strictly less than 1.

$$\begin{array}{lll}
(\mathbf{X1}) \frac{true(\beta \preceq \gamma, w, \tilde{F})}{\mathbf{X}} & (\mathbf{X2}) \frac{false(\beta \preceq \gamma, w, \tilde{F})}{\mathbf{X}} & (\mathbf{X3}) \frac{false(0 \preceq \psi, w, \tilde{F})}{\mathbf{X}} \\
\text{where } \beta \not\preceq \gamma & \text{where } \beta \succ 0, \gamma \prec 1, \text{ and } \beta \preceq \gamma & \\
\\
(\mathbf{X4}) \frac{false(\psi \preceq 1, w, \tilde{F})}{\mathbf{X}} & (\mathbf{X5}) \frac{false(\beta \preceq \psi, w, \tilde{F}')}{\mathbf{X}} & (\mathbf{X6}) \frac{Q(\cdot, \cdot, \tilde{F})}{\mathbf{X}} \\
& \text{where } \beta \preceq \gamma & \text{where } \tilde{F} \text{ is inconsistent}
\end{array}$$

Fig. 6. Branch closing rules. Every branch $B \in \mathcal{B}$ containing the node v which is being expanded must be closed.

Despite their limited use as initial decorations, these reversed decorations, in conjunction with the reverse rule facilitate reducing the number of necessary rules. For instance, a decoration including the judgment $true(p \preceq \beta, w, \tilde{F})$ appearing in a branch of a tableau starting with the judgment $true(\alpha \preceq \varphi, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)$ informs us that in any model validating α -satisfiability of φ , the propositional variable p must take a value less than or equal to β . For any given formula φ and value α , a tableau starting with $true(\alpha \preceq \varphi, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)$ (resp., $false(\alpha \preceq \varphi, w_0, \langle \{w_0\}, \emptyset, \dots, \emptyset \rangle)$) is termed SAT-tableau (resp., VAL-tableau).

Two distinct notions of soundness and completeness are applicable: one for the SAT-tableau system and another for the VAL-tableau system. Although the foundational arguments for both systems bear strong similarities, our statement is focused on the former.

LEMMA 1. *The tableau system for many-valued multi-modal logic of finite FL_{ew} -algebras is sound for proving α -satisfiability, that is, given a finite FL_{ew} -algebra $\mathbf{A}$, an FL_{ew} - K_n -formula φ , if φ is α -satisfiable then there exists an open SAT-tableau for φ and α .*

PROOF. Consider a formula φ . Assume that τ is the tableau for φ and $\alpha \in \mathbf{A}$, where $\mathbf{A}$ is a finite FL_{ew} -algebra. We proceed contrapositively to prove that if τ is closed then φ is not α -satisfiable. Given a node v in τ such that $\tilde{F} = \langle W, \tilde{R}_1, \dots, \tilde{R}_n \rangle$ is the frame in its decoration,

$$S(v) = \{v' \mid v' \triangleleft v\}$$

and we say that $S(v)$ is α -satisfiable if and only if there is a structure based on $\mathbf{A}$ of the type

$$\tilde{M} = \langle \tilde{F}^*, \tilde{V} \rangle,$$

where $\tilde{F}^* = \langle W^*, \tilde{R}_1^*, \dots, \tilde{R}_n^* \rangle$ is a structure that extends $\tilde{F}$, such that

- for each node $v' \in S(v)$ such that $d(v') = true(\beta \preceq \psi, w, \tilde{F}')$ (resp., $false(\psi \preceq \beta, w, \tilde{F}')$), it is the case that $\tilde{V}(\psi, w) \succeq \beta$ (resp., $\tilde{V}(\psi, w) \succeq \gamma$, for some minimal γ not below β), and
- for each node $v' \in S(v)$ such that $d(v') = true(\psi \preceq \beta, w, \tilde{F}')$ (resp., $false(\beta \preceq \psi, w, \tilde{F}')$), it is the case that $\tilde{V}(\psi, w) \preceq \beta$ (resp., $\tilde{V}(\psi, w) \preceq \gamma$ for some maximal γ not above β).

Observe that $\tilde{M}$ depends on α since $v_0 \in S(v)$, for every v . Moreover, if $S(v_0)$ is α -satisfiable, then φ is α -satisfiable. Now, we prove the following stronger statement: if every branch containing a node v is closed, then the set $S(v)$ is not α -satisfiable. Let us proceed by induction on the height h of the node v .

- If $h = 0$, then there is exactly one branch that contains it. Since such branch is closed, one of the following must hold. Firstly, for some $v' \in S(v)$, $d(v') = true(\beta \preceq \gamma, w, \tilde{F}')$ but $\beta \not\preceq \gamma$ (condition $(\mathbf{X1})$), or $d(v') = false(\beta \preceq \gamma, w, \tilde{F}')$ but $\beta \preceq \gamma$ (condition $(\mathbf{X2})$). Secondly, for some $v' \in S(v)$, $d(v') = false(0 \preceq \psi, w, \tilde{F}')$

(condition (X3)), or $false(\psi \preceq 1, w, \tilde{F}')$ (condition (X4)). Or, thirdly, for some $v', v'' \in S(v)$, $d(v') = false(\beta \preceq \psi, w, \tilde{F}')$ and $d(v'') = true(\gamma \preceq \psi, w, \tilde{F}'')$, but $\gamma \not\preceq \beta$ (condition (X5)). In all such cases, $\tilde{M}$ cannot be realized, so $S(v)$ is not α -satisfiable, as we wanted.

- Suppose, now, that $h > 0$. First, observe that if every branch that contains v is closed, then every branch that contains any of its successors must be closed too, so that the inductive hypothesis applies to them. Then, consider the node $v' \in S(v)$ that has been expanded when v was a leaf, and let us analyze the possible rules that have been applied at v' . If $d(v') = true(\psi \preceq \beta, w, \tilde{F}')$, then the immediate successor v'' of v is such that $d(v'') = false(\gamma \preceq \psi, w, \tilde{F}'')$, where γ is some minimal element of A such that $\gamma \not\preceq \beta$ (rule $(true \preceq)$); by inductive hypothesis, $S(v'')$ is not α -satisfiable, but this implies that $S(v)$ cannot be α -satisfiable either. If $d(v') = false(\beta \preceq \psi, w, \tilde{F}')$, then all immediate successors v_j of v are such that $d(v_j) = true(\psi \preceq \gamma_j, w, \tilde{F}_j)$, where γ_j is a maximal element of A such that $\gamma_j \not\preceq \beta$ (rule $(false \succeq)$); by inductive hypothesis, $S(v_j)$ is not α -satisfiable for any j , but this implies that $S(v)$ cannot be α -satisfiable either. The cases in which another reverse rule has been applied to v are similar. If $d(v') = true(\beta \preceq (\psi \wedge \xi), w, \tilde{F}')$, then v has an immediate successor v_1 with $d(v_1) = true(\beta \preceq \psi, w, \tilde{F}_1)$, which in turn has an immediate successor v_2 with $d(v_2) = true(\beta \preceq \xi, w, \tilde{F}_2)$ (rule $(true \wedge)$). By inductive hypothesis, $S(v_2)$, in particular, is not α -satisfiable, but this implies that $S(v)$ is not α -satisfiable either. If $d(v') = false(\beta \preceq (\psi \wedge \xi), w, \tilde{F}')$, then v has two immediate successors v_1 and v_2 with $d(v_1) = false(\beta \preceq \psi, w, \tilde{F}_1)$ and $d(v_2) = false(\beta \preceq \xi, w, \tilde{F}_2)$ (rule $(false \wedge)$). By inductive hypothesis, both $S(v_1)$ and $S(v_2)$ are not α -satisfiable, but this implies that $S(v)$ is not α -satisfiable either. The cases in which another propositional rule has been applied to v are similar. If $d(v') = true(\beta \preceq [X_i]\psi, w, \tilde{F}')$ then v has a chain of successors $v_1, \dots, v_m$, such that $d(v_j) = true((\beta \cdot \gamma_j) \preceq \psi, w_j, \tilde{F}_j)$ and $\gamma_j = \tilde{R}_i(w, w_j)$, for all $w_j \in o(c(B))$, where $1 \leq j \leq m$ (rule $(true \Box)$). Observe that asking that the evaluation of $[X_i]\psi$ is above β is equivalent to asking that the evaluation of ψ is above $\beta \cdot \gamma_i$ on every world w_j . Since, in particular, $S(v_m)$ is not α -satisfiable by inductive hypothesis, $S(v)$ is not α -satisfiable as well. The case in which $(true \Diamond)$ has been applied to v is similar. Finally, if $d(v') = false(\langle X_i \rangle \psi \preceq \beta, w, \tilde{F}')$, then every immediate successor v_j of v is such that $d(v_j) = false(\psi \preceq (\gamma_j \hookrightarrow \beta), w_j, \tilde{F}_j)$ and $\gamma_j = \tilde{R}_i(w, w_j)$, for all $w_j \in o(c(B)) \cup n(c(B))$, where $1 \leq i \leq m$ (rule $(false \Diamond)$). Observe that asking that the evaluation of $\langle X_i \rangle \psi$ is below β is equivalent to asking that the evaluation of ψ is below $\gamma_j \hookrightarrow \beta$ on some world w_j . Since all $S(v_j)$ are not α -satisfiable by inductive hypothesis, $S(v)$ is not α -satisfiable as well. The case in which $(false \Box)$ has been applied to v is similar.

□

LEMMA 2. *The tableau system for many-valued multi-modal logic of finite FL_{ew} -algebras is complete for proving α -satisfiability, that is, given a finite FL_{ew} -algebra A , an FL_{ew} - K_n -formula φ , and a value $\alpha \in A$, if there is an open SAT-tableau for φ and α , then φ is α -satisfiable.*

PROOF. Consider formula φ , and assume that τ is a SAT-tableau for φ and α . Consider an open branch B in τ and let $\tilde{F}^* = \bigcup_{v \in B} \tilde{F}_v$, where $\tilde{F}_v$ is the frame in $d(v)$. Then, consider the structure

$$\tilde{M} = \langle \tilde{F}^*, \tilde{V} \rangle,$$

where $\tilde{V}$ is any valuation function that respects the following for every propositional letter p and world w in $\tilde{F}^*$:

$$\tilde{V}(p, w) = \begin{cases} \beta & \text{if } d(v) = true(\beta \preceq p, w, \tilde{F}_v), \text{ for some } v \in B; \\ \gamma & \text{if } d(v) = false(\beta \preceq p, w, \tilde{F}_v), \text{ for some } v \in B \text{ and some maximal } \gamma \not\preceq \beta. \end{cases}$$

The structure $\tilde{M}$ is the direct translation of the branch B ; in particular, it holds a coherent assignment of truth values of all propositional letters on all worlds. As far as the case of the judgment $false(\beta \preceq p, w, \tilde{F})$ is concerned, we need to associate any truth value γ such that $\gamma \not\preceq \beta$.

$\cdot_{G3}$	0	α	1	$\cdot_{MV3}$	0	α	1	$\hookrightarrow_{G3}$	0	α	1	$\hookrightarrow_{MV3}$	0	α	1
0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1
α	0	α	α	α	0	0	α	α	0	1	1	α	α	1	1
1	0	α	1	1	0	α	1	1	0	α	1	1	0	α	1

Fig. 7. Truth tables for G3 and MV3 (i.e., 3-valued algebras with $0 \prec \alpha \prec 1$) t-norms and their relative implications; the values which vary from G3 to MV3 are highlighted in red, while the columns used to expand the *false* $\diamond$ rules in the G3 and MV3 tableaux in Fig. 9 and 10 are highlighted in light gray.

We want to prove that, for every $v \in B$,

- if $d(v) = \text{true}(\beta \preceq \psi, w, \tilde{F}_v)$, then $\tilde{V}(\psi, w) \succeq \beta$,
- if $d(v) = \text{true}(\psi \preceq \beta, w, \tilde{F}_v)$, then $\tilde{V}(\psi, w) \preceq \beta$,
- if $d(v) = \text{false}(\beta \preceq \psi, w, \tilde{F}_v)$, then $\tilde{V}(\psi, w) \not\succeq \beta$, and
- if $d(v) = \text{false}(\psi \preceq \beta, w, \tilde{F}_v)$, then $\tilde{V}(\psi, w) \not\preceq \beta$.

Observe that the above implies that φ is α -satisfiable on w_0 in $\tilde{M}$, that is, it is α -satisfiable. Also, observe that $\tilde{M}$ is constructible and well-defined because B is open. Consider a node $v \in B$ such that $d(v)$ is a decoration with a judgment for a formula ψ on some world w . We proceed by structural induction on ψ .

- If $\psi = p$ or $\psi = \beta$, then the claim is trivial.
- If $\psi = \xi \wedge \chi$, then suppose, first, that $d(v) = \text{true}(\beta \preceq (\xi \wedge \chi), w, \tilde{F}_v)$. Since τ is fully expanded, rule (*true* $\wedge$) has been applied to v . It follows that B contains two nodes v_1 and v_2 such that $d(v_1) = \text{true}(\beta \preceq \xi, w, \tilde{F}_{v_1})$ and $d(v_2) = \text{true}(\beta \preceq \chi, w, \tilde{F}_{v_2})$. By inductive hypothesis, $\tilde{V}(\xi, w) \succeq \beta$ and $\tilde{V}(\chi, w) \succeq \beta$, which is equivalent to $\tilde{V}(\psi, w) \succeq \beta$. Suppose, now, that $d(v) = \text{false}(\beta \preceq (\xi \wedge \chi), w, \tilde{F}_v)$. Since τ is fully expanded, rule (*false* $\wedge$) has been applied to v . It follows that B contains a node v' such that $d(v') = \text{false}(\beta \preceq \xi, w, \tilde{F}_{v'})$ or $d(v') = \text{false}(\beta \preceq \chi, w, \tilde{F}_{v'})$. By inductive hypothesis, $\tilde{V}(\xi, w) \not\succeq \beta$ or $\tilde{V}(\chi, w) \not\succeq \beta$, which is equivalent to $\tilde{V}(\psi, w) \not\succeq \beta$. The remaining propositional cases are similar.
- Finally, if $\psi = [X_i]\xi$, then suppose, first, that $d(v) = \text{true}(\beta \preceq [X_i]\xi, w, \tilde{F}_v)$. Since τ is fully expanded, rule (*true* $\Box$) has been applied to v . This entails that, for every world w_j in $\tilde{F}^*$, B contains a node v_j such that $d(v_j) = \text{true}((\beta \cdot \gamma_j) \preceq \xi, w_j, \tilde{F}_{v_j})$, that is, $d(v_j) = \text{true}((\beta \cdot \tilde{R}_i(w, w_j)) \preceq \xi, w_j, \tilde{F}_{v_j})$; observe that this is guaranteed by the fact that the rule has been applied at a certain point of the construction on the m possible worlds that were in the frame at that point, but then an additional node was also created at the end of the branch with the same decoration, ensuring that, should any more worlds be added at some later moment, the rule is applied, again, on them. By inductive hypothesis, $\tilde{V}(\xi, w_j) \succeq (\beta \cdot \gamma_j)$. Therefore, $\tilde{V}(\psi, w) = \bigcap_{w_j} \{\tilde{R}_i(w, w_j) \hookrightarrow \tilde{V}(\xi, w_j)\} \succeq \beta$. Suppose, now, that $d(v) = \text{false}(\beta \preceq [X_i]\xi, w, \tilde{F}_v)$. Since τ is fully expanded, rule (*false* $\Box$) has been applied to v . This entails that, for some world w_j in $\tilde{F}^*$, B contains a node v_j such that $d(v_j) = \text{false}((\beta \cdot \gamma_j) \preceq \xi, w_j, \tilde{F}_{v_j})$, that is, $d(v_j) = \text{false}((\beta \cdot \tilde{R}_i(w, w_j)) \preceq \xi, w_j, \tilde{F}_{v_j})$. By inductive hypothesis, $\tilde{V}(\xi, w_j) \not\succeq \beta \cdot \gamma_j$. Therefore, $\tilde{V}(\psi, w) = \bigcap_{w_j} \{\tilde{R}_i(w, w_j) \hookrightarrow \tilde{V}(\xi, w_j)\} \not\succeq \beta$. The remaining cases are similar.

□

THEOREM 3. *The tableau system for FL_{ew}-K_n is sound and complete.*

Figs. 7, 8, 9 and 10 serve as complete examples of an application of the tableau system to find if the formula $\diamond(p \wedge \alpha) \wedge \alpha$ of FL_{ew}-K (i.e., comprising only one $\diamond$ and one $\Box$ modality) is α -satisfiable in G3 and MV3, respectively.

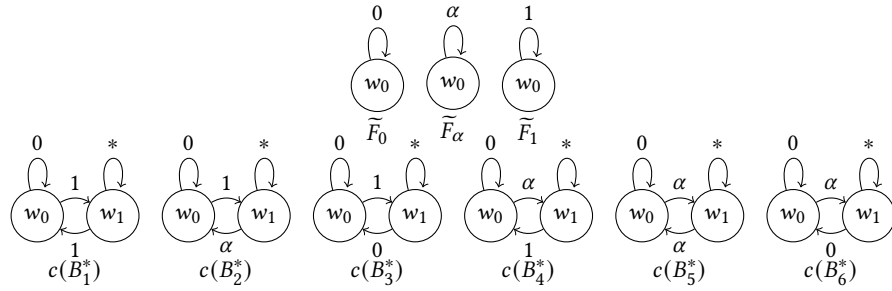


Fig. 8. On startup, 3 tableaux will be instantiated with frames $\tilde{F}_0$, $\tilde{F}_\alpha$ and $\tilde{F}_1$ respectively. When expanding the node $false(\diamond(p \wedge \alpha) \preceq 0, w_0, \tilde{F}_0)$ in Fig. 9, 18 new branches are generated with frames $c(B_1^0), c(B_1^\alpha), c(B_1^1), \dots, c(B_6^1), (B_6^\alpha), c(B_6^1)$ respectively, since we are taking all the relations with value γ such that $\gamma \hookrightarrow 0 < 1$ (see $\hookrightarrow_{G3}$ in Fig. 7). When expanding $false(\diamond(p \wedge \alpha) \preceq \alpha, w_0, \tilde{F}_0)$ in Fig. 10, 9 new branches are generated with frames $c(B_1^0), c(B_1^\alpha), c(B_1^1), \dots, c(B_3^0), c(B_3^\alpha), c(B_3^1)$ respectively, since we are taking all the relations with value γ such that $\gamma \hookrightarrow \alpha < 1$ (see $\hookrightarrow_{MV3}$ in Fig. 7).

3 Applications

In this section, we introduce various modal temporal and spatial logics, we provide a uniform procedure for their manification, and we discuss the implementation of a reasoning tool based on the tableau from Sec. 2.2 for such logics, which is thoroughly tested, also against a first-order translation leveraging a state of the art SMT solver.

3.1 Modal Spatial and Temporal Logic

The literature on modal spatial and temporal logic is very wide, going back to the seventies. Temporal and spatial logic languages have been introduced in several forms, taking points or intervals as primitive objects in the temporal case, and points or extended areas in the spatial case. The typical choices in terms of relations include directional relations (e.g., in the future, or upwards), as well as topological ones (e.g., during, or inside). Spatial and temporal logic languages have been proposed for knowledge representation in several application areas, although they are usually not examined together. However, some of them share a common multi-modal nature, and it is sometimes convenient to study them as particular cases of a generic multi-modal language. In fact, multi-modal logic is representative of a wide range of spatial and temporal logic languages (although this is not true for all of them, e.g., LTL with ‘next’ and ‘until’). A modal spatial and temporal logic is typically obtained by selecting worlds from a linear order or a Euclidean space, and constraining the accessibility relations accordingly.

Linear Temporal Logic with Future and Past. Linear temporal logic (LTL) (Pnueli 1977) is a temporal logic proposed by Amir Pnueli in 1977 for formal software and hardware verification. Classically, it is defined using the operators for ‘next’ and ‘until’, as well as their past counterparts, that require the underlying linear order to be discrete, and do not have an immediate modal interpretation. However, in LTL one can immediately define qualitative operators for ‘sometime in the future’ and ‘sometime in the past’; historically, such a fragment is denoted by LTL[F,P]. The fragment itself allows for a natural modal interpretation, and it is referred to as the basic temporal language (Blackburn et al. 2001), reflecting the observation that, given a linearly ordered set, there are exactly three ways in which two points can be placed relative to each other: they are either the same point, or one is in the future (resp., in the past) with respect to the other. This fragment can be interpreted as a Multi-Modal Logic as follows.

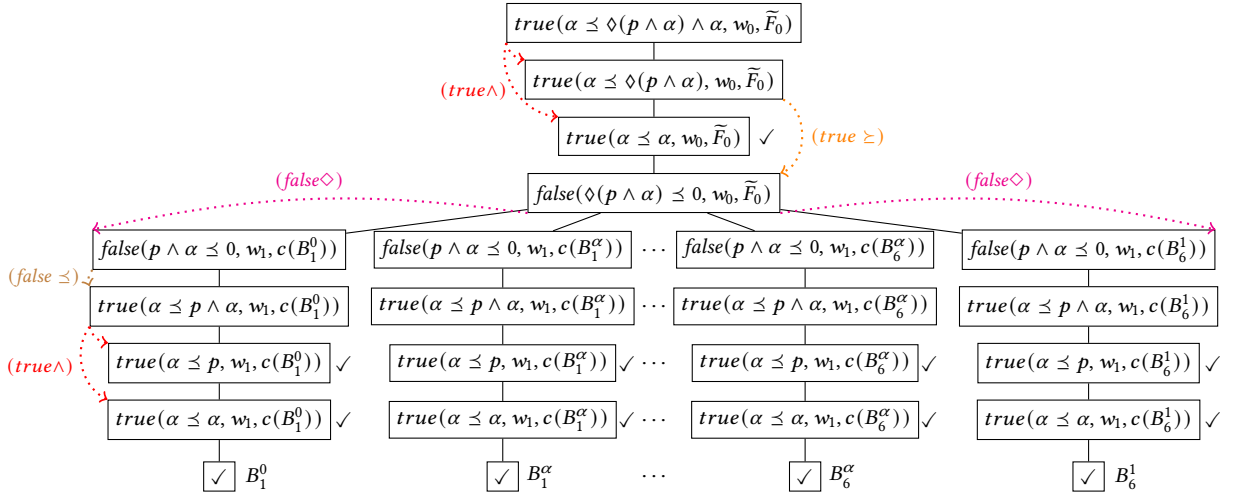


Fig. 9. Fully evaluated SAT-tableau for $\alpha \leq \Diamond(p \wedge \alpha) \wedge \alpha$ in G3, starting with frame $\tilde{F}_0$ (tableaux starting from $\tilde{F}_\alpha, \tilde{F}_1$ look very similar). Note that we only needed to fully expand one branch (e.g., B_1^0), to prove α -satisfiability; here, we choose to show all possible branches for example purposes. Frames $\tilde{F}_0, c(B_1^0), c(B_1^\alpha), c(B_1^1), \dots, c(B_6^0), c(B_6^\alpha), c(B_6^1)$ are represented in Fig. 8.

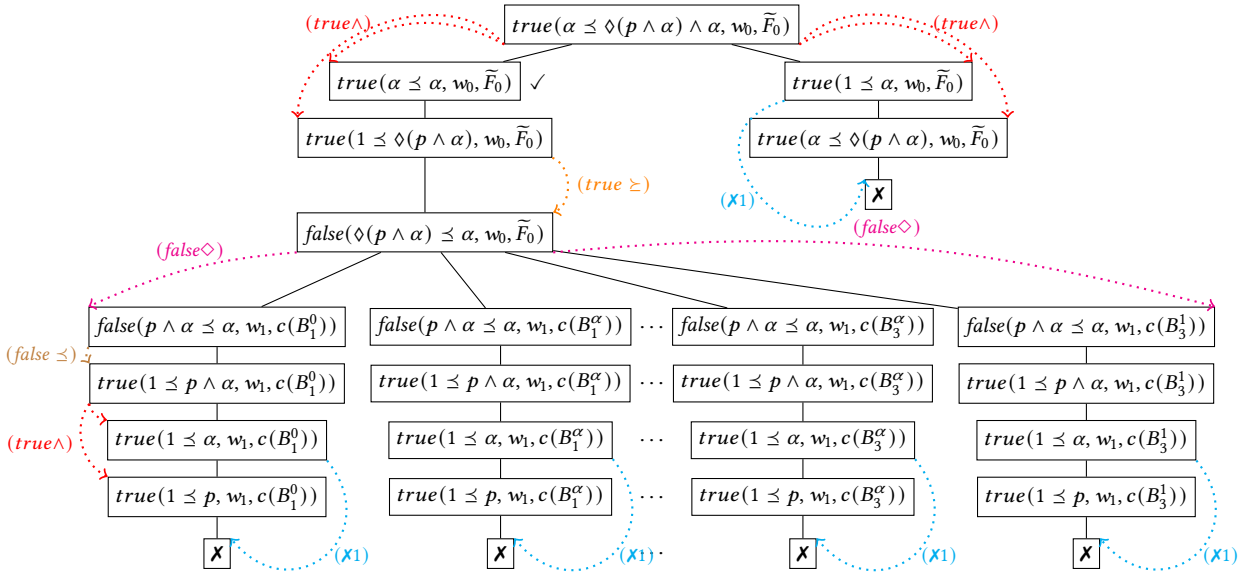


Fig. 10. Fully evaluated SAT-tableau for $\alpha \leq \Diamond(p \wedge \alpha) \wedge \alpha$ in MV3, starting with frame $\tilde{F}_0$. In this example, applying the rules of the tableau system results in closing all branches (tableaux starting with $\tilde{F}_\alpha, \tilde{F}_1$ are similarly closed), proving that the formula is not α -satisfiable in MV3. Frames $\tilde{F}_0, c(B_1^0), c(B_1^\alpha), c(B_1^1), \dots, c(B_3^0), c(B_3^\alpha), c(B_3^1)$ are represented in Fig. 8.

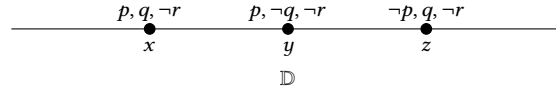


Fig. 11. An example of a point-based temporal model with three points x, y and z that satisfy $\{p, q\}$, $\{p\}$, and $\{q\}$ respectively. In this model, for instance, $\langle F \rangle q$ holds at the point x , $[P]p$ holds at the point z , and $\langle F \rangle \langle P \rangle q$ holds at the point y .

Let $\mathbb{D} = \langle D, < \rangle$ be a (strict) linear order with domain D . In $\text{LTL}[F, P]$, points, denoted here by $x, y, \dots$, play the role of worlds. Consider the two following relations on $\mathbb{D} \times \mathbb{D}$:

$$\begin{aligned} R_F &= \{ \langle x, y \rangle \mid x < y \}, \\ R_P &= R_F^{-1}, \end{aligned}$$

called, respectively, *future* and *past*. For each binary relation R_X , where $X \in \{F, P\}$, there is a corresponding unary modality $\langle X \rangle$. Thus, well-formed $\text{LTL}[F, P]$ -formulas are obtained by a straightforward adaptation of Definition 1, and interpreted on point-based temporal structures of the type $M = \langle F, V \rangle$, where F is a point-based temporal frame $F = \langle \mathbb{D}, R_X \rangle$, with $X \in \{F, P\}$. To obtain a notion of $\text{LTL}[F, P]$ -formula satisfied on a point-based temporal structure at a point x , Definition 2 is immediately adapted to the case of $\text{LTL}[F, P]$ with a clause

$$M, x \models \langle X \rangle \psi \quad \text{iff} \quad \text{there is a } y \text{ s.t. } xR_X y \text{ and } M, y \models \psi,$$

for $X \in \{F, P\}$. An example of a point-based temporal model is shown in Fig. 11.

The satisfiability problem for LTL and its fragments is well known to be decidable: starting with the study of its complexity for future-only fragments of LTL (Sistla and Clarke 1985), this analysis continued with the addition of past operators (Lichtenstein and Pnueli 2000), and culminated in a wide range of available reasoners (Geatti et al. 2024; Li et al. 2013). Unfortunately, none of these tools are able to deal with the many-valued interpretation of LTL and its fragments.

Compass logic. Compass Logic (CL) (Blackburn et al. 2001) is a two-dimensional modal logic introduced by Venema and later studied by Marx and Reynolds (Marx and Reynolds 1999), which has been applied in database systems, natural language processing, and computational planning. CL has an obvious geometric interpretation, and can be considered the immediate two-dimensional generalization of $\text{LTL}[F, P]$, featuring operators for ‘somewhere upward’, ‘somewhere to the right’, and their reversed counterparts.

A modal interpretation for CL can be given as follows. Let $\mathbb{D}_1 = \langle D_1, <_1 \rangle, \mathbb{D}_2 = \langle D_2, <_2 \rangle$ be two (strict) linear orders with domains D_1, D_2 . In CL, we interpret such linear orders as linear spatial dimensions of a Cartesian plane $\mathbb{D}_1 \times \mathbb{D}_2$. By convention, we use $\mathbb{D}_1$ (resp., $\mathbb{D}_2$) as the horizontal (resp., vertical) axis. In CL, each pair of points in $\mathbb{D}_1 \times \mathbb{D}_2$, denoted here by $P, Q, \dots$, plays the role of a modal world. Consider the following relations on $(\mathbb{D}_1 \times \mathbb{D}_2) \times (\mathbb{D}_1 \times \mathbb{D}_2)$, where $P = \langle x_1, x_2 \rangle$ and $Q = \langle y_1, y_2 \rangle$:

$$\begin{aligned} R_U &= \{ \langle P, Q \rangle \mid x_1 = y_1 \wedge x_2 < y_2 \}, \\ R_R &= \{ \langle P, Q \rangle \mid x_2 = y_2 \wedge x_1 < y_1 \}, \\ R_D &= R_U^{-1}, \\ R_L &= R_R^{-1}, \end{aligned}$$

called, respectively, *up*, *right*, *down*, and *left*, as well as their corresponding modalities $\langle X \rangle$, with $X \in \{U, R, D, L\}$. An immediate adaptation of Definition 1 produces well-formed CL-formulas, which are interpreted over point-based spatial structures of the type $M = \langle F, V \rangle$, where F is a point-based spatial frame of the type $F = \langle (\mathbb{D}_1 \times \mathbb{D}_2), R_X \rangle$, $X \in \{U, R, D, L\}$. A notion of CL-formula satisfied on a point-based spatial structure at a point P is obtained by adapting Definition 2 with a clause

$$M, P \models \langle X \rangle \psi \quad \text{iff} \quad \text{there is a } Q \text{ s.t. } PR_X Q \text{ and } M, Q \models \psi,$$

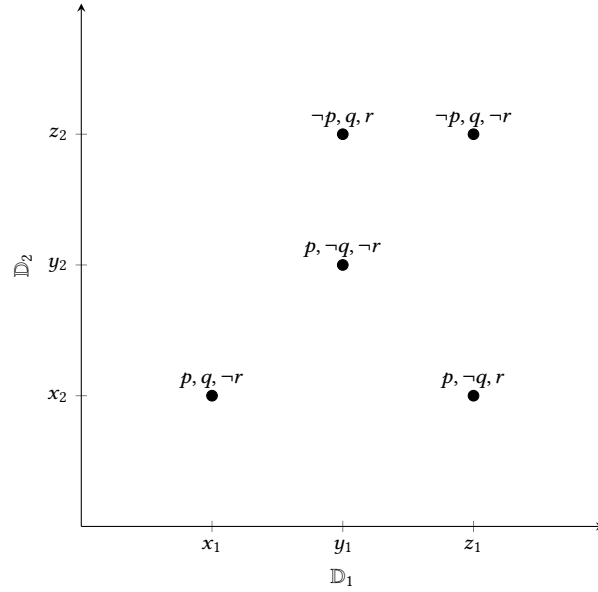


Fig. 12. In this example, a point-based spatial model with nine points. In particular, the points with a non-null valuation function are $P = \langle x_1, x_2 \rangle$, which satisfies $\{p, q\}$, $Q = \langle y_1, y_2 \rangle$, which satisfies $\{p\}$, $R = \langle z_1, x_2 \rangle$, which satisfies $\{p, r\}$, $S = \langle y_1, z_2 \rangle$, which satisfies $\{q, r\}$, and $T = \langle z_1, z_2 \rangle$, which satisfies $\{q\}$. For instance, in this model, $\langle U \rangle \langle R \rangle q$ is satisfied at P (e.g., q is true at R , which is ‘right’ of $\langle x_1, x_2 \rangle$, which is ‘up’ of P), while $[D][L]\neg r$ is satisfied at Q (in fact, r is false at P , which is the only point ‘left’ of $\langle y_1, x_2 \rangle$, which is the only point ‘down’ of Q).

for $X \in \{U, R, D, L\}$. An example of a point-based spatial model is shown in Fig. 12.

The satisfiability problem for CL is undecidable, and no reasoning tools exist that directly or indirectly allow one to deal with CL-formulas, except via standard translation to first-order logic, and –needless to say– such a strategy would make it very hard to reason in the many-valued context.

Halpern and Shoham’s Modal Logic of Time Intervals. In the context of interval temporal logic, the modal language introduced by Halpern and Shoham (HS) (Halpern and Shoham 1991) is probably the one that received the most attention. As reported in (Goranko, Montanari, and Sciavicco 2004), HS is one of three common interval languages, the other two being Venema’s CDT (Venema 1991), and Moszkowski’s ITL (Moszkowski 1985). ITL, just like the original version of LTL, lacks a clear modal interpretation (though successive variants/fragments of it can be considered as fragments of CDT); this is not the case for HS and CDT. From an algebraic point of view, both HS and CDT are based on a notion of jointly exhaustive and mutually exclusive set of relations in an interval domain; the main difference between them is that CDT uses three ternary relations, while HS uses thirteen binary ones. Binary relations correspond to unary modal operators and can be considered closer to the human usage of language, making HS more intuitive than CDT. In a sense, HS is the interval counterpart of LTL[F,P], including relations between extended events such as ‘during’ and ‘overlaps’.

Let $\mathbb{D} = \langle D, < \rangle$ be a linear order with domain D . A strict interval over $\mathbb{D}$ is an ordered pair $[x, y]$, where $x, y \in \mathbb{D}$ and $x < y$. In HS, the role of worlds is played by intervals; let $\mathbb{I}(\mathbb{D}) = \{[x, y] \mid x, y \in \mathbb{D}, x < y\}$, and let us denote intervals using $I, J, \dots$. Excluding the identity relation, there are 12 different binary ordering relations between two strict intervals on a linear order, often called Allen’s interval relations (Allen 1983); they are defined as follows, where $I = [x_1, y_1]$ and $J = [x_2, y_2]$:

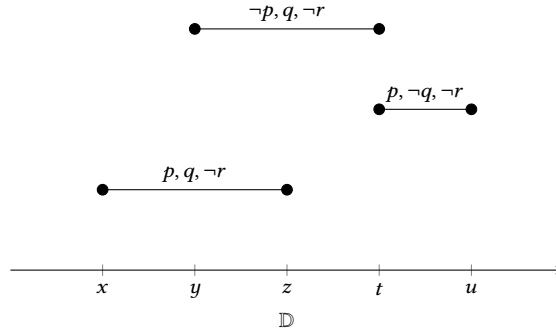


Fig. 13. The interval model shown in the example has five points and ten intervals. The intervals with a non-null valuation function are $I = [x, z]$, satisfying $\{p, q\}$, $J = [y, t]$, satisfying $\{q\}$, and $K = [t, u]$, satisfying $\{p\}$. Among many possible formulas, in this model $\langle O \rangle q$ is true at I (e.g., q is true at J , which is ‘overlapping’ with I), while $[\bar{L}] \neg r$ is true at K (in fact, r is *false* at I , which is the only interval with a non-null valuation function accessible with the ‘before’ relation from K).

$$\begin{aligned}
 R_A &= \{\langle I, J \rangle \mid y_1 = x_2\}, \\
 R_L &= \{\langle I, J \rangle \mid y_1 < x_2\}, \\
 R_B &= \{\langle I, J \rangle \mid x_1 = x_2 \wedge y_2 < y_1\}, \\
 R_E &= \{\langle I, J \rangle \mid y_1 = y_2 \wedge x_1 < x_2\}, \\
 R_D &= \{\langle I, J \rangle \mid x_1 < x_2 \wedge y_2 < y_1\}, \\
 R_O &= \{\langle I, J \rangle \mid x_1 < x_2 \wedge x_2 < y_1 \wedge y_1 < y_2\}, \\
 R_{\bar{A}} &= R_A^{-1}, \\
 R_{\bar{L}} &= R_L^{-1}, \\
 R_{\bar{B}} &= R_B^{-1}, \\
 R_{\bar{E}} &= R_E^{-1}, \\
 R_{\bar{D}} &= R_D^{-1}, \\
 R_{\bar{O}} &= R_O^{-1},
 \end{aligned}$$

which are named, respectively, *meets*, *later*, *starts*, *finishes*, *during*, *overlaps*, *met by*, *before*, *started by*, *finished by*, *contains*, and *overlapped by*. In HS too, to each binary relation R_X , with $X \in \{A, \bar{A}, L, \bar{L}, B, \bar{B}, E, \bar{E}, D, \bar{D}, O, \bar{O}\}$, corresponds a unary modal operator $\langle X \rangle$, and well-formed formulas come from readily adapting Definition 1. Formulas of HS are interpreted over interval structures of type $M = \langle F, V \rangle$, F being an interval frame of type $F = \langle \mathbb{I}(\mathbb{D}), R_X \rangle$, where $X \in \{A, \bar{A}, L, \bar{L}, B, \bar{B}, E, \bar{E}, D, \bar{D}, O, \bar{O}\}$.

The notion of a HS-formula being satisfied on an interval structure at an interval I is obtained by adapting Definition 2 with a clause

$$M, I \models \langle X \rangle \psi \quad \text{iff} \quad \text{there is a } J \text{ s.t. } IR_X J \text{ and } M, J \models \psi,$$

for $X \in \{A, \bar{A}, L, \bar{L}, B, \bar{B}, E, \bar{E}, D, \bar{D}, O, \bar{O}\}$. An example of an interval model is shown in Fig. 13.

The satisfiability problem for HS is undecidable; yet much work has been done to study the complexity of its fragments, which ranges from NP to EXPSpace and beyond (Bresolin, Della Monica, Montanari, Sala, et al. 2019, 2014). While a few attempts exist to provide a reasoning system for HS, none is currently available. A many-valued generalization of HS leveraging only Heyting algebras has been presented in (Conradie, Della Monica, et al. 2023), with a first attempt at a tableau system presented in (Badia, Noguera, et al. 2024; Conradie, Monego, et al. 2023).

Lutz and Wolter's Modal Logic of Topological Relations. In qualitative spatial reasoning, typical approaches are usually directional or topological. Directional relations follow some underlying order, as in the case of 'to the right', or 'on top of', while topological ones encode order-symmetric situations, such as 'within'.

The separation between these choices is not always perfectly sharp; interval relations (time intervals can be seen as one-dimensional areas) can be considered partially topological and partially directional at the same time. Moreover, spatial reasoning has been sometimes addressed from the viewpoint of extended areas with no specific shape, maybe assuming convexity alone, while other approaches require areas to be, for example, rectangular. In the case of the Modal Logic of Topological Relations (Lutz and Wolter 2006), extended areas may be of any shape, and two areas can be related by exactly one of eight so-called RCC8 relations (Randell et al. 1992). The version of Lutz and Wolter's language that actually assumes areas to be rectangular and axes-aligned has been studied in the same paper (Lutz and Wolter 2006). With rectangular areas, however, one may study every binary relation that emerges from a pair of interval relations (one per each axis); the logic so obtained can be regarded as the bi-dimensional version of HS, and encompasses as many as 169 binary (directional) relations. Reasoning with a language with such high expressive power may not be very natural; for simplicity, we confine ourselves to the language of L_{RCC8}^{Rec} , with rectangular, axes-aligned areas, and eight topological relations, namely L_{RCC8}^{Rec} .

Let $\mathbb{D}_1 = \langle D_1, <_1 \rangle$, $\mathbb{D}_2 = \langle D_2, <_2 \rangle$ be two (strict) linear orders with domains D_1, D_2 . In L_{RCC8}^{Rec} , we interpret such linear orders as linear spatial dimensions of a Cartesian plane $\mathbb{D}_1 \times \mathbb{D}_2$. As it was for CL, by convention we use $\mathbb{D}_1$ (resp., $\mathbb{D}_2$) as the horizontal (resp., vertical) axis. In L_{RCC8}^{Rec} , each rectangle $R, S, \dots$ plays the role of a world; let $\mathbb{R}(\mathbb{D}) = \{([x_1, y_1], [x_2, y_2]) \mid x_1, y_1 \in \mathbb{D}_1, x_2, y_2 \in \mathbb{D}_2, x_1 < y_1, x_2 < y_2\}$. Consider the following topological relations on $\mathbb{R}(\mathbb{D}) \times \mathbb{R}(\mathbb{D})$, where $R = ([x_1, y_1], [x_2, y_2])$ and $S = ([x'_1, y'_1], [x'_2, y'_2])$:

$$\begin{aligned}
R_{DC} &= \{ \langle R, S \rangle \mid y'_1 < x_1 \vee y_1 < x'_1 \vee y'_2 < x_2 \vee y_2 < x'_2 \}, \\
R_{EC} &= \{ \langle R, S \rangle \mid y'_1 = x_1 \vee y_1 = x'_1 \vee y'_2 = x_2 \vee y_2 = x'_2 \} \setminus R_{DC}, \\
R_{NTPP} &= \{ \langle R, S \rangle \mid x_1 < x'_1 \wedge y'_1 < y_1 \wedge x_2 < x'_2 \wedge y'_2 < y_2 \}, \\
R_{TPP} &= \{ \langle R, S \rangle \mid \\
&\quad x_1 = x'_1 \wedge (y'_1 < y_1 \vee y'_1 = y_1) \wedge (x_2 < x'_2 \vee x_2 = x'_2) \wedge (y'_2 < y_2 \vee y'_2 = y_2) \vee \\
&\quad y_1 = y'_1 \wedge (x_1 < x'_1 \vee x_1 = x'_1) \wedge (x_2 < x'_2 \vee x_2 = x'_2) \wedge (y'_2 < y_2 \vee y'_2 = y_2) \vee \\
&\quad x_2 = x'_2 \wedge (x_1 < x'_1 \vee x_1 = x'_1) \wedge (y'_1 < y_1 \vee y'_1 = y_1) \wedge (y'_2 < y_2 \vee y'_2 = y_2) \vee \\
&\quad y_2 = y'_2 \wedge (x_1 < x'_1 \vee x_1 = x'_1) \wedge (y'_1 < y_1 \vee y'_1 = y_1) \wedge (x_2 < x'_2 \vee x_2 = x'_2) \} \\
&\quad \setminus (R_{DC} \cup R_{EC}), \\
R_{PO} &= \{ \langle R, S \rangle \mid x_1 < x'_1 < y_1 < y'_1 \vee x'_1 < x_1 < y'_1 < y_1 \vee x_2 < x'_2 < y_2 < y'_2 \vee \\
&\quad x'_2 < x_2 < y'_2 < y_2 \vee (x_1 < x'_1 < y'_1 < y_1 \wedge x'_2 < x_2 < y_2 < y'_2) \vee \\
&\quad (x'_1 < x_1 < y_1 < y'_1 \wedge x_2 < x'_2 < y'_2 < y_2) \} \setminus (R_{DC} \cup R_{EC}), \\
R_{\overline{TPP}} &= R_{TPP}^{-1}, \\
R_{\overline{NTPP}} &= R_{NTPP}^{-1},
\end{aligned}$$

respectively called *disconnected*, *externally connected*, *non-tangential proper part*, *tangential proper part*, *partially overlapping*, *tangential proper part inverse*, and *non-tangential proper part inverse*; $R_{=}$ denotes the identity.

As in all other cases, for each relation $X \in \{DC, EC, TPP, NTPP, \overline{TPP}, \overline{NTPP}, PO\}$ there is a corresponding modal operator $\langle X \rangle$, and well-formed formulas are obtained by instantiating Definition 1. A formula of L_{RCC8}^{Rec} is interpreted over a topological spatial structure of the type $M = \langle F, V \rangle$, where $F = \langle \mathbb{R}(\mathbb{D}), R_X \rangle$, with $X \in \{DC, EC, TPP, NTPP, \overline{TPP}, \overline{NTPP}, PO\}$, is a topological frame. The notion of an L_{RCC8}^{Rec} -formula being satisfied on a topological structure at a rectangle R is obtained by adapting Definition 2 with a clause

$$M, R \models \langle X \rangle \psi \quad \text{iff} \quad \text{there is an } S \text{ s.t. } RR_X S \text{ and } M, S \models \psi,$$

for $X \in \{DC, EC, TPP, NTPP, \overline{TPP}, \overline{NTPP}, PO\}$. An example of a topological model is shown in Fig. 14.

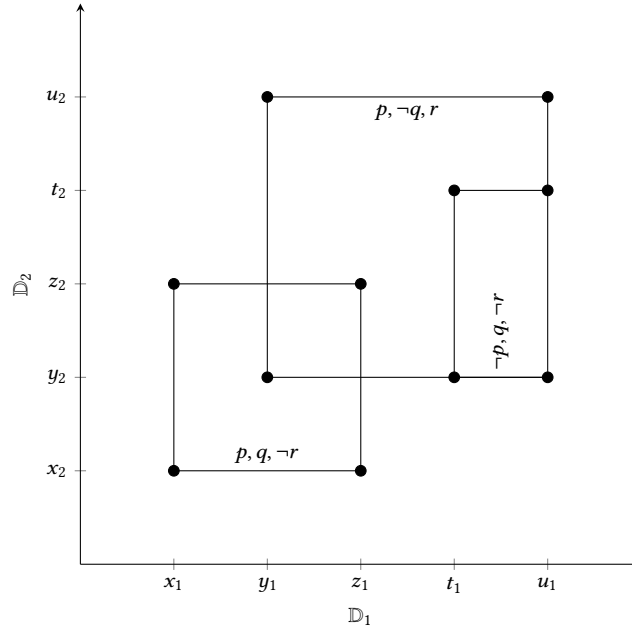


Fig. 14. An example of a topological model over 25 points. Three rectangles have a non-null valuation function in it, namely $R = ([x_1, z_1], [x_2, z_2])$, that satisfies $\{p, q\}$, $S = ([y_1, u_1], [y_2, u_2])$, that satisfies $\{p, r\}$, and $T = ([t_1, u_1], [y_2, t_2])$, that satisfies $\{q\}$. Among many other formulas, we notice that $\langle TPP \rangle q$ is true at S (e.g., q is true in T , which is accessible through the ‘tangential proper part’ relation from S), while $[PO] \neg q$ is true at R (there is no rectangle ‘partially overlapping’ with R such that q is true in that rectangle).

Logic languages for spatial reasoning tend to be undecidable, and L_{RCC8}^{Rec} is no exception. To the best of our knowledge, no reasoning tool is available for dealing with extended areas spatial reasoning.

Summarizing the discussion above, it is important to underline that there are several ways to qualitatively and quantitatively reason about time and space, many more than we have mentioned. The four described frameworks, however, share a number of qualities: they are very intuitive and applicable to different situations, they share the modal nature, and they are all based on the notion of a linear order. The fact that their satisfiability problem is computationally hard in some cases, undecidable in other, and unknown in the rest of them should not prevent us to pursue a reasoning tool that exploits their similarities.

In addition, the mentioned spatial and temporal reasoning frameworks can all benefit from being generalized to the many-valued case; their common modal nature, as well as their common underlying notion of linear order, allow us to study a uniform generalization strategy, as we shall see.

3.2 Many-Valued Modal Spatial and Temporal Logic

Many-Valued Linearly Ordered Set. While previous fuzzy generalizations of spatial and temporal logic focused specifically on the propositional aspects of the language, towards a fully many-valued approach to spatial and temporal reasoning we follow Fitting’s schema, in which both the propositional side and the accessibility relations are generalized. In this sense, having identified the modal nature of interesting spatial and temporal languages helps towards a uniform generalization technique. Since (our) spatial and temporal accessibility relations are order-based, having a many-valued version of a linear order is a necessary step.

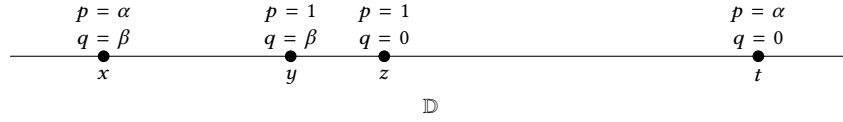


Fig. 15. An example of a point-based temporal model with four points, $P_1 = x$ with $p = \alpha, q = \beta$, $P_2 = y$ with $p = 1, q = \beta$, $P_3 = z$ with $p = 1, q = 0$, and $P_4 = t$ with $p = \alpha, q = 0$, where y, z are so close to each other we do not know which in reality comes before the other. Let $R_F(x, y) = \alpha$ and $R_F(x, z) = \beta$, while $R_F(x, t) = 1$ (we know for sure that x comes before t), where α and β are both less than 1 and greater than 0 but non-comparable in an FL_{ew} -algebra with 4 values and $\cap$ as t-norm. In this model, for example, $[F]q \succeq \alpha$ at the point x , but $[F]q \not\geq \beta$ at the same point.

DEFINITION 9. Let $A = \langle A, \cap, \cup, \cdot, 0, 1 \rangle$ a complete FL_{ew} -algebra. An FL_{ew} -linear order is a structure of the type

$$\tilde{D} = \langle D, \tilde{<}, \tilde{=}, \tilde{\geq} \rangle,$$

where D is a domain enriched with two functions $\tilde{<}, \tilde{=}: D \times D \rightarrow A$, for which the following conditions apply for every x, y , and z in the domain D :

$$\begin{aligned} \tilde{=}(x, y) &= 1 \text{ iff } x = y, \\ \tilde{=}(x, y) &= \tilde{=}(y, x), \\ \tilde{<}(x, x) &= 0, \\ \tilde{<}(x, z) &\succeq \tilde{<}(x, y) \cdot \tilde{<}(y, z), \\ \text{if } \tilde{<}(x, y) > 0 \text{ and } \tilde{<}(y, z) > 0, &\text{ then } \tilde{<}(x, z) > 0, \\ \text{if } \tilde{<}(x, y) = 0 \text{ and } \tilde{<}(y, x) = 0, &\text{ then } \tilde{=}(x, y) = 1, \\ \text{if } \tilde{=}(x, y) > 0, &\text{ then } \tilde{<}(x, y) < 1. \end{aligned}$$

There are several possible definitions of non-crisp linear orders. Zadeh (Zadeh 1971) defines a similarity relation in a set, imposing that it is reflexive, symmetric, and transitive, as well as a notion of many-valued ordering, anti-symmetry, and totality. Similarly, Bodenhofer (Bodenhofer 2003) advocates for the use of similarity-based many-valued orderings, in which the linearity is in a strong form; the same notion is also used in (Kundu 2000). Ovchinnikov (Ovchinnikov 1991) proposes a notion of many-valued ordering with a non-strict ordering relation. A common denominator to all such proposals is the definition of a very weak version of the transitivity property, which allows to obtain very general definitions. A many-valued linear order defined as above is similar to Zadeh's, modified to take into account both the many-valued linear order and the many-valued equality in the same structure. This is motivated by the fact that spatial and temporal relations involve both equality and linear order.

Since spatial and temporal logic has been defined, in each case, as a particular case of multi-modal logic, its many-valued generalization is in fact immediate. Spatial and temporal worlds (i.e., temporal and spatial points, temporal intervals, and extended areas) are defined on top of the notion of linear order. Given an FL_{ew} -algebra A and a logic language $L \in \{\text{LTL}[F, P], \text{CL}, \text{HS}, L_{\text{RCC8}}^{\text{Rec}}\}$, we proceed to obtain the corresponding FL_{ew} -L logic by simply substituting each linear set with a FL_{ew} -linear order, and every $<$ with $\tilde{<}$ and every $=$ with $\tilde{=}$ in the relevant definitions (in the case of $L_{\text{RCC8}}^{\text{Rec}}$, however, it is required to have an explicit definition of each topological relation in order to mechanically apply such a generalization). Examples for the many-valued version of $\text{LTL}[F, P]$, CL , HS , and $L_{\text{RCC8}}^{\text{Rec}}$ are shown in Fig. 15, Fig. 16, Fig. 17 and Fig. 18 respectively. In particular, the Compass Logic case is quite insightful: in fact, note how in classical logic one point can only be *up*, *right*, *down*, or *left* with respect to another, meaning that it cannot be, for example, both *up* and *right* at the same time, which is rather counterintuitive (a possible solution would be to extend the language with four more relations for *up-left*, *up-right*, *down-left*, *down-right*, which would not be mutually exclusive with respect to the other relations); however, leveraging many-valued logic, we can state that a point can be both *up* and *left* of another at the same time.

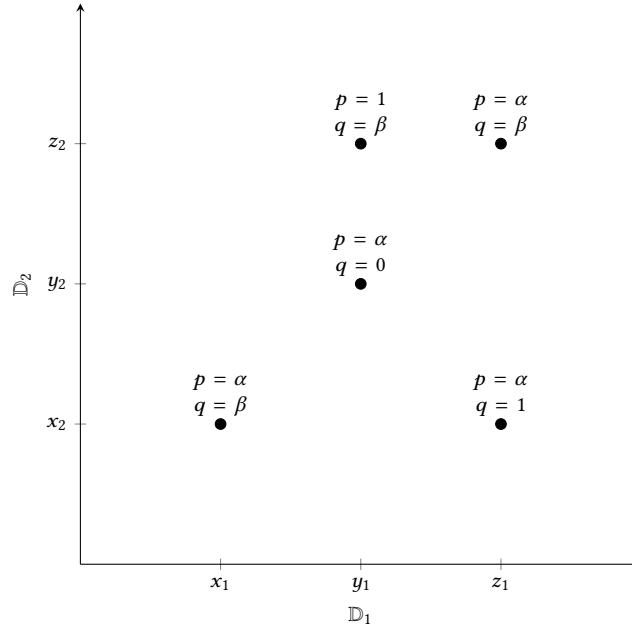


Fig. 16. In this example, a many-valued version of the example in Fig. 12. In each point, we show the truth value of each letter; for points that are not in the picture, we assume that every letter is evaluated to 0. In particular, the points with non-null valuation functions are $P = \langle x_1, x_2 \rangle$, with $p = \alpha$ and $q = \beta$, $Q = \langle y_1, y_2 \rangle$, with $p = \alpha$ and $q = 0$, $R = \langle z_1, x_2 \rangle$, with $p = \alpha$ and $q = 1$, $S = \langle y_1, z_2 \rangle$, with $p = 1$ and $q = \beta$, and $T = \langle z_1, z_2 \rangle$, with $p = \alpha$ and $q = \beta$. Differently from the crisp case, where $R_U(P, P_i) = 0$ for each point P_i with non-null valuation function, here we can assume $R_U(P, S) = \alpha$, where $0 \prec \alpha \prec \beta \prec 1$ in an algebra with 4 values with $\cap$ as monoid. In this model, $\langle U \rangle p \succeq \alpha$ at point P .

3.3 A Tableau System for Many-Valued Spatial and Temporal Logic

The entire logic underlying tableau rules, expansion, open and closed branches defined in Section 2.2 can be readily adapted to deal with worlds based on linear order(s). Therefore, we now have a reasoning system for very natural temporal and spatial logical languages interpreted on some arbitrary finite FL_{ew} -algebra. In this section, we discuss a possible implementation for such system, as well as a broad set of experiments aiming to assess both its correctness and its performance. The tableau system has been implemented using the Julia programming language as part of an open-source project for representing, reasoning, and learning from structured and unstructured data, namely *Sole.jl* (Manzella, Pagliarini, Paparella, et al. 2025), specifically in the *SoleReasoners.jl* (Paparella and Sciavicco 2025) package.

In our implementation, formulas are represented as syntax trees with leaves consisting of either propositional letters or algebra values (Fig. 19). Finite FL_{ew} -algebras are configured by establishing a finite domain, defining operators through their tables, and ensuring, via a one-time check, that all axioms proper to an FL_{ew} -algebra are satisfied. The tableau itself is structured as a rooted tree with variable arity. Each node in the tableau has a decoration and maintains two pointers, one to its parent node and the other to the head of a list of its children, which can be empty. Nodes also carry two flags to track whether they have been expanded (or not) and whether they are open (or closed). Specifically, a closed node indicates that all branches containing it are also closed. The tableau is linked to an array of priority queues, implemented as binary heaps, where each queue node references

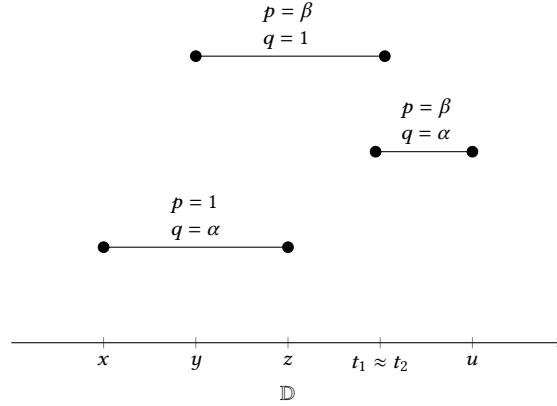


Fig. 17. The interval model shown in the example has six points and thirty intervals. The intervals with a non-null valuation function are $I = [x, z]$, with $p = 1$, $q = \alpha$, $J = [y, t_1]$, with $p = \beta$, $q = 1$, and $K = [t_2, u]$, with $p = \beta$, $q = \alpha$, where t_1 and t_2 are slightly apart. Hence, for example, $R_O(J, K) = R_A(J, K) = \alpha$, where $0 < \alpha < \beta < 1$ in an FL_{ew} -algebra with 4 elements and $\cap$ as t-norm. In this model, for example, $\langle O \rangle(p \wedge q) \succeq \alpha$ at interval J .

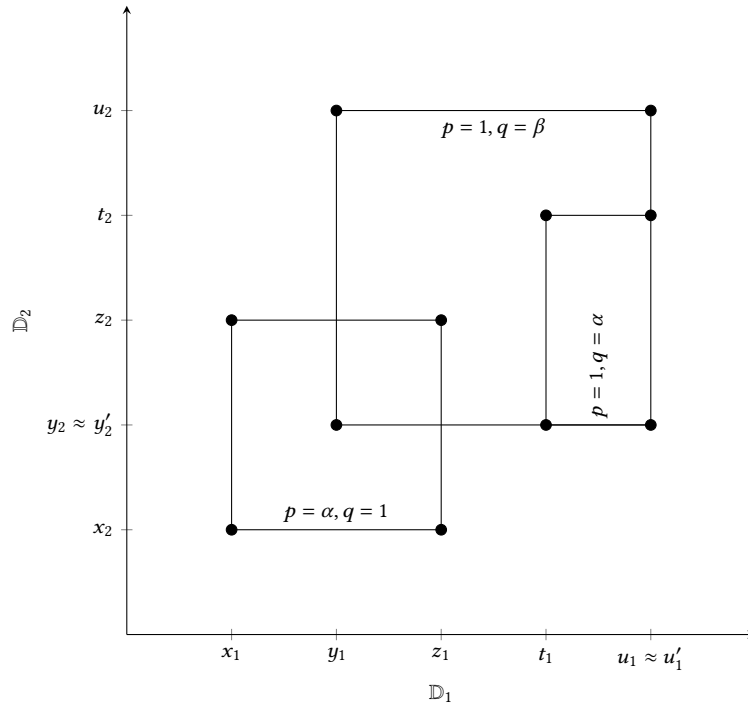


Fig. 18. A topological model over 36 points. The rectangles with a non-null valuation function are $R = ([x_1, z_1], [x_2, z_2])$, with $p = \alpha$ and $q = 1$, $S = ([y_1, u_1], [y_2, u_2])$, with $p = 1$ and $q = \beta$, and $T = ([t_1, u'_1], [y'_2, t_2])$, with $p = 1$ and $q = \alpha$, where $u_1, u'_1 \in \mathbb{D}_1$ and $y_2, y'_2 \in \mathbb{D}_2$ are slightly apart. Hence, for example, $R_{TPP}(S, T) = R_{NTPP}(S, T) = \alpha$, where $0 < \alpha < \beta < 1$ in an FL_{ew} -algebra with 4 elements and $\cap$ as t-norm. In this model, for example, $\langle NTPP \rangle(p \rightarrow q) \succeq \alpha$ at rectangle S .

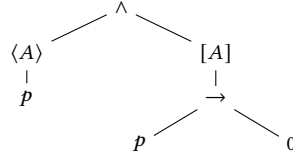
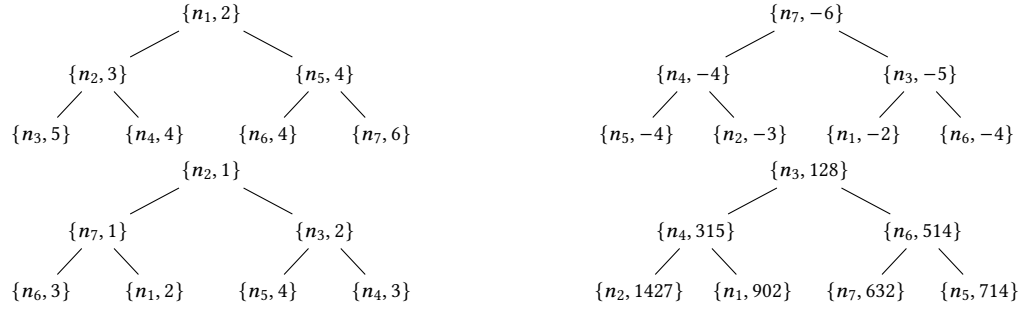
Fig. 19. Syntax tree representation for the formula $\langle A \rangle p \wedge [A](p \rightarrow 0)$.

Fig. 20. Examples of different kind of priority queues, represented as min-heap structures, to choose which branch to expand: (*top-left*) distance from the root (*dfr*); (*top-right*) inverse distance from root (*idfr*); (*bottom-left*) formula height (*fh*); (*bottom-right*) random branch (*rb*). Each search strategy proposes the head of its relative heap as candidate node, which branch will be expanded: *dfr* will propose node n_1 , as it's the nearest from the root; conversely, *idfr* will propose n_7 ; *fh* will propose n_2 , which only contains either a proposition or a truth value; finally, *rb* will propose n_3 , as it randomly assigned the least value to it. A meta-policy will then be employed to choose which node between n_1, n_7, n_2, n_3 to expand.

a tableau node and carries a value determined by a priority function. This priority setting is crucial during the expansion phase to determine the next branch for expansion.

All standard search strategies can be used within our system, and some examples are shown in Fig. 20; different search strategies can be used at the same time, in a round-robin policy, or even with a further level of parallelization, in order to implement a virtual best solver policy. Currently supported search strategies are: (i) give priority to nodes near the root (*dfr*), similarly to a breadth-first search; (ii) give priority to nodes far from the root (*idfr*), similarly to a deep-first search; (iii) give priority to nodes containing shorter formulas (*fh*); (iv) assign a random priority to each node (*rb*). Those four strategies, as well as some of their combinations considering a round-robin scheduling policy, will be compared in the next paragraphs. However, the package makes it very easy for the user to implement customized search strategies.

The construction of the tableau follows a standard search algorithm, beginning with an empty node to which several immediate children are attached; each child corresponds to a different algebra value for the relation $\preceq$ between the initial points. At each cycle, we evaluate all nodes at the top of any priority queue. We select one node from this group based on a predefined meta-policy (e.g., round-robin or majority). The selected node, immediately removed from other queues where it was at the top, identifies a set of branches chosen for expansion. We first verify whether this node v is already closed; if so, the cycle proceeds without action. Next, if v has already been expanded, the process skips to the next cycle unless v is a leaf, signaling an open branch in a fully expanded tableau. For a SAT-tableau, such an open branch identifies a model that α -satisfies the starting formula; for a VAL-tableau, it represents a counterexample to its α -validity. If v has not been expanded, we identify the nearest unexpanded ancestor v' of v and mark it as expanded. The subsequent steps are to apply the closing rules, thereby

closing ν' and its descendants (including ν). If this is not applicable, the propositional rules extend the tree from all leaves descending from ν' and insert the fresh nodes into all priority queues checking that each fresh node was not already present in the branch; in such case, if no node is added to that branch, a virtual node containing $true(1 \preceq 1, w, c(B))$ is inserted to keep track of the presence of a branch. If no propositional rule is applicable, the modal rules are applied, with the only difference that when applying $(true \Box)$ or $(true \Diamond)$ rules, if at least one node has been added to the branch before $true(\beta \preceq [X_i]\psi, w, c(B))$ (resp. $true(\langle X_i \rangle \psi \preceq \beta, w, c(B))$), we still add the latter, as we may have new worlds in $c(B)$ w.r.t. $\tilde{F}$; otherwise, we end the procedure, as we have found an open branch with no contradictions within that is no longer expandable. Again, if this is not applicable, the reverse rules are applied with implications for all leaves descending from ν' , inserting all fresh nodes in all priority queues with the same precautions taken for the expansion rules. Finally, if none of the above applies, ν is reinserted into the priority queues from which was extracted. The process concludes when all priority queues are empty, indicating that no open branches remain: the tableau is fully expanded and declared closed. For a SAT-tableau, this proves the α -unsatisfiability of the starting formula; for a VAL-tableau, it demonstrates α -validity.

Fig. 21 and Fig. 23 serve as an example of execution of the tableau to find if the formula $\langle B \rangle \langle B \rangle p \rightarrow \langle B \rangle p$, which is a known tautology of Halpern and Shoham's Interval Temporal Logic, is still valid when considering a three-valued logic. Note that an open branch for a VAL-tableau like the one in Fig. 23 hints to a counter-example for it: in fact, from such branch we can infer that given the FL_{ew} -linear order $c(B'')$ in Fig. 22, there is a model which doesn't satisfy the given formula, e.g., when p is at least α in the interval $[e, f]$ and 0 everywhere else.

To improve both time and space efficiency of our implementation, we periodically perform a cleaning operation on all priority queues. For a parameter $K \in \mathbb{N}$, every K cycles all nodes within a queue that have already been expanded (unless they are leaves, corresponding to open branches) or that are already closed are removed. This operation helps managing the computational overhead and optimizes the performance of the tableau construction. Moreover, since the creation of each new branch in the modal rules $(false \Box)$ and $(false \Diamond)$ is independent from the others, as for each new node we have to check that no inconsistencies are introduced, this process has been parallelized, introducing a locking system to manage the concurrent writing to both the tableau structure and the heaps. This is crucial, since modal rules (which in some scenarios, like HS, are known to bring undecidability to the tableau already in the crisp case) are undoubtedly the most demanding expansions, as we have to consider each possible new world, and so each possible new linear order extension, hence, each combination of true values for $\equiv$ and $\prec$ involving the new points added to the linear order. Finally, we provide a timeout parameter to automatically stop the tableau execution in case no solution is reached before a certain time (at the moment of writing, the tableau does not implement any other termination, blocking, nor bounding strategy).

3.4 Experiments

To assess the performance of our implementation, we run an extensive set of experiments investigating the impact of different finite FL_{ew} -algebras, depicted in Fig. 24, different search strategies, shown in Fig. 20, as well as growth in formula height, for both SAT-tableaux and VAL-tableaux. All tests were conducted on a machine equipped with 2 Intel Xeon Gold 28-Core CPUs and 724GB of RAM, and the code is provided in *SoleReasoners.jl* for reproducibility purposes (a fixed seed is employed when randomness is involved).

At the time of writing, and to the best of the authors' knowledge, there is no available benchmark suitable to test reasoning tools working at this level of generality, i.e., encompassing both the many-valued and the multi-modal components. To fill this gap, we employed the following algorithm to produce a benchmark of many-valued multi-modal formulas for the different kinds of algebras and temporal and spatial logics investigated. Formula generation was governed by a weighted selection process, choosing with equal probability a connective $c \in \{\wedge, \vee, \rightarrow\}$ or a modality $m \in \{\langle X_i \rangle, [X_i]\}$, for each $X_i \in X$, where X depends on the chosen logic, until a

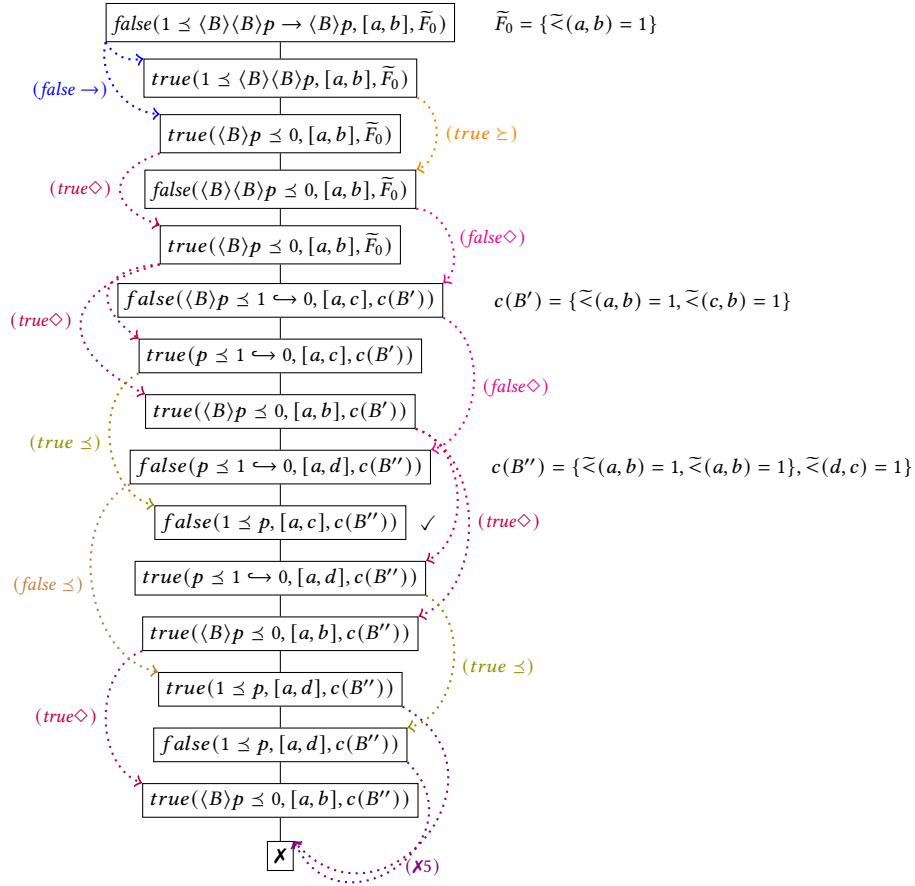


Fig. 21. Fully evaluated VAL-tableau for $1 \leq \langle B \rangle \langle B \rangle p \rightarrow \langle B \rangle p$ in BA. FL_{ew} -Kripke frames $\widetilde{F}_0$, $c(B')$ and $c(B'')$ are represented in an implicit way through constraints on the FL_{ew} -linear order. In this example, applying the rules of the tableau system results in closing all branches, proving that the given formula is, in fact, valid in BA.

chosen height is reached; similarly, algebra values $\alpha \in A$ and propositional letters $p \in \mathcal{P}$ were chosen with equal probability as leaves. When randomness is involved, the Xoshiro algorithm was employed.

For the first batch of experiments, we generated 200 random formulas for each algebra and for each height up to 6 (i.e., up to 64 symbols). The performance of each formula in terms of α -satisfiability and α -validity, with $\alpha \succ 0$ chosen randomly, was analyzed; the results for the presented temporal and spatial logics are depicted in Fig. 25, Fig. 26, Fig. 27 and Fig. 28. Throughout these experiments, the branch priority policy was kept random. For each logic, we consider how many formulas are solved with a timeout of 60 seconds for each formula. As one may expect, there is a notable difference in performance between point-wise temporal and spatial logics (Fig. 25 and Fig. 26) and interval temporal (Fig. 27) and especially spatial (Fig. 28) logics. While increasing the number of the elements in the algebra is also challenging for the implementation, it is interesting to notice how most of the time H4 is less demanding than other 4-valued algebras for VAL-tableaux, suggesting better performance when

$\widetilde{F}_0^1$:	$\cong$	a	b	$\widetilde{<}$	a	b	$\widetilde{F}_0^2$:	$\cong$	a	b	$\widetilde{<}$	a	b	$\widetilde{F}_0^3$:	$\cong$	a	b	$\widetilde{<}$	a	b																			
	a	1	0		a	0		α	a	1		α	a		0	α	a		1	α	a	0	α																
	b	0	1		b	0	1		b	0	1		b	0	0		b	α	1		b	0	0		b	0	0		b	0	0								
$c(B')$:		$\cong$	a	c	d	b		$\widetilde{<}$	a	c	d	b		$\widetilde{<}$	a	c	d	b		$\widetilde{<}$	a	c	d	b		$\widetilde{<}$	a	c	d	b									
	a	1	α	0	0		a	0	α	1	1		a	0	α	1	1		a	0	α	1	1		a	0	α	1	1		a	0	α						
	c	α	1	0	0		c	0	0	1	1		c	0	0	1	1		c	0	0	1	1		c	0	0	1	1		c	0	0	1	1				
	d	0	0	1	α		d	0	0	0	α		d	0	0	0	α		d	0	0	0	α		d	0	0	0	α		d	0	0	0	α				
	b	0	0	α	1		b	0	0	0	0		b	0	0	0	0		b	0	0	0	0		b	0	0	0	0		b	0	0	0	0				
$c(B'')$:		$\cong$	a	c	e	f	d	b		$\widetilde{<}$	a	c	e	f	d	b		$\widetilde{<}$	a	c	e	f	d	b		$\widetilde{<}$	a	c	e	f	d	b							
	a	1	α	0	0	0	0		a	0	α	1	1	1	1		a	0	α	1	1	1	1		a	0	α	1	1	1	1		a	0	α	1	1	1	1
	c	α	1	α	0	0	0		c	0	0	α	1	1	1		c	0	0	α	1	1	1		c	0	0	α	1	1	1		c	0	0	α	1	1	1
	e	0	α	1	α	0	0		e	0	0	0	α	1	1		e	0	0	0	α	1	1		e	0	0	0	α	1	1		e	0	0	0	α	1	1
	f	0	0	α	1	α	0		f	0	0	0	0	α	1		f	0	0	0	0	α	1		f	0	0	0	0	α	1		f	0	0	0	0	α	1
	d	0	0	0	α	1	α		d	0	0	0	0	0	α		d	0	0	0	0	0	α		d	0	0	0	0	0	α		d	0	0	0	0	0	α
b	0	0	0	0	α	1		b	0	0	0	0	0	0		b	0	0	0	0	0	0		b	0	0	0	0	0	0		b	0	0	0	0	0	0	

Fig. 22. Implicit representation of the FL_{ew} -Kripke frames mentioned in the tableau in Fig. 23 through their respective associated FL_{ew} -linear order: $\widetilde{F}_0^1$, $\widetilde{F}_0^2$ and $\widetilde{F}_0^3$ represent all the possible FL_{ew} -linear orders with two points using G3; $c(B')$ (resp., $c(B'')$) is one of the many possible FL_{ew} -linear orders expanding $\widetilde{F}_0^1$ (resp., $c(B')$) with up to two other points.

working with partial orders considering the same number of elements when proving α -validity (the reverse is true when proving α -satisfiability); at the same time, Gödel and MV algebras usually share similar performance.

The second batch of experiments aimed at comparing different search metrics on the same set of formulas. For this experiment, we chose to stick with many-valued linear temporal logic with future and past, since it was the least computational demanding, as proved by the previous batch of experiments. Results are depicted in Fig. 29, zooming in to provide a greater granularity: generally, search strategies involving *distance from root* as a metric proved to give the best performance for both SAT-tableaux and VAL-tableaux, while strategies leveraging *inverse distance from root* gave the worst results.

3.5 Comparison with first-order translation

One natural way to address the satisfiability/validity evaluation problem for many-valued modal logics is that of translating it to a pure satisfiability problem in first-order logic and then using a first-order logic reasoner. This has been proposed by other authors, although here we consider a translation with the same level of generality as our tableau method. Following our previous presentation, we detail here a translation to first-order logic of the problem *is φ α -satisfiable?*, which in fact is the same question answered by a SAT-tableau; a mirror-image translation can be devised to mimic the behavior of a VAL-tableau. Coherently with our previous presentation, we focus on $\text{FL}_{\text{ew}}\text{-K}_n$ interpreted in some finite FL_{ew} -algebra A .

We use the following conventions. We translate the α -satisfiability problem of a formula φ interpreted in a given FL_{ew} -algebra A as first-order formula with variables $x, y, \dots$ to denote worlds, $a, b, \dots$ to denote algebra values, t to denote either a world or an algebra value, constants $0, 1, \alpha, \beta, \dots$ taking algebra values (these are also the constants in the modal language), two unary predicates $A()$ (true only on algebra values) and $W()$ (true only on Kripke worlds), a binary predicate $\preceq()$ (such that, for any two given algebra values a, b , $\preceq(a, b)$ if and only if $a \preceq b$), three ternary predicates $\cdot()$, $\cup()$, and $\hookrightarrow()$ (such that, for any three algebra values a, b, c , it is the case that $\cdot(a, b, c)$, $\cup(a, b, c)$, and $\hookrightarrow(a, b, c)$ is true if and only if c is the result of applying the operator $\cdot$, $\cup$, and $\hookrightarrow$ to a and b respectively), n ternary predicates $R_1(), \dots, R_n()$ (true only on triples x, y, a where both x, y are Kripke

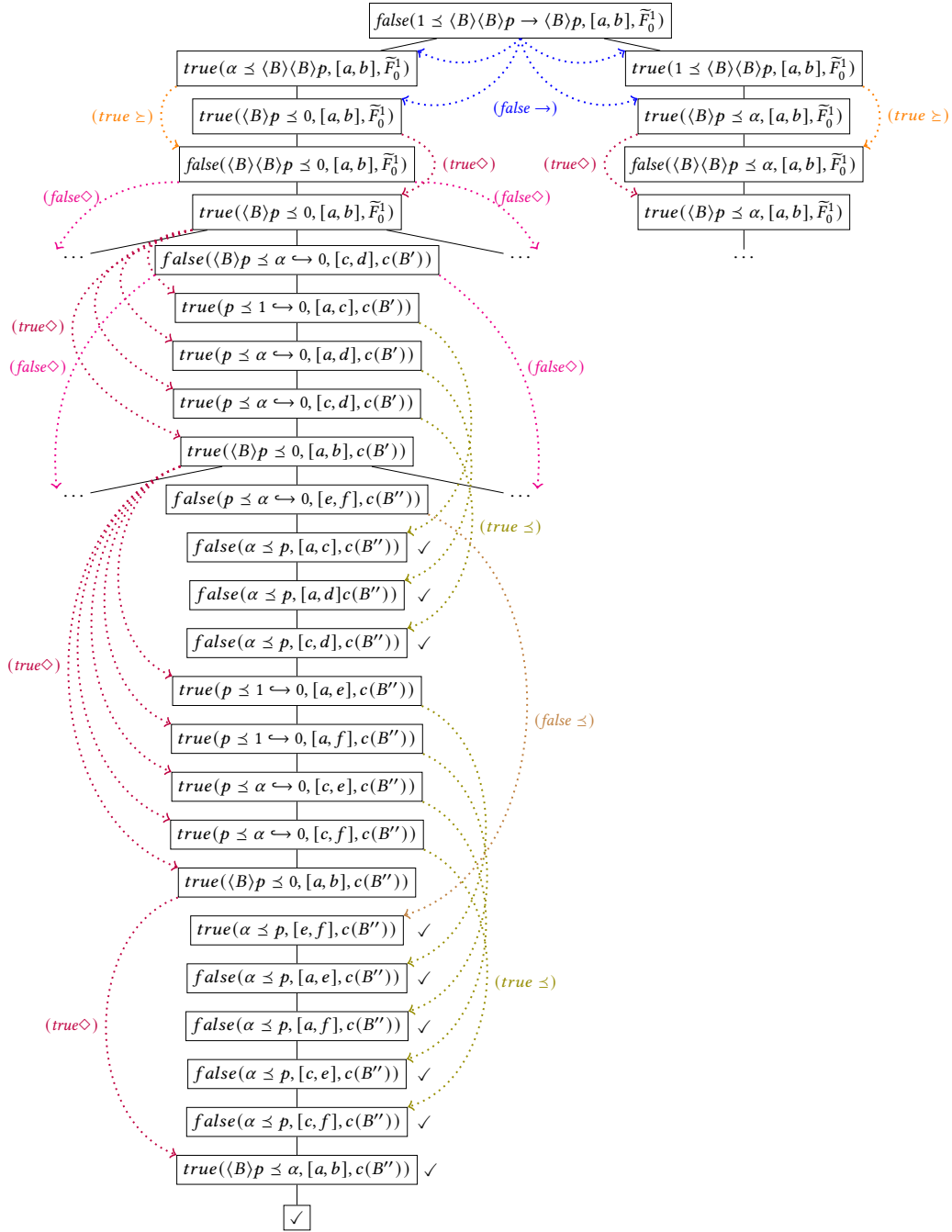


Fig. 23. Partially evaluated VAL-tableau for $1 \leq \langle B \rangle \langle B \rangle p \rightarrow \langle B \rangle p$ in G3, starting with frame $\tilde{F}_0^1$ (note that on startup, two other analogous tableaux are setup starting with $\tilde{F}_0^2$ and $\tilde{F}_0^3$ respectively). FL_{ew}-Kripke frames $\tilde{F}_0^1, \tilde{F}_0^2, \tilde{F}_0^3, c(B')$ and $c(B'')$ are represented in Fig. 22. In this example, applying the rules of the tableau system results in at least one open branch, which serves as a counter-example for the validity of the given formula, proving that it is not valid in G3.

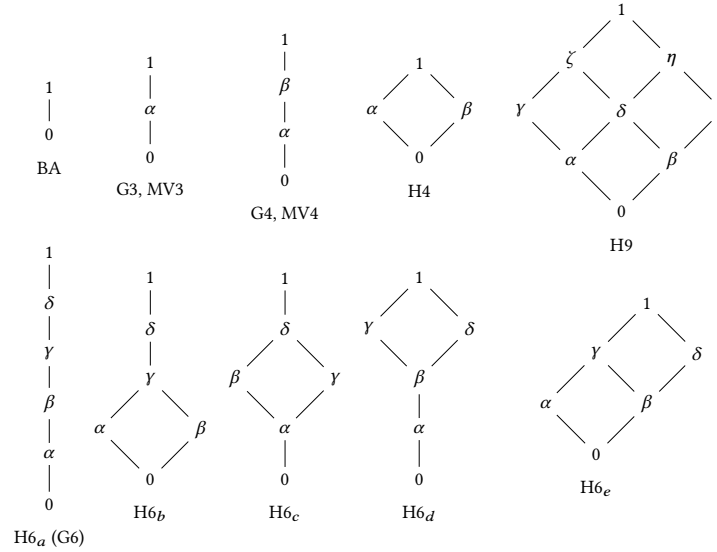


Fig. 24. Set of algebras used in our experiments. Notice how G3 and MV3 (resp. G4 and MV4) only differ for the monoid.

worlds and a is an algebra value if and only if a if the value of the relation between x and y), and one binary predicate $P()$ for each propositional letter $p \in \mathcal{P}$ (true only on pairs x, a , where x is a Kripke world and a is an algebra value, if and only if a is the value of p at world x). Now, we provide the following translation:

$$ST(\varphi, \alpha, L, \mathbf{A}),$$

(a quaternary translation) which returns a first-order formula f in the above language that is true if and only if φ is α -satisfiable in $\mathbf{A}$ given the class of frames L . In the following, the class of frames will be the class of all frames, that is, $L = K$; our translation needs further sets of conjuncts for more specific classes of frames, and when we want to translate many-valued formulas of some temporal or spatial logic it may become convenient to use a translation into a suitable SMT first-order language, based on the same principles as the one proposed here. To save space, in the following, variables of the sort $a, b, \dots$ will always be assumed to satisfy the predicate $A()$, and variables of the sort $x, y, \dots$ will always be assumed to satisfy the predicate $W()$. Thus, we have:

$$ST(\varphi, \alpha, L, \mathbf{A}) = \exists x \exists a (ST(\varphi, x, a) \wedge f_{\mathbf{A}} \wedge f_L \wedge \preceq(\alpha, a)),$$

where $ST(\varphi, x, a)$ (a ternary translation) is the actual translation, $f_{\mathbf{A}}$ is a first-order formula that describes the actual finite algebra $\mathbf{A}$, and f_L describes the class of frames on which φ is interpreted. In the case of K , for example, f_L is $f_{K_n} = 1$; as a different example, in the case of a temporal logic, f_L should define complex worlds (such as intervals, for example) and describe an adequate many-valued linear order, as in Section 3.2. The formula $f_{\mathbf{A}}$ depends on the actual finite algebra $\mathbf{A}$ and it generally comprises five conjuncts:

$$f_{\mathbf{A}} = f_{\mathbf{A},d} \wedge f_{\mathbf{A},\cdot} \wedge f_{\mathbf{A},\cup} \wedge f_{\mathbf{A},\hookrightarrow} \wedge f_{\mathbf{A},\preceq},$$

height	BA	G3	MV3	G4	MV4	H4	H6 _a	H6 _b	H6 _c	H6 _d	H6 _e	H9
α -satisfiability												
1	200	200	200	200	200	200	200	200	200	200	200	176
2	198	199	199	199	199	185	198	195	188	190	181	157
3	191	195	196	194	191	171	190	183	178	180	168	134
4	179	182	181	164	165	140	158	145	139	142	131	112
5	168	165	157	147	136	128	136	126	123	126	111	100
6	153	137	129	115	111	104	111	109	100	101	99	95
α -validity												
1	200	200	200	200	200	200	200	200	200	200	200	200
2	198	194	194	194	194	194	193	193	193	193	193	182
3	181	183	184	184	184	182	174	175	173	172	172	151
4	174	165	161	142	141	145	129	136	134	131	138	123
5	151	124	115	102	99	108	95	94	91	87	95	101
6	119	95	87	86	80	91	78	79	75	79	78	86

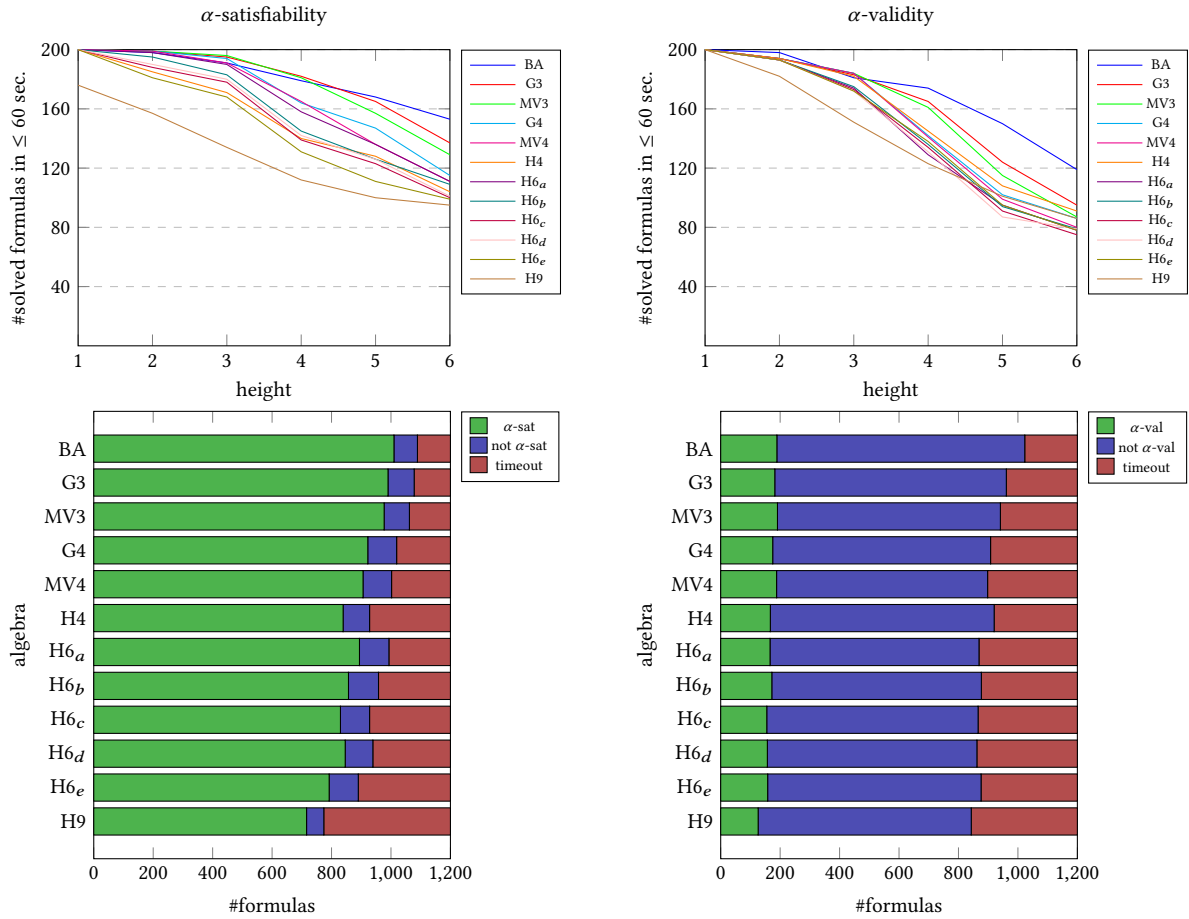


Fig. 25. Estimation of how the computation time for *many-valued linear temporal logic with future and past* is influenced by the height of the formula and the type of algebra. *Top*: how many formulas can be computed within a 60-second timeout. *Bottom*: number of α -sat (resp. α -val) and not α -sat (resp. not α -val) formulas over 1200 formulas (200 for each eight from 1 to 6) for algebras BA, G3, MV3, G4, MV4, H4, H6_a (G6), H6_b, H6_c, H6_d, H6_e, and H9.

height	BA	G3	MV3	G4	MV4	H4	H6 _a	H6 _b	H6 _c	H6 _d	H6 _e	H9
α -satisfiability												
1	200	200	200	200	200	200	200	195	190	189	179	176
2	198	199	199	197	198	179	189	178	177	172	160	148
3	192	192	192	185	188	159	167	158	152	153	136	124
4	185	171	174	155	158	128	128	127	123	116	115	104
5	168	152	148	134	132	118	113	114	106	103	103	92
6	150	129	130	104	111	94	104	98	99	95	93	83
α -validity												
1	200	200	200	200	200	200	200	200	200	200	200	200
2	197	194	194	190	193	191	175	175	176	175	181	174
3	180	182	183	163	176	179	147	147	144	145	150	136
4	166	144	137	122	119	135	106	108	103	105	115	103
5	137	109	102	98	93	102	86	85	84	84	85	91
6	116	88	83	88	83	93	75	77	78	76	79	80

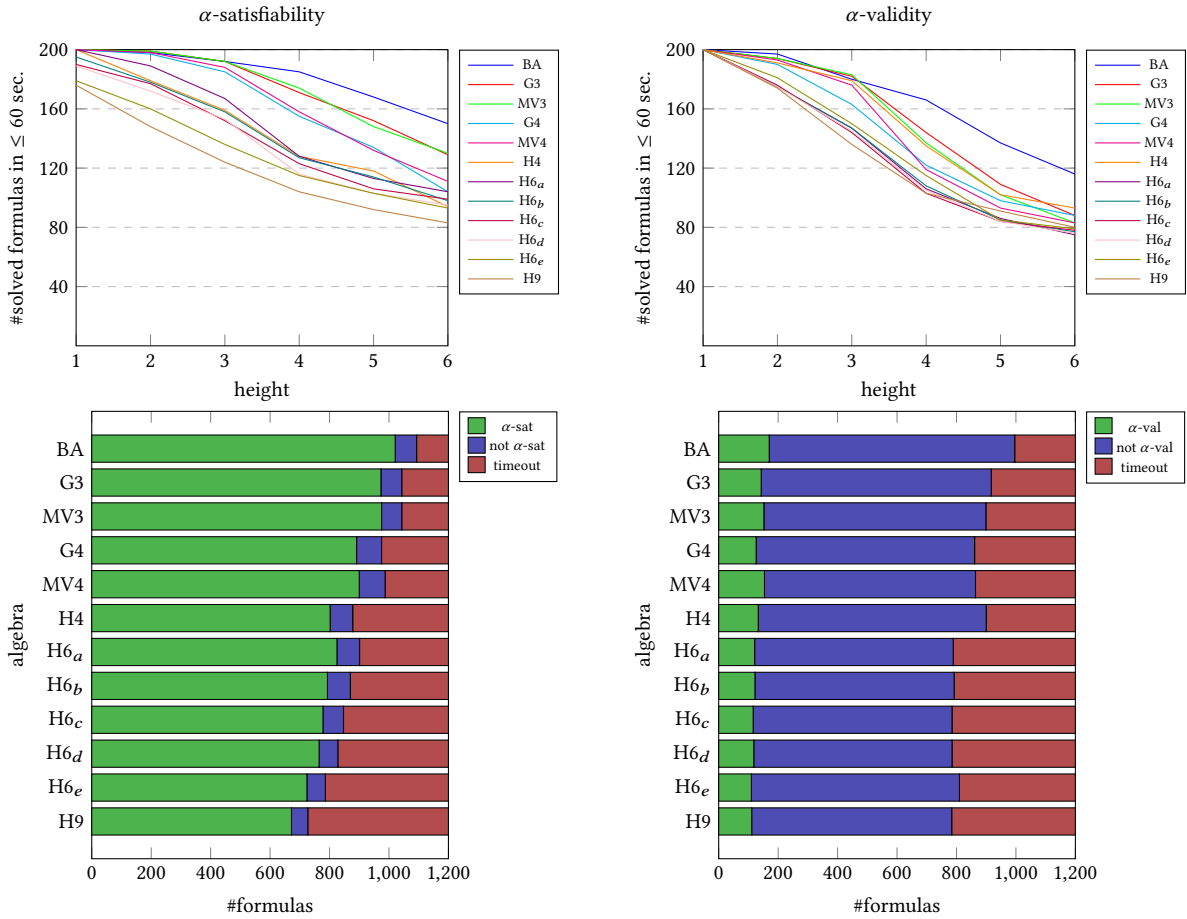


Fig. 26. Estimation of how the computation time for *many-valued compass logic* is influenced by the height of the formula and the type of algebra. *Top*: how many formulas can be computed within a 60-second timeout. *Bottom*: number of α -sat (resp. α -val) and not α -sat (resp. not α -val) formulas over 1200 formulas (200 for each eight from 1 to 6) for algebras BA, G3, MV3, G4, MV4, H4, H6_a (G6), H6_b, H6_c, H6_d, H6_e, and H9.

height	BA	G3	MV3	G4	MV4	H4	H6 _a	H6 _b	H6 _c	H6 _d	H6 _e	H9
α -satisfiability												
1	200	200	200	147	147	147	152	152	152	152	152	146
2	198	183	183	117	116	113	117	117	114	116	112	106
3	186	156	156	100	103	97	100	99	99	96	97	91
4	161	142	141	88	85	83	88	84	89	83	80	79
5	141	124	121	89	86	79	82	82	83	82	80	72
6	125	116	112	82	80	77	81	80	79	75	79	73
α -validity												
1	200	200	200	140	140	140	144	144	144	144	144	144
2	197	171	172	92	90	94	88	90	87	87	89	94
3	177	138	139	80	75	78	73	76	70	72	71	77
4	142	104	104	66	66	68	66	66	64	66	66	70
5	114	87	90	64	64	64	60	64	62	63	58	68
6	97	80	83	69	66	67	59	61	58	61	61	65

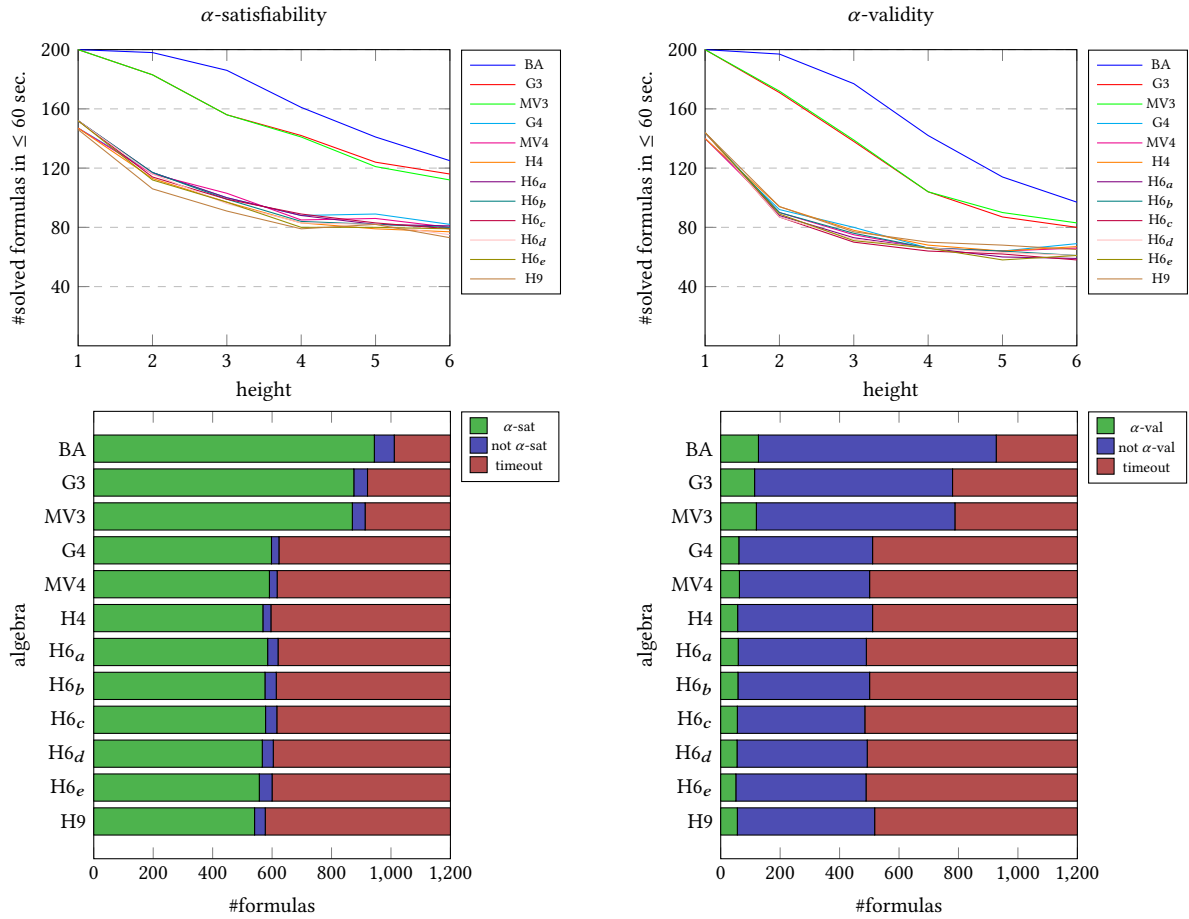


Fig. 27. Estimation of how the computation time for *many-valued Halpern and Shoham's interval temporal logic* is influenced by the height of the formula and the type of algebra. *Top*: how many formulas can be computed within a 60-second timeout. *Bottom*: number of α -sat (resp. α -val) and not α -sat (resp. not α -val) formulas over 1200 formulas (200 for each eight from 1 to 6) for algebras BA, G3, MV3, G4, MV4, H4, H6_a (G6), H6_b, H6_c, H6_d, H6_e, and H9.

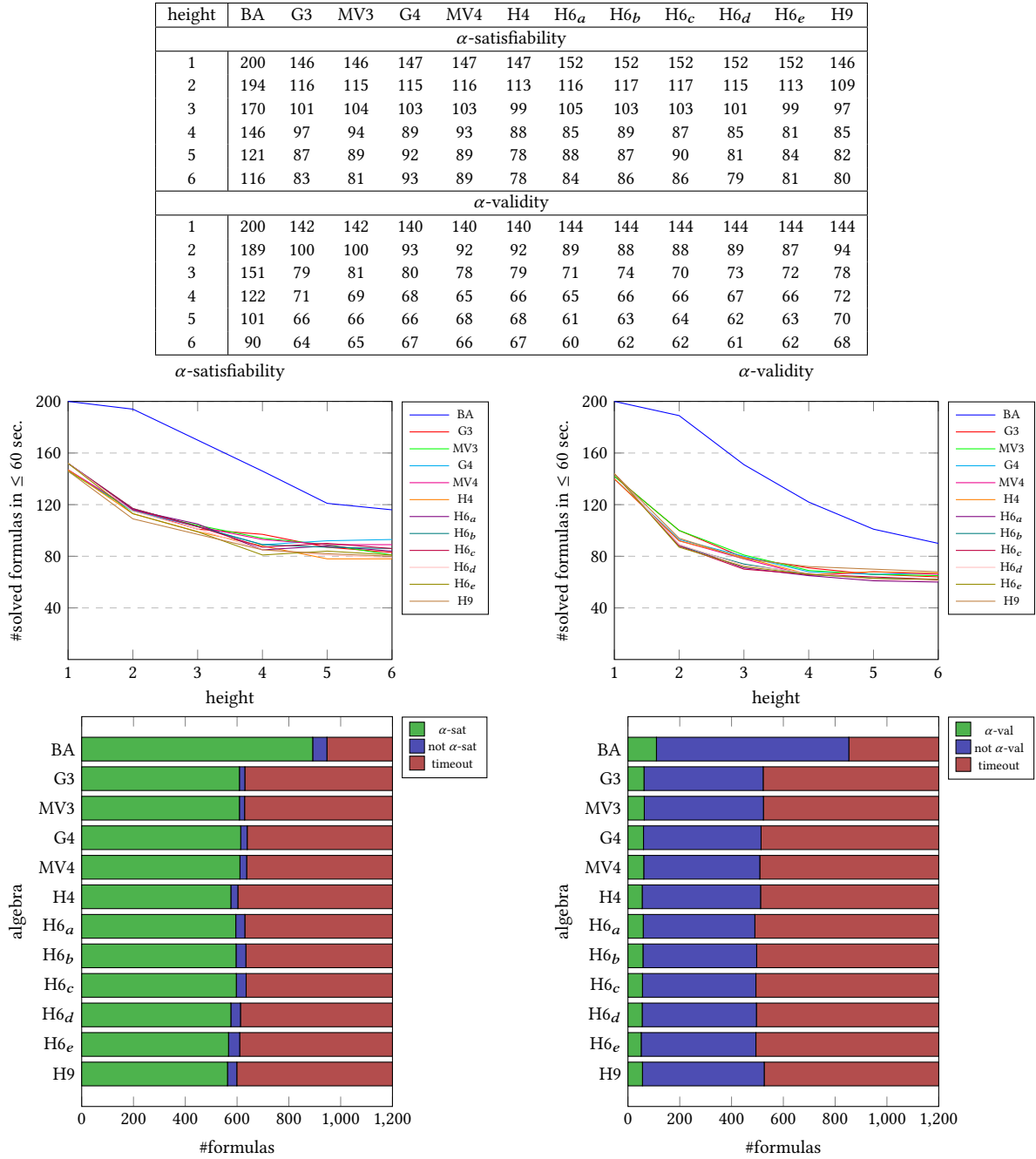


Fig. 28. Estimation of how the computation time for *many-valued linear Lutz and Wolter's modal logic of topological relations* is influenced by the height of the formula and the type of algebra. *Top*: how many formulas can be computed within a 60-second timeout. *Bottom*: number of α -sat (resp. α -val) and not α -sat (resp. not α -val) formulas over 1200 formulas (200 for each eight from 1 to 6) for algebras BA, G3, MV3, G4, MV4, H4, H6_a (G6), H6_b, H6_c, H6_d, H6_e, and H9.

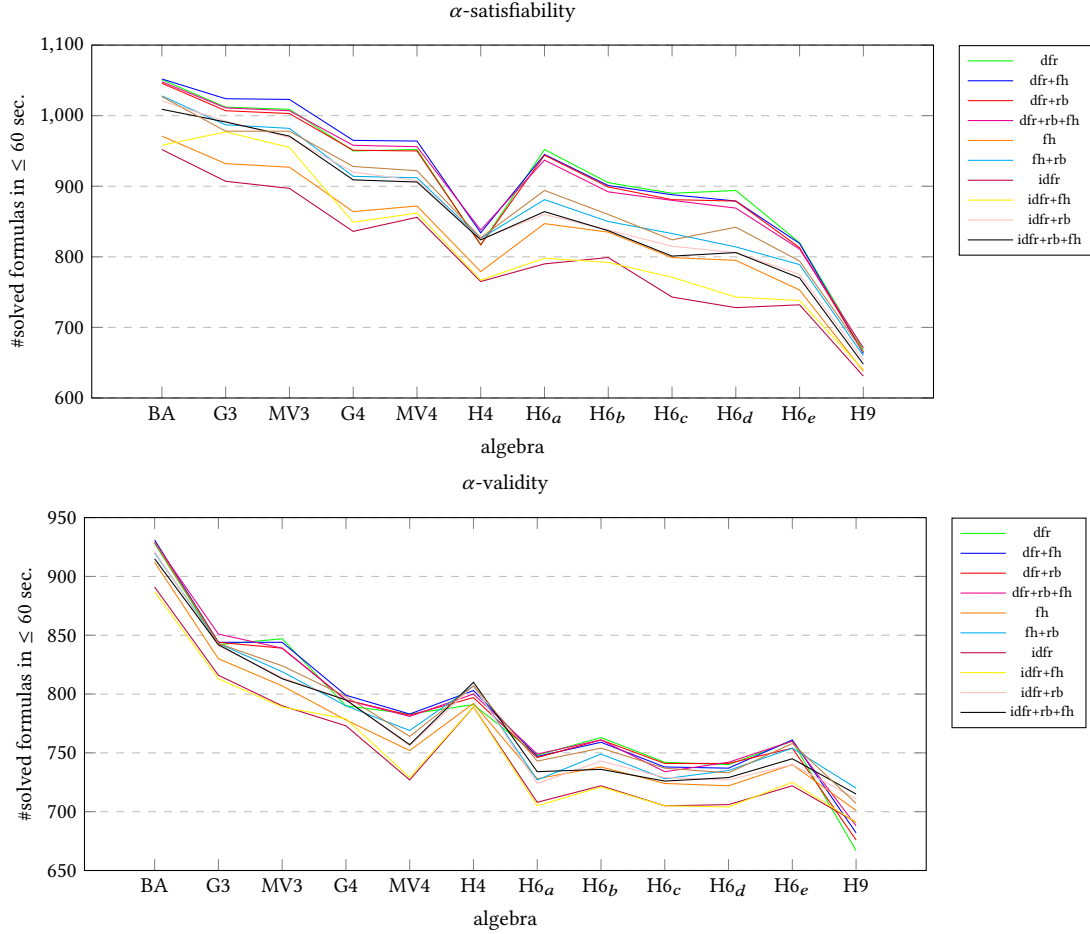


Fig. 29. Estimation of how the computation time for *many-valued linear temporal logic with future and past* is influenced by the chosen metrics (*dfr* - distance from root, *idfr* - inverse distance from root, *fh* - formula height, *rb* - random branch; when using more than one metric, round-robin policy is used to choose the node from the different metric heaps); specifically, how many formulas can be computed within a 60-second timeout over 1200 formulas (200 for each eight from 1 to 6) for algebras BA, G3, MV3, G4, MV4, H4, H6_a (G6), H6_b, H6_c, H6_d, H6_e, and H9. For instance, *dfr* seems to be a good metric to consider, especially when evaluating α -satisfiability.

$f_{A,d}$ lists the domain of A , and each $f_{A,*}$, where $*$ $\in \{., \cup, \hookrightarrow, \preceq\}$ is simply the collection of clauses that describes the corresponding table. Thus, in the case of $A = G3$, for example, we have:

$$\begin{aligned}
 f_{G3,d} &= \forall a((a = 0) \vee (a = 1) \vee (a = \alpha)) \\
 f_{(G3,.)} &= \forall a, b, c((a, b, c) \rightarrow ((c = 1) \leftrightarrow ((a = 1) \wedge (b = 1)))) \wedge \\
 &\quad \forall a, b, c((a, b, c) \rightarrow (c = \alpha) \leftrightarrow (((a = 0) \wedge (b \neq 0)) \vee ((b = 0) \wedge (a \neq 0)) \vee ((a = \alpha) \wedge (b = \alpha)))) \\
 f_{(G3,\cup)} &= \forall a(\cup(a, a, a)) \wedge \forall a, b, c(\cup(a, b, c) \leftrightarrow \cup(b, a, c)) \wedge \cup(0, \alpha, \alpha) \wedge \cup(0, 1, 1) \wedge \cup(\alpha, 1, 1) \\
 f_{(G3,\hookrightarrow)} &= \forall a, b, c(\hookrightarrow(a, b, c) \rightarrow ((c = \alpha) \leftrightarrow ((a = 1) \wedge (b = \alpha)))) \wedge \\
 &\quad \forall a, b, c(\hookrightarrow(a, b, c) \rightarrow ((c = 0) \leftrightarrow ((b = 0) \wedge \preceq(\alpha, a)))) \\
 f_{(G3,\preceq)} &= (\preceq(0, \alpha) \wedge \preceq(\alpha, 1) \wedge \preceq(0, 1) \wedge \forall a(\preceq(a, a))).
 \end{aligned}$$

Finally, translating φ amounts to the following. First, recall that $ST(\varphi, x, a)$ should be satisfiable if and only if φ can be evaluated to some value greater or equal to a on the world x (which, for us, will be $ST_{\geq}(\varphi, x, a)$), but for us to be able to establish that, in general, we also need to be able to establish, for some formula ψ , if ψ cannot be evaluated to some value strictly greater than a given b ($ST_{\leq}(\psi, x, b)$); moreover, whenever convenient, we shall use the shortcut $\prec (a, b)$ for two variables a, b of the algebraic sort to denote $\leq (a, b) \wedge a \neq b$. Also, for the sake of clarity, we omit a $(a = 0)$ (resp., $(a = 1)$) disjunct in all $ST_{\geq}(\varphi, x, a)$ (resp., $ST_{\leq}(\varphi, x, a)$) translations, as every formula φ is always evaluated at a world w to at least 0 and no greater than 1. Thus, we say:

$$ST(\varphi, x, a) = ST_{\geq}(\varphi, x, a),$$

and:

$$\begin{aligned} ST_{\geq}(\alpha, x, a) &= \leq (a, \alpha), \\ ST_{\geq}(p, x, a) &= \exists b(P(x, b) \wedge \leq (a, b)), \\ ST_{\geq}(\varphi \wedge \psi, x, a) &= \exists b, c, d(ST_{\geq}(\varphi, x, b) \wedge ST_{\geq}(\psi, x, c) \wedge \cdot(b, c, d) \wedge \leq (a, d)), \\ ST_{\geq}(\varphi \vee \psi, x, a) &= \exists b, c, d(ST_{\geq}(\varphi, x, b) \wedge ST_{\geq}(\psi, x, c) \wedge \cup(b, c, d) \wedge \leq (a, d)), \\ ST_{\geq}(\varphi \rightarrow \psi, x, a) &= \exists b, c, d(ST_{\leq}(\varphi, x, b) \wedge ST_{\geq}(\psi, x, c) \wedge \hookrightarrow(b, c, d) \wedge \leq (a, d)), \\ ST_{\geq}(\langle X_i \rangle \varphi, x, a) &= \exists y, c(R_i(x, y, c) \wedge \forall b(\neg(\leq (a, b)) \rightarrow \neg ST_{\leq}(\langle X_i \rangle \varphi, x, b))), \\ ST_{\geq}([X_i] \varphi, x, a) &= \forall y, b, c, d((ST_{\geq}(\varphi, x, b) \wedge R_i(x, y, c) \wedge \hookrightarrow(b, c, d)) \rightarrow \leq (a, d)), \\ ST_{\leq}(\alpha, x, a) &= \leq (\alpha, a), \\ ST_{\leq}(p, x, a) &= \forall b(P(x, b) \rightarrow \leq (b, a)), \\ ST_{\leq}(\varphi \wedge \psi, x, a) &= \forall b, c, d((ST_{\leq}(\varphi, x, b) \wedge ST_{\leq}(\psi, x, c) \wedge \cdot(b, c, d)) \rightarrow \leq (d, a)), \\ ST_{\leq}(\varphi \vee \psi, x, a) &= \forall b, c, d((ST_{\leq}(\varphi, x, b) \wedge ST_{\leq}(\psi, x, c) \wedge \cup(b, c, d)) \rightarrow \leq (d, a)), \\ ST_{\leq}(\varphi \rightarrow \psi, x, a) &= \forall b, c, d((ST_{\leq}(\varphi, x, b) \wedge ST_{\leq}(\psi, x, c) \wedge \hookrightarrow(b, c, d)) \rightarrow \leq (d, a)), \\ ST_{\leq}(\langle X_i \rangle \varphi, x, a) &= \forall y, b, c, d((ST_{\leq}(\varphi, y, b) \wedge R_i(x, y, c) \wedge \cdot(b, c, d)) \rightarrow \leq (d, a)), \\ ST_{\leq}([X_i] \varphi, x, a) &= \exists y, c(R_i(x, y, c) \wedge \forall b(\neg(\leq (b, a)) \rightarrow \neg ST_{\leq}([X_i] \varphi, x, b))). \end{aligned}$$

As we have observed, the above translation can be customized for an arbitrary finite FL_{ew} -algebra and, at least, all temporal and spatial multi-modal languages that we have discussed in this paper. However, we limit ourselves to prove the following theorem in the case of K_n .

THEOREM 4. *Let A be a finite FL_{ew} -algebra, φ an FL_{ew} - K_n -formula φ in the vocabulary $\mathcal{P}$, and $\alpha \in A$. Then, it holds that $ST(\varphi, \alpha, K_n, A)$ is true if and only if there exists an FL_{ew} -Kripke structure $\tilde{M} = \langle \tilde{F}, \tilde{V} \rangle$ and some w in it such that $\tilde{V}(\varphi, w) \succeq \alpha$.*

PROOF. Let A be a finite FL_{ew} -algebra, φ an FL_{ew} - K_n -formula φ , and $\alpha \in A$. We want to prove that $ST(\varphi, \alpha, K_n, A)$ is true if and only if there exists an FL_{ew} -Kripke structure $\tilde{M} = \langle \tilde{F}, \tilde{V} \rangle$ and a world w in it such that $\tilde{V}(\varphi, w) \succeq \alpha$. Let $\mathcal{I}$ be a first-order interpretation of the language of $ST()$. In any such interpretation, the elements that satisfy $A()$ form a FL_{ew} -algebra, since f_A must hold. Let $\tilde{F} = \langle W, \tilde{R}_1, \dots, \tilde{R}_n \rangle$ be the many-valued frame that emerges naturally from $\mathcal{I}$ by identifying the elements for which $W()$ holds as worlds and the relations $R_i()$ as accessibility relations, and let $\tilde{M} = \langle \tilde{F}, \tilde{V} \rangle$ the corresponding many-valued model obtained by evaluating a letter $p \in \mathcal{P}$ as α on a world w if and only if α is the maximal (w.r.t. $\leq ()$) value for which $P(w, \alpha)$ holds in $\mathcal{I}$. Since $ST(\varphi, \alpha, K, A)$ implies $\exists x(W(x))$, $\mathcal{I}$ (and therefore W) must contain at least one world w . Let us interpret x as w and a as a value β in A such that $\alpha \leq \beta$. We want to prove that $ST(\varphi, x/w, a/\beta) = ST_{\geq}(\varphi, x/w, a/\beta)$ is true if and only if $\tilde{M}$ evaluates φ to at least β on w . We proceed by induction on the complexity of φ ; because of how the translation is designed, this also requires to show that $ST_{\leq}(\varphi, x/w, a/\beta)$ is true if and only if the FL_{ew} -Kripke structure $\tilde{M}$ cannot evaluate φ at any value greater than β on w , but since the two arguments are very similar to each other, we shall omit it.

- If $\varphi = \gamma$, for some value $\gamma \in \mathbf{A}$, then

$$ST_{\geq}(\gamma, w, \beta) = \preceq (\beta, \gamma),$$

but $\preceq (\beta, \gamma)$ can be true in the first-order domain if and only if, in $\mathbf{A}$, $\beta \preceq \gamma$, so γ is at least β .

- If $\varphi = p$, for some propositional letter p , then

$$ST_{\geq}(p, w, \beta) = \exists b(P(w, b) \wedge \preceq (\beta, b)),$$

which can be satisfied if and only if in the first-order domain we are able to find a value γ to which b can be assigned such that $\beta \preceq \gamma$ and that $P(w, \gamma)$ is true.

- If $\varphi = \psi \wedge \xi$, then

$$ST_{\geq}(\psi \wedge \xi, w, \beta) = \exists b, c, d(ST_{\geq}(\psi, w, b) \wedge ST_{\geq}(\xi, w, c) \wedge (b, c, d) \wedge \preceq (\beta, d)).$$

By inductive hypothesis it is the case that, in $\widetilde{F}$, ψ (resp., ξ) is evaluated, on w , at least γ (resp., δ), for some γ, δ such that $\gamma \cdot \delta \succeq \beta$. But this happens if and only if $\psi \wedge \xi$ is evaluated at least β on w .

- If $\varphi = \psi \vee \xi$, then

$$ST_{\geq}(\psi \vee \xi, w, \beta) = \exists b, c, d(ST_{\geq}(\psi, w, b) \wedge ST_{\geq}(\xi, w, c) \wedge \cup(b, c, d) \wedge \preceq (\beta, d)).$$

By inductive hypothesis it is the case that, in $\widetilde{F}$, ψ (resp., ξ) is evaluated, on w , at least γ (resp., δ), for some γ, δ such that $\gamma \cdot \delta \succeq \beta$. But this happens if and only if $\psi \vee \xi$ is evaluated at least β on w .

- If $\varphi = \psi \rightarrow \xi$, then

$$ST_{\geq}(\psi \rightarrow \xi, w, \beta) = \exists b, c, d(ST_{\leq}(\psi, w, b) \wedge ST_{\geq}(\xi, w, c) \wedge \hookrightarrow (b, c, d) \wedge \preceq (\beta, d)).$$

By inductive hypothesis it is the case that, in $\widetilde{F}$, ψ (resp., ξ) is evaluated, on w , at most γ (resp., at least δ), and that $\gamma \hookrightarrow \delta \succeq \beta$. By definition, $\beta \preceq \gamma \hookrightarrow \delta$ if and only if $\beta \cdot \gamma \preceq \delta$. Again by definition, if $\gamma' \preceq \gamma$ and $\delta' \succeq \delta$, then $\beta \cdot \gamma' \preceq \delta'$, that is, $\beta \preceq \gamma' \hookrightarrow \delta'$. Thus, if ψ is evaluated on w to some value $\gamma' \preceq \gamma$, ξ is evaluated on w to some value $\delta' \succeq \delta$, and $\beta \preceq \gamma \hookrightarrow \delta$, then $\psi \rightarrow \xi$ is evaluated at least β on w .

- If $\varphi = \langle X_i \rangle \psi$, then

$$ST_{\geq}(\langle X_i \rangle \psi, w, \beta) = (\beta = 0) \vee \exists y, c(R_i(w, y, c) \wedge \forall b(\neg(\preceq (\beta, b)) \rightarrow \neg ST_{\leq}(\langle X_i \rangle \psi, y, b))).$$

This happens if and only if first, in $\widetilde{F}$, w has at least one reachable successor, and second, for any value γ strictly less than or incomparable to β , it is the case that $\langle X_i \rangle \psi$ cannot be constrained to be evaluated at a value less than or equal to γ . Since we are assuming that the translation $ST_{\leq}$ is correct, we have the thesis.

- Finally, if $\varphi = [X_i] \psi$, then

$$\forall y, b, c, d((ST_{\geq}(\varphi, y, b) \wedge R_i(w, y, c) \wedge \hookrightarrow (c, b, d)) \rightarrow \preceq (\beta, d)).$$

This is equivalent to say that for every algebra values γ, δ and world w' , if (by virtue of the inductive hypothesis) φ is evaluated at least γ on w' and w' is reachable from w with value δ , and $\delta \rightarrow \gamma$ is ϵ , then ϵ is at least β ; thus, the infimum of all such values ϵ must be β too. Hence, the thesis. $\square$

For the purpose of example, we considered the particular case of the many-valued version of the interval temporal logic HS, using the corresponding adaptation of the above translation. After generating 200 random formulas for each algebra and for each height up to 7 (i.e., up to 128 symbols) using the procedure described in Section 3.4, and setting a timeout of 60 seconds, we compared the output of the tableau system and the translation, assessing that for all formulas for which neither of them timed out they agreed on the α -satisfiability of the formula. Statistics of how many formulas timed out can be seen in Fig. 30.

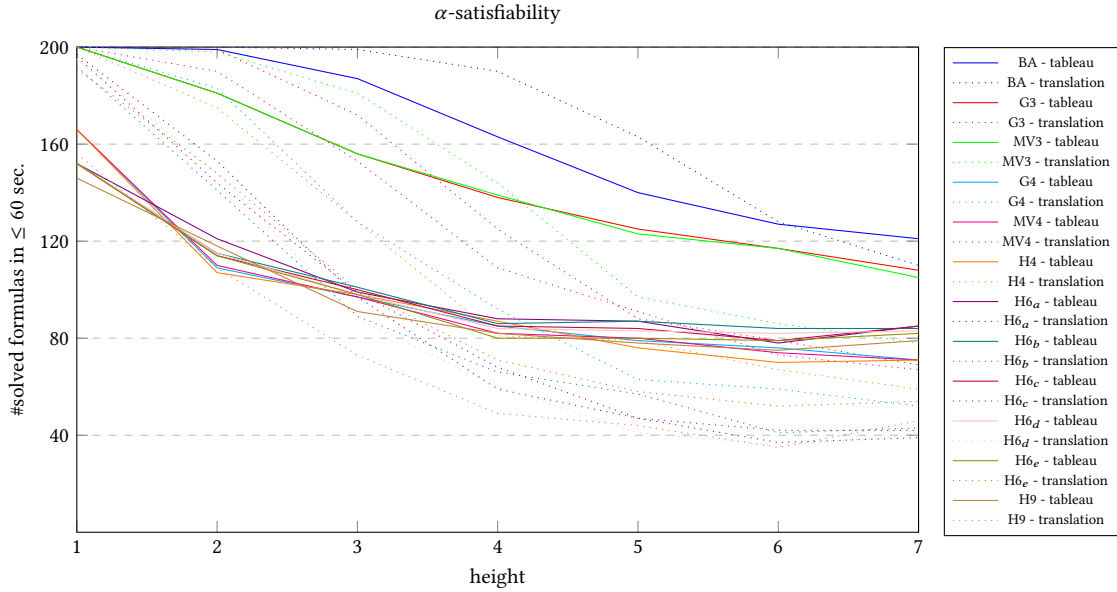


Fig. 30. Tableau (strict lines) vs translation (dotted lines) performance for solving α -satisfiability for *many-valued Halpern and Shoham's interval temporal logic*: how many formulas can be computed within a 60-second timeout over 1400 formulas (200 for each eight from 1 to 7) for algebras BA, G3, MV3, G4, MV4, H4, H6_a (G6), H6_b, H6_c, H6_d, H6_e, and H9. While translation performs better for smaller formulas, performance degrade much faster increasing the height of the formula due to alternating $\forall$ and $\exists$ operators in the rules, a known challenge for first-order SAT and SMT solvers. Moreover, the cardinality of the domain of the algebra has a much greater impact on the translation performance.

As it can be seen, other than the boolean case for which it is generally convenient to leverage a first-order translation and call an SMT-solver, the tableau system performs generally better. This is probably due to a known constraint of first order SAT and SMT solvers when dealing with alternating between $\forall$ and $\exists$ operators.

4 Conclusions and Future Work

In this paper, we proposed a general framework for many-valued temporal and spatial modal logic and we provided a reasoning tool for it based on analytic tableau technique. In order to do so, we first introduced a common framework to model well-known temporal and spatial logics, leveraging multi-modal logic and based on the notion of a linear order. Then, we proposed an easy and uniform procedure to obtain a many-valued version for these logics leveraging FL_{ew}-algebras, hence generalizing both fuzzy logics and Heyting algebras. Finally, we provided a sound and complete tableau system to reason with many-valued temporal and spatial logic, solving both the α -satisfiability and α -validity problems, also providing an open-source implementation and an extensive set of tests to assess both its performance for various logics, algebras, and search methods.

Although much effort was spent on assessing the correctness of the reasoning tool, we believe that a thorough evaluation is still impossible without an agreed benchmark for many-valued multi-modal logic or other reasoning tools aiming to solve the same problems, which are, needless to say, outside the scope of this publication. At the same time, while we provide a modular structure that is easy to extend mainly designed around search strategies, there is still much room for improvement left (e.g., employing termination, blocking, and bounding strategies). Furthermore, the generality of the tableau has a significant impact on performance: when reasoning

with specific logics (e.g., when the order of the algebra is total), it is advised to design a specific tableau for such logics, especially in real-world applications where performance is critical.

We believe that this contribution is a valuable first step in fulfilling the need for reasoning tools for many-valued and multi-modal logics: in fact, as we have mentioned throughout the paper, very few attempts have been made in providing reasoning systems for only some of the mentioned spatial and temporal logics, let alone the many-valued case, for which, to the best of our knowledge, no implementation is currently available. We strongly believe in the importance of these tools in many real-world applications, from formal verification, to planning, to symbolic learning. Furthermore, this tool is integrated into a framework for symbolic learning, enhancing the strong correlation between symbolic models and the underlying logic and making it unique in this sense.

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