

Variety-Seeking Jump Games on Graphs

LATA NARAYANAN, Concordia University, Canada

JAROSLAV OPATRNY, Concordia University, Canada

SHANMUKHA TUMMALA, Concordia University, Canada

ALEXANDROS A. VOUDOURIS*, University of Southern Denmark, Denmark

We consider a class of jump games in which agents of different types occupy the nodes of a graph aiming to maximize the *variety of types* in their neighborhood. In particular, each agent derives a utility equal to the *number of types different from its own* in its neighborhood. We show that the jump game induced by the strategic behavior of the agents (who aim to maximize their utility) may in general have improving response cycles, but is a potential game under *any* of the following four conditions: there are only two types of agents; or exactly one empty node; or the graph is of degree at most 2; or the graph is 3-regular and there are two empty nodes. Additionally, we show that on trees, cylinder graphs, and tori, there is always an equilibrium. Finally, we show tight bounds on the price of anarchy with respect to two different measures of diversity: the social welfare (the total utility of the agents) and the number of colorful edges (that connect agents of different types).

JAIR Associate Editor: Reshef Meir

JAIR Reference Format:

Lata Narayanan, Jaroslav Opatrny, Shanmukha Tummala, and Alexandros A. Voudouris. 2026. Variety-Seeking Jump Games on Graphs. *Journal of Artificial Intelligence Research* 86, Article 50 (August 2026), 27 pages. doi: [10.1613/jair.1.21746](https://doi.org/10.1613/jair.1.21746)

1 Introduction

While analyzing residential segregation, [Schelling \(1971, 1969\)](#) imagined a simple scenario in which agents of two different types are randomly assigned to the nodes of a graph representing a city. The agents are allowed to randomly *jump* to other available locations or *swap* locations with other agents whenever this increases the fraction of same-type neighbors they have, up to a threshold. Schelling experimentally showed that, in most cases, this random behavior of the agents leads to segregated neighborhoods consisting only of agents of one type. His work inspired researchers in many different disciplines to further study this model and generalize it to capture more complicated dynamics between agents of different types. Recent work within the multi-agent literature has considered game-theoretic variants of Schelling’s model in which agents act *selfishly* rather than randomly by aiming to maximize a *utility* function. This strategic behavior of the agents then defines a *game* between them and the objective is to analyze whether *equilibria* (stable assignments where agents simultaneously achieve the maximum possible utility they can) exist and what properties (related to segregation) they have.

*Corresponding Author.

Authors’ Contact Information: Lata Narayanan, ORCID: [0000-0002-3875-0371](https://orcid.org/0000-0002-3875-0371), lata.narayanan@concordia.ca, Concordia University, Montreal, Canada; Jaroslav Opatrny, ORCID: [0000-0001-6149-003X](https://orcid.org/0000-0001-6149-003X), jaroslav.opatrny@concordia.ca, Concordia University, Montreal, Canada; Shanmukha Tummala, shanmukha.tummala@outlook.com, Concordia University, Montreal, Canada; Alexandros A. Voudouris, ORCID: [0000-0003-1105-3856](https://orcid.org/0000-0003-1105-3856), voudouris@imada.sdu.dk, University of Southern Denmark, Vejle, Denmark.



This work is licensed under a Creative Commons Attribution International 4.0 License.

© 2026 Copyright held by the owner/author(s).

doi: [10.1613/jair.1.21746](https://doi.org/10.1613/jair.1.21746)

With few exceptions, most of the utility functions that have been proposed and studied over the years can be described as *similarity-seeking* or *homophilic* in the sense that agents prefer to be close to other agents of the same type (Agarwal et al. 2021; Bullinger, Lenzner, et al. 2025; Bullinger, Suksompong, et al. 2021; Chauhan et al. 2018; Echzell et al. 2019; Kanellopoulos et al. 2021). Such a behavior is well-justified in scenarios where agents aim to form groups that share similar interests or require similar skill sets to complete tasks. On the other hand, however, there are many other important applications where the desideratum is *diversity*. For example, data from the General Social Survey (Smith et al. 2019) (conducted in the US since 1950) show that the percentage of survey respondents answering yes to whether their neighborhood includes people of different races has increased from 23.9% in 1972 to 70.6% in 2024. This would support a conclusion that a steadily increasing percentage of people prefer to live in, or at least have no objection to living in diverse neighborhoods. In addition, it is well-documented that, nowadays, many governments, businesses, and other institutions are actively promoting increased diversity as being beneficial to both societal harmony, efficiency and financial performance (for example, by attracting external talent to reflect the diverse customer base and bring different problem-solving perspectives) (Lorenzo and Reeves 2018; Ntantamis and Zhou 2025; Rock and Grant 2016).

Motivated by applications like those mentioned above, some recent papers have considered utility functions which aim to model diversity as being beneficial for the agents. In particular, Bilò et al. (2022a) and Friedrich et al. (2023) studied games with two types of agents and a single-peaked utility function which increases monotonically with the fraction of same-type neighbors in the interval $[0, \Lambda]$ for some $\Lambda \in (0, 1)$, and then decreases monotonically. A different model was proposed by Kanellopoulos et al. (2023) who focused on a utility function that assigns different weights to different types according to their position on a line, indicating different preferences over the types. More recently, Narayanan, Sabbagh, et al. (2025) focused on jump games where the utility of an agent is the fraction of its *different-type* neighbors; essentially, this function is the complement of the one implied by Schelling’s original model and used in a plethora of subsequent works.

1.1 Our Contribution

An important aspect of the models studied in the aforementioned papers is that the agents are mainly concerned with the type difference between themselves and their neighbors, and not the difference or diversity *among* their neighbors. For example, according to the utility function considered by Narayanan, Sabbagh, et al. (2025), an agent is fully satisfied even if all its neighbors are of one type, as long as this type is different than its own. Whether this is a truly diverse neighborhood is up to debate. In this paper¹, we study jump games where agents are seeking *variety*. An agent’s utility is defined as the *number of types different from its own* among its neighbors. Such a utility function was very recently studied for swap games by Li et al. (2025). With this utility function, the maximum possible utility of an agent is the maximum between the number of types and the degree of the underlying graph.

To be more specific, we consider *jump* games with n agents that are partitioned into $k \geq 2$ different types and occupy the nodes of a graph $G = (V, E)$, where $|V| < n$. Each agent aims to maximize the number of different types in its neighborhood. We first show that such games may have an *improving response cycle*, even when G is 3-regular and $|V| = n + 3$. This means that there is an initial assignment of the agents to the nodes of the graph so that if the agents jump one by one to empty nodes of the graph, they will eventually cycle back to that initial assignment; in other words, starting from such an assignment, an equilibrium cannot be reached. In spite of this impossibility, we show that the game is *potential* (that is, the Nash dynamics converges to an equilibrium) under

¹The preliminary version of our paper appeared in the proceedings of the *34th International Joint Conference on Artificial Intelligence (IJCAI)* (Narayanan, Opatrny, et al. 2025); this full version contains a more extensive discussion of the model, the full proofs of all statements, and refined results for non-connected graphs.

any of the following conditions: $k = 2$; $|V| = n + 1$; the graph is of degree at most 2; the graph is 3-regular and $|V| = n + 2$. Additionally, we show that there is always an equilibrium when the graph is a tree, or a cylinder, or a torus by carefully constructing one for these cases.

We next switch to analyzing the quality of equilibria under two different objectives: The *social welfare* (defined as the total utility of the agents) and the *number of colorful edges* (defined as the number of edges between agents of different types). Both objectives are different measures of diversity (though not necessarily of variety), and have been considered before by (Li et al. 2025) for the case of swap games. We show bounds on the *price of anarchy*, which quantifies the loss in the social welfare or the number of colorful edges in the worst equilibrium. In particular, we first observe that the price of anarchy is unbounded if the graph is disconnected, for any of the two objectives. We then focus on connected graphs and show that the price of anarchy with respect to the social welfare is $\Theta(n)$ for general games, and $\Theta(k)$ for balanced types (that is, when all types are of the same cardinality). For colorful edges, we show that the price of anarchy is $\Theta(n)$ even for balanced types, and $\Theta(\delta)$ when the graph is δ -regular and the types are balanced. In addition, for both objectives, we provide a lower bound on the *price of stability*, showing that there are instances in which the optimal assignment is not necessarily an equilibrium.

1.2 Related Work

As already mentioned, many generalizations and variations of Schelling’s random model have been studied via agent-based simulations in many different disciplines, including sociology (Clark and Fossett 2008), economics (Pancs and Vriend 2007; Zhang 2004), and physics (Vinković and Kirman 2006). In computer science, most of the work has focused on probabilistic analyses of Schelling’s model (Immorlica2017exponential; Barmpalias et al. 2014; Bhakta et al. 2014; Bläsius et al. 2023; Brandt et al. 2012) and also on the computational complexity of assignments with certain efficiency properties (Bullinger, Suksompong, et al. 2021; Deligkas et al. 2024).

Our work is related to a recent stream of papers that consider Schelling games induced when the agents do not act randomly but strategically by maximizing a utility function. The majority of papers in this literature study jump and swap games with similarity-seeking utility functions that one way or another depend on the ratio of same-type agents in the neighborhood, and consider questions related to the existence of equilibria, their computational complexity, and their quality as measured by the price of anarchy and the price of stability. Some of the first papers in this area include the works of Chauhan et al. (2018) and Echzell et al. (2019) who studied the convergence of the best response dynamics to equilibrium assignments, and the work of Agarwal et al. (2021) who showed that equilibria may not always exist and, even when they do, they might be hard to compute. Much of the follow-up work provided stronger hardness results (Kreisel et al. 2024), studied different utility functions aiming to model other natural agent behaviors (Bilò et al. 2022b; Kanellopoulos et al. 2021), and made different assumptions about how agents are related to each other (Bilo2023continuous; Chan et al. 2020).

Some recent work on Schelling games has deviated from the standard assumption that agents aim to be close to their own type and have proposed different models in which the agents derive utility from other types of agents as well (Bilò et al. 2022a; Friedrich et al. 2023; Kanellopoulos et al. 2023; Li et al. 2025; Narayanan, Sabbagh, et al. 2025). The paper most related to ours is that of (Li et al. 2025) who studied swap games with several different diversity-seeking utility functions, including the one we consider here, that is the number of different types in an agent’s neighborhood. Among other results, in contrast to our work here where we show that there might exist improving response cycles, they showed that swap games are always potential no matter the structure of the underlying graph. They also showed tight bounds on the price of anarchy for objectives such as the number of colorful edges that we also consider, as well as some other ones. Most of their bounds indicate that the price of anarchy depends on the degree of the graph and in many case tends to 1 as the number of agents n or the number of types k grows; in contrast, our price of anarchy bounds suggest a linear dependency on n or k in most cases.

Schelling games are related to *hedonic games* (Aziz and Savani 2016), and in particular, to the so-called *fractional hedonic games* (Aziz, Brandl, et al. 2019). These are coalition formation games that cover settings in which the agents classify the other agents as friends or non-friends, and are partitioned into groups aiming to maximize the fraction of friends in their group. In Schelling games, the underlying graph whose nodes the agents occupy, can lead to overlapping coalitions and different dynamics. More related to the variety-seeking games we consider in this work is the *Bakers and Millers hedonic game* (Aziz, Brandl, et al. 2019; Krogmann et al. 2025) in which there are two types of agents (bakers and millers) that aim to maximize the number of agents of the other type in their neighborhoods, and other *hedonic diversity games* in which agents are also partitioned into multiple types and variety-seeking preferences can be expressed (Boehmer and Elkind 2020a; Brederbeck et al. 2019; Darmann 2023; Ganian et al. 2023). Diversity preferences have also been studied in the context of the *stable roommate problem*, where the goal is to pair the agents so that no two of them want to deviate and pair with each other instead (Boehmer and Elkind 2020b; Ge and Itoh 2025), and *refugee housing*, where agents with potentially variety-seeking preferences must be allocated to empty nodes of a graph (Knop and Schierreich 2023; Schierreich 2023).

2 Preliminaries

There is a set N of $n \geq 2$ agents who are partitioned into $k \geq 2$ types $\mathcal{T} = \{T_1, \dots, T_k\}$. We denote by $t(i)$ the type of agent i ; that is, $t(i) = T$ if $i \in T$. We denote by n_T the number of agents of type T ; if $n_T = n/k$ for each type T , then the types are called *balanced*. Sometimes, for simplicity, we will also use colors (such as red or blue) and different letter (such as R or B) to refer to different types. Each agent i occupies a node of a graph $G = (V, E)$ with $|V| > n$. An assignment $\mathbf{v} = (v_i)_{i \in N}$ specifies the node v_i that each agent i occupies in G , such that $v_i \neq v_j$ for different agents i and j . Given an assignment $\mathbf{v}$, we denote by $N_i(\mathbf{v})$ the *neighbors* of i , which is the set of all agents that occupy nodes adjacent to v_i in G . Also, for any node $v \in V$ and assignment $\mathbf{v}$, we denote by $v(\mathbf{v})$ the agent that occupies v (if no agent occupies v , then $v(\mathbf{v}) = \emptyset$) and by $T_v(\mathbf{v})$ the set of agent types located at nodes adjacent to v . Given this, the *type-count* of node v is $\tau_v(\mathbf{v}) = |T_v(\mathbf{v})|$. Depending on the formation of its neighborhood, each agent i gains a *utility* $u_i(\mathbf{v})$ equal to the number of types different than $t(i)$ in i 's neighborhood, i.e., $u_i(\mathbf{v}) = |T_{v_i}(\mathbf{v}) \setminus \{t(i)\}|$.

An assignment $\mathbf{v}$ is a *pure Nash equilibrium* (or, simply, *equilibrium*) if no agent can strictly increase its utility by unilaterally jumping to an *empty* node in the graph. That is, for every agent i and empty node v (according to $\mathbf{v}$), $u_i(\mathbf{v}) \geq u_i(\mathbf{v}')$, where $\mathbf{v}'$ is the same as $\mathbf{v}$ with the only exception that $v'_i = v$. For an instance $I = (N, \mathcal{T}, G)$, let $\text{NE}(I)$ be the set of equilibrium assignments; note that this set might be empty.

We are interested in characterizing the classes of graphs for which equilibrium assignments are guaranteed to exist. To do so, we will in many cases show that the best-response Nash dynamics converges to an equilibrium by identifying an *ordinal potential function* $\Phi(\mathbf{v})$. Such a function has the following property: For any two assignments $\mathbf{v}$ and $\mathbf{v}'$ that differ only on the node occupied by an agent i , it holds that

$$(u_i(\mathbf{v}) - u_i(\mathbf{v}')) \cdot (\Phi(\mathbf{v}) - \Phi(\mathbf{v}')) > 0.$$

Essentially, this property requires that the function is strictly increasing whenever there is an agent with a deviating strategy that leads to strictly larger utility. Note that a strictly decreasing function can be also used as a potential since its monotonicity can be switched by changing its sign. A potential function can be defined only if there is no *improving response cycle* (IRC) in the Nash dynamics (that is, a sequence of assignments in which the first and last assignments are the same, and two consecutive assignments differ only on the node occupied by an agent who has larger utility in the latter assignment of the two).

When equilibrium assignments exist, we are also interested in measuring their quality in terms of achieved diversity. To do this, we consider two objectives functions:

- The *social welfare* (SW), defined as the total utility of the agents:

$$SW(\mathbf{v}) = \sum_{i \in N} u_i(\mathbf{v}).$$

- The *number of colorful edges* (CE) which is the number of edges whose endpoints are occupied by agents of different types:

$$CE(\mathbf{v}) = |(x, y) \in E : x(\mathbf{v}) \neq \emptyset \wedge y(\mathbf{v}) \neq \emptyset \wedge t(x(\mathbf{v})) \neq t(y(\mathbf{v}))|.$$

For each objective $f \in \{SW, CE\}$, we define the *price of anarchy* as the worst-case ratio (over a class of instances $\mathcal{I}$ in which at least one instance admits an equilibrium) between the maximum possible f -value over all assignments and the *minimum* f -value over all equilibrium assignments:

$$PoA_f = \sup_{I \in \mathcal{I} : NE(I) \neq \emptyset} \frac{\max_{\mathbf{v}} f(\mathbf{v})}{\min_{\mathbf{v} \in NE(I)} f(\mathbf{v})}.$$

Similarly, the *price of stability* is defined as the worst-case ratio (over a class of instances $\mathcal{I}$ in which at least one instance admits an equilibrium) between the maximum possible f -value over all assignments and the *maximum* f -value over all equilibrium assignments:

$$PoS_f = \sup_{I \in \mathcal{I} : NE(I) \neq \emptyset} \frac{\max_{\mathbf{v}} f(\mathbf{v})}{\max_{\mathbf{v} \in NE(I)} f(\mathbf{v})}.$$

Observe that, by definition, $PoA_f \geq PoS_f \geq 1$.

3 Existence of Equilibria

In this section, we focus the existence of equilibrium assignments. We start with an impossibility: There exists a game with an improving response cycle in its Nash dynamics. This implies that the jump game we consider is not always a potential game, and thus the existence of an equilibrium assignment cannot always be shown by constructing a potential function.

THEOREM 3.1. *There exists a game with an improving response cycle in the Nash dynamics, even when the graph is 3-regular and there are three empty nodes.*

PROOF. Consider the improving response cycle shown in Figure 1. In the first assignment in the left column, the red agent occupying node a has utility 1 (as it only has blue neighbors) and prefers to jump to node d to improve its utility to 2. This leads to the second assignment in the left column where the blue agent at node b has now utility 1 (as it only has green neighbors) and incentive to jump to node e to increase its utility to 2. This leads to the third assignment in the left column where the green agent at node c has utility 1 (as it only has red neighbors) and incentive to jump to node f to increase its utility to 2. This leads to the third assignment in the right column where the red agent at node d has utility 1 again, and would now prefer to jump back to node a where it can get utility 2. This leads to the second assignment in the right column where the blue agent at node e has utility 1 and would also now prefer to go back to node b to get utility 2. This leads to the first assignment in the right column where the green agent at node f has been left with only red neighbors and now has incentive to jump to node c again to get utility 2, thus completing the cycle. $\square$

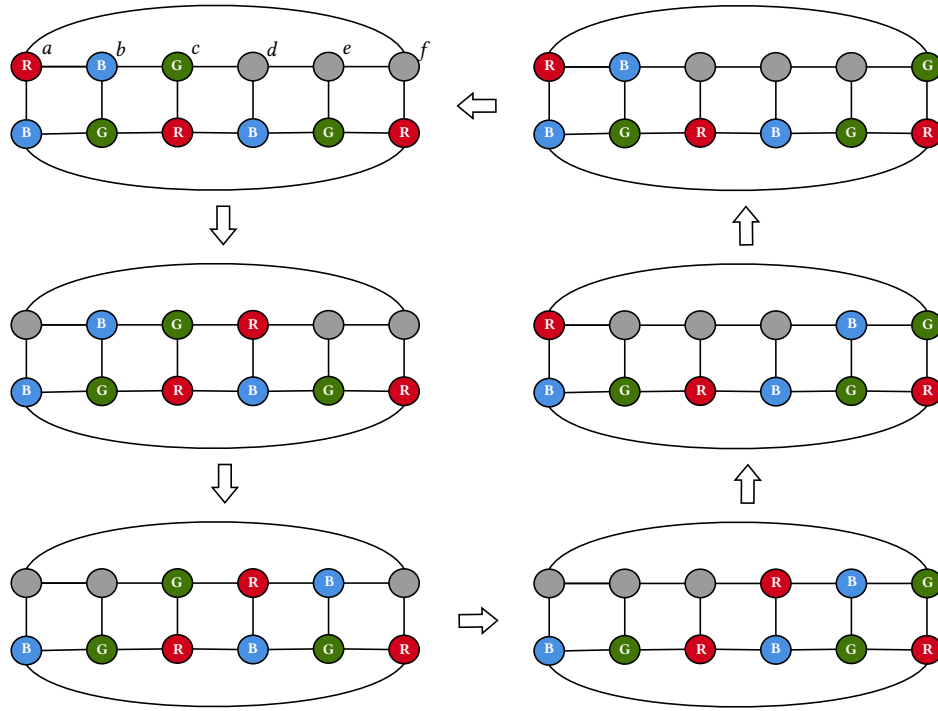


Fig. 1. A game with an improving response cycle involving agents of three types: red (R), blue (B), and green (G).

Next, we identify several cases where the game is indeed potential, and thus there is always at least one equilibrium assignment. In particular, we show that this is true when (1) there are only two types of agents, (2) there is a single empty node, (3) the graph is of degree at most 2, and (4) there are two empty nodes and the graph is 3-regular. We start with the first case.

THEOREM 3.2. *The game is potential when there are only two types of agents.*

PROOF. We argue that the social welfare (the total utility of the agents) is a potential function. Observe that, if some agent i jumps to an empty node, then it does so because its utility increases from 0 to 1. Since i has only neighbors of its own type (so that its utility is 0) in the initial assignment, the deviation of i does not alter the utility of those agents. In addition, the utility of each of i 's new neighbors in the new assignment does not decrease in comparison to how much it was before; in particular, it either remains the same or increases from 0 to 1. Hence, the social welfare overall increases by at least 1 after i jumps, and is thus a potential function, leading to an equilibrium when it is maximized. $\square$

Next, we consider the second case where the graph has only one empty node.

THEOREM 3.3. *The game is potential when the graph contains just one empty node.*

PROOF. We will prove by induction that, for every $\ell \in [k]$, there is no IRC containing an assignment $\mathbf{v}$ in which the empty node, say u , has agents of ℓ different types surrounding it, that is, $\tau_u(\mathbf{v}) = \ell$. Consequently, no IRC can exist in the Nash dynamics, thus showing that the game is potential.

Base case: $\ell = 1$. Suppose for the purpose of contradiction that there exists an IRC containing an assignment $\mathbf{v}$, in which $|T_u(\mathbf{v})| = 1$ for the empty node u . Without loss of generality, suppose all the agents adjacent to u are of type R . Consider the next assignment $\mathbf{v}'$ in the IRC, and suppose agent i is located at node u in $\mathbf{v}'$. Let v be the node that i occupies in $\mathbf{v}$; that is, agent i jumps from node v to node u , implying that $u_i(\mathbf{v}') > u_i(\mathbf{v})$. Note that v is the empty node in the assignment $\mathbf{v}'$.

Agent i cannot be of type R since it gets utility 1 after jumping to u , and it must have had utility 0 by occupying v in assignment $\mathbf{v}$. If i had no neighbors in the assignment $\mathbf{v}$, then either v is not connected to any node or the only neighbor of v in the graph is u , the empty node.

- If v is not connected to any node, then no one wants to jump to v in $\mathbf{v}'$. So, this is an equilibrium assignment and there is no IRC, a contradiction.
- If the only neighbor of v in the graph is u , the empty node, then u has neighbors of two types, a contradiction.

Therefore, agent i must have had only neighbors of its own type when occupying v in $\mathbf{v}$. So, i was segregated in the assignment $\mathbf{v}$ and $|T_v(\mathbf{v}')| = 1$. Applying the same argument again, it follows that every assignment in the IRC has an empty node in which all neighboring agents are of the same type, and every jump is made by a segregated agent. Notice however, that each such jump reduces the number of segregated agents, leading to a contradiction.

Induction hypothesis: Assume there is no IRC containing an assignment where the empty node has neighbors of p different types, for all $p \in [\ell]$ for some $\ell < k$.

Induction step: We will show that the hypothesis remains true for $\ell + 1$. Suppose instead that there is an IRC containing an assignment $\mathbf{v}$ where the empty node u has neighbors of exactly $\ell + 1$ types, that is, $\tau_u(\mathbf{v}) = \ell + 1$. Consider the next assignment $\mathbf{v}'$ in the IRC that is the result of agent i jumping from node v to u . Then, the empty node in $\mathbf{v}'$ is v and, by the inductive hypothesis, $\tau_v(\mathbf{v}') > \ell$. Consider the following three cases:

- $t(i) \in T_u(\mathbf{v})$. Then $u_i(\mathbf{v}') = \ell$ and $u_i(\mathbf{v}) \geq \ell$, which contradicts the fact that i is motivated to jump.
- $t(i) \notin T_u(\mathbf{v})$ and $t(i) \notin T_v(\mathbf{v})$. Then $u_i(\mathbf{v}') = \ell + 1$ and $u_i(\mathbf{v}) > \ell$, which contradicts the fact that i is motivated to jump.
- $t(i) \notin T_u(\mathbf{v})$ and $t(i) \in T_v(\mathbf{v})$. Then $u_i(\mathbf{v}') = \ell + 1$, and $u_i(\mathbf{v}) \geq \ell$. For i to be motivated to jump, it must be that $u_i(\mathbf{v}) = \ell$. Since $t(i) \in T_v(\mathbf{v})$, this implies that $\tau_v(\mathbf{v}') = \ell + 1$. In other words, the empty node v in assignment $\mathbf{v}'$ has $\ell + 1$ types in its neighborhood.

From the above we see that the *number of monochromatic edges* in $\mathbf{v}'$ is less than that in $\mathbf{v}$. Applying the same argument repeatedly, it follows that for every assignment in the IRC, (a) the empty node has $\ell + 1$ types in its neighborhood, and (b) the agent that deviates has at least one neighbor of its own type in its neighborhood, but no neighbors of its type at the empty node it jumps to, causing the number of monochromatic edges to decrease in successive assignments, which contradicts the existence of the IRC. $\square$

Next, we focus on the third case where the graph is of degree at most 2.

THEOREM 3.4. *The game is potential when the graph is of degree at most 2.*

PROOF. For any assignment $\mathbf{v}$, recall that $\text{CE}(\mathbf{v})$ is the number of colorful edges that connect agents of different types. In addition, let $c(\mathbf{v})$ be the number of empty nodes that are adjacent to at most one type of agents, that

is $c(\mathbf{v}) = |\{v \in V : v \text{ empty in } \mathbf{v} \text{ and } \tau_v(\mathbf{v}) \leq 1\}|$. We will show that the function $\Phi(\mathbf{v}) = 2 \cdot \text{CE}(\mathbf{v}) + c(\mathbf{v})$ is a potential one.

We will focus on two assignments $\mathbf{v}$ and $\mathbf{v}'$ that differ on the node occupied by a single red agent i . In particular, agent i occupies a source node s in $\mathbf{v}$ and jumps to a destination node d in $\mathbf{v}'$. Note that d is empty in $\mathbf{v}$ and s is empty in $\mathbf{v}'$. We will show that $\Delta\Phi = \Phi(\mathbf{v}') - \Phi(\mathbf{v}) > 0$ when $u_i(\mathbf{v}') > u_i(\mathbf{v})$. For simplicity, further define $\Delta\text{CE} = \text{CE}(\mathbf{v}') - \text{CE}(\mathbf{v})$ and $\Delta c = c(\mathbf{v}') - c(\mathbf{v})$. We consider the following two cases depending on the utility that agent i derives in assignment $\mathbf{v}$.

Case 1: $u_i(\mathbf{v}) = 0$. Since i has only neighbors of its own type or no neighbors at all, the source node s it occupies does not participate in any colorful edges. Since i jumps from s to d , d must be adjacent to at least one non-red agent for i to increase its utility. Hence, the number of colorful edges increases by at least 1 when i jumps, and thus $\Delta\text{CE} \geq 1$. In $\mathbf{v}'$, node s is empty and is surrounded by at most one type of agents. Hence, Δc may increase by 1 due to s . If d is surrounded by only one type of agents in $\mathbf{v}$, then Δc may decrease by 1. Since the graph is of degree at most 2 and d is adjacent to a non-red agent, d may be adjacent to at most one empty node, say v , and Δc might further decrease by 1 due to v (when i becomes adjacent to v by jumping to d). Overall, $\Delta c \geq -1$, and thus $\Delta\Phi = 2\Delta\text{CE} + \Delta c \geq 1$.

Case 2: $u_i(\mathbf{v}) = 1$. In this case, the node d where i jumps to from s must be adjacent to two agents of different types, say blue and green. This further implies that d is not adjacent to s .

- If i has only one non-red neighbor in $\mathbf{v}$, then there will be one more colorful edge in $\mathbf{v}'$, and thus $\Delta\text{CE} = 1$. Now observe that Δc is not affected due to d since it is adjacent to a blue and a green agent in $\mathbf{v}$ where it is empty. Due to s , Δc might increase by 1 in case s is adjacent to a non-red agent only, or not at all in case s is adjacent to a red and a non-red agent. So, overall, $\Delta c \geq 0$, and $\Delta\Phi = 2\Delta\text{CE} + \Delta c \geq 2$.
- Otherwise, i has two non-red neighbors of the same type in $\mathbf{v}$. Hence, there is no change in the number of colorful edges when i jumps from s to d , that is, $\Delta\text{CE} = 0$. Similarly to before, there is no change in Δc due to d , but there is an increase of 1 in Δc due to s , and thus $\Delta c = 1$. Overall, $\Delta\Phi = 2\Delta\text{CE} + \Delta c = 1$.

This completes the proof. $\square$

Before proceeding with our next result, we first prove a lemma that will be useful.

LEMMA 3.5. *Let $\mathbf{v}$ be a non-equilibrium assignment and suppose an agent i jumps from node s to node d leading to assignment $\mathbf{v}'$. Then,*

- (1) $\tau_d(\mathbf{v}) \geq \tau_s(\mathbf{v}')$, and
- (2) if $\tau_d(\mathbf{v}) = \tau_s(\mathbf{v}')$, then the jumping agent i 's utility increases exactly by 1, the number of monochromatic edges decreases, and no new monochromatic edges are created.

PROOF. Without loss of generality, let the jumping agent i be red. Suppose there are agents of x non-red types that are neighbors to s in $\mathbf{v}$. Then there must be agents of at least $x + 1$ non-red types that are neighbors to d in $\mathbf{v}$, as otherwise i is not motivated to jump. So, we have $\tau_d(\mathbf{v}) \geq x + 1$ and $\tau_s(\mathbf{v}') \leq x + 1$, since s could have red neighbors in $\mathbf{v}'$. Therefore, $\tau_d(\mathbf{v}) \geq \tau_s(\mathbf{v}')$.

To show (2), suppose $\tau_d(\mathbf{v}) = \tau_s(\mathbf{v}')$. Then, since $\tau_d(\mathbf{v}) \geq x + 1$ and $\tau_s(\mathbf{v}') \leq x + 1$, it must be the case that $\tau_d(\mathbf{v}) = \tau_s(\mathbf{v}') = x + 1$. This shows that $u_i(\mathbf{v}) = x$ and $u_i(\mathbf{v}') = x + 1$, that is, the utility of i increases exactly by 1, as claimed. Since s had x non-red types as neighbors in $\mathbf{v}$, $\tau_s(\mathbf{v}') = x + 1$ implies that s has at least one red neighbor in $\mathbf{v}'$. Similarly, $\tau_d(\mathbf{v}) = x + 1$ implies that d has no red neighbor in $\mathbf{v}$. This implies that s cannot be a neighbor of d . So, s must have some neighbor p other than d with a red agent. When agent i jumps, the

monochromatic edge between s and p is destroyed, and further, the jump does not create new monochromatic edges incident to d , since d had no red neighbors. This proves (2). $\square$

We are now ready to handle the final case, where the graph has two empty nodes and is 3-regular.

THEOREM 3.6. *The game is potential when the graph has two empty nodes and is 3-regular.*

PROOF. For any assignment $\mathbf{v}$, let $\text{TE}(\mathbf{v}) = \tau_{v_1}(\mathbf{v}) + \tau_{v_2}(\mathbf{v})$ be the total type-count of the two empty nodes v_1 and v_2 . Also, let $M(\mathbf{v})$ be the number of monochromatic edges in $\mathbf{v}$, and

$$B(\mathbf{v}) = \begin{cases} 1, & \text{if the empty nodes are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

We will show that the function

$$\Phi(\mathbf{v}) = 2(\text{TE}(\mathbf{v}) + M(\mathbf{v})) + B(\mathbf{v})$$

is a potential one.

Let $\mathbf{v}$ be an arbitrary non-equilibrium assignment, and suppose red agent i jumps from a node s to node d which leads to the next assignment $\mathbf{v}'$. For any function f of an assignment, let $\Delta f = f(\mathbf{v}') - f(\mathbf{v})$. We will show that

$$\Delta\Phi = \Phi(\mathbf{v}') - \Phi(\mathbf{v}) < 0.$$

Observe that d is an empty node in $\mathbf{v}$ and s is an empty node in $\mathbf{v}'$. Since there are two empty nodes, there is another node, say o , that is empty in both $\mathbf{v}$ and $\mathbf{v}'$. We consider the following 4 cases for the location of o in the neighborhood of s and d .

Case 1: o is neither a neighbor of s nor of d . In this case, there is no edge between the two empty nodes in either $\mathbf{v}$ or $\mathbf{v}'$, that is, $\Delta B = 0$. Also, o 's type-count remains unchanged, that is, $\tau_o(\mathbf{v}) = \tau_o(\mathbf{v}')$. So, $\Delta\text{TE} = \tau_s(\mathbf{v}') - \tau_d(\mathbf{v})$. Therefore,

$$\Delta\Phi = 2(\Delta\text{TE} + \Delta M) + \Delta B = 2(\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M)$$

From Lemma 3.5(1) we have $\tau_d(\mathbf{v}) \geq \tau_s(\mathbf{v}')$. Also, from Lemma 3.5(2), if $\tau_d(\mathbf{v}) = \tau_s(\mathbf{v}')$, then $\Delta M \leq -1$ which implies that $\Delta\Phi \leq 2(0 - 1) = -2$. Therefore we assume that $\tau_d(\mathbf{v}) > \tau_s(\mathbf{v}')$ and will show that $\Delta\Phi \leq -1$, by showing that $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M \leq -1$, for all possible jumps.

- $u_i(\mathbf{v}) = 0$ and $u_i(\mathbf{v}') \geq 1$: If $u_i(\mathbf{v}) = 0$, then all the agents surrounding s are of type red. This implies that there are at least 2 red agents adjacent to s (as d , which is empty before the jump, could also be a neighbor of s). So, at least 2 monochromatic red edges are destroyed because of the jump. After the jump, the utility of i increases by at least 1, which implies there is at least 1 non-red agent that is a neighbor of d . So, at most 2 new monochromatic red edges could be created after the jump. Therefore, $\Delta M \leq 0$ and $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M \leq -1$ is satisfied.
- $u_i(\mathbf{v}) = 1$ and $u_i(\mathbf{v}') \geq 2$: In this case, there is at least 1 non-red agent that is a neighbor of s and at most 1 red agent that is a neighbor of d . $M(\mathbf{v}')$ is always less than or equal to $M(\mathbf{v})$ except when there are no monochromatic red edges incident on s in $\mathbf{v}$ and exactly 1 monochromatic red edge incident on d in $\mathbf{v}'$. Therefore, $\Delta M = 1$ implies there are 2 agents of different non-red types and one red agent surrounding d , and s is not a neighbor of d . So $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M = 1 - 3 + 1 = -1$. When $\Delta M \leq 0$, $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M \leq -1 + 0 = -1$.
- $u_i(\mathbf{v}) = 2$ and $u_i(\mathbf{v}') = 3$: No new monochromatic red edges will be created since all the neighbors of d must be non-red to give agent i utility 3. So $\Delta M \leq 0$ and $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M \leq -1$.

Case 2: o is a neighbor of s but not d . Since s is an empty node in $\mathbf{v}'$, we have $B(\mathbf{v}') = 1$. As d is an empty node in $\mathbf{v}$, we have $B(\mathbf{v}) = 0$. Hence $\Delta B = 1$. Since o gains no new neighbor and loses a red neighbor in $\mathbf{v}'$ compared to $\mathbf{v}$, we have $\tau_o(\mathbf{v}') \leq \tau_o(\mathbf{v})$ and $\Delta\tau_o \leq 0$. So, $\Delta\text{TE} = (\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}')) + \Delta\tau_o \leq \tau_s(\mathbf{v}') - \tau_d(\mathbf{v})$. Therefore,

$$\Delta\Phi \leq 2(\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M) + 1$$

From Lemma 3.5(1) we have $\tau_d(\mathbf{v}) \geq \tau_s(\mathbf{v}')$. Also, from Lemma 3.5(2), if $\tau_d(\mathbf{v}) = \tau_s(\mathbf{v}')$, then $\Delta M \leq -1$ which implies that $\Delta\Phi \leq 2(0 - 1) + 1 = -1$. Therefore we assume that $\tau_d(\mathbf{v}) > \tau_s(\mathbf{v}')$, that is, $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) \leq -1$. We will show that $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M \leq -1$, for all possible jumps of the agent i .

- $u_i(\mathbf{v}) = 0$ and $u_i(\mathbf{v}') \geq 1$: There is at least 1 non-red agent that is adjacent to d . This implies that there can be at most 2 red agents that are neighbors of d . Also, $u_i(\mathbf{v}) = 0$ implies that all the agents surrounding s are red agents. When d is not a neighbor of s , there must be 2 red agents connected to s in $\mathbf{v}$. So, 2 monochromatic red edges are destroyed when i jumps, and at most 2 monochromatic red edges are created. Hence, $\Delta M \leq 0$. When d is a neighbor to s , exactly 1 red agent must be adjacent to s in $\mathbf{v}$. After the jump d is connected to the empty s and at least 1 non-red neighbor. So 1 monochromatic red edge is destroyed and at most 1 monochromatic red edge is created after the jump. Hence, $\Delta M \leq 0$. Therefore, $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M \leq -1$.
- $u_i(\mathbf{v}) \geq 1$: Using the same analysis as in Case 1 we can show that $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M \leq -1$.

Case 3: o is a neighbor of d but not s . Since s is an empty node in $\mathbf{v}'$ and d is an empty node in $\mathbf{v}$, we have $\Delta B = -1$. Therefore, $\Delta\Phi = 2(\Delta\text{TE} + \Delta M) - 1$. To show that $\Delta\Phi \leq -1$, it suffices to show that $\Delta\text{TE} + \Delta M \leq 0$.

For i to be motivated to jump to d , since d is adjacent to the empty node o , it must be that d has at least 1 non-red neighbor and therefore at most one red neighbor in $\mathbf{v}$. If d has no red neighbors in $\mathbf{v}$, then clearly no new monochromatic red edges will be created when i jumps, so $\Delta M \leq 0$. Suppose instead that d has one non-red neighbor and one red neighbor in $\mathbf{v}$. If s is adjacent to d , then that red neighbor is in fact agent i , and when i jumps to d , then no monochromatic red edges would be created, and $\Delta M \leq 0$. If s is not adjacent to d , it must be that $u_i(\mathbf{v}) = 0$, since i gets utility 1 after jumping to d , and so s must have 3 monochromatic edges incident on it that will be destroyed when i jumps from s to d , and only one new monochromatic edge incident on d will be created. Therefore, $\Delta M \leq 0$.

After the jump, a new red agent becomes a neighbor to o , as o is connected to d . So, $0 \leq \Delta\tau_o \leq 1$. From Lemma 3.5(1) we know that $\tau_d(\mathbf{v}) \geq \tau_s(\mathbf{v}')$. If $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) \leq -1$, then $\Delta\text{TE} = (\tau_s(\mathbf{v}') - \tau_d(\mathbf{v})) + \Delta\tau_o \leq -1 + 1 = 0$. So, $\Delta\text{TE} + \Delta M \leq 0$. Also, from Lemma 3.5(2), if $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) = 0$, then $\Delta M \leq -1$, and

$$\Delta\text{TE} = (\tau_s(\mathbf{v}') - \tau_d(\mathbf{v})) + \Delta\tau_o \leq 0 + 1 = 1.$$

Therefore,

$$\Delta\text{TE} + \Delta M \leq 1 - 1 = 0.$$

Case 4: o is a neighbor of both s and d . Here, $\tau_o(\mathbf{v}') = \tau_o(\mathbf{v})$, so $\Delta\tau_o = 0$. There is an edge between the 2 empty nodes, d and o , in $\mathbf{v}$ and there is an edge between the 2 empty nodes, s and o , in $\mathbf{v}'$. So, $\Delta B = 0$. Therefore,

$$\Delta\Phi = 2(\Delta\text{TE} + \Delta M) + \Delta B = 2(\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M).$$

From Lemma 3.5(1) we have $\tau_d(\mathbf{v}) \geq \tau_s(\mathbf{v}')$. Also, from Lemma 3.5(2), if $\tau_d(\mathbf{v}) = \tau_s(\mathbf{v}')$, then $\Delta M \leq -1$ which implies that $\Delta\Phi = 2(\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) + \Delta M) \leq 2(0 + (-1)) = -2$. Therefore we assume that $\tau_d(\mathbf{v}) > \tau_s(\mathbf{v}')$ that is, $\tau_s(\mathbf{v}') - \tau_d(\mathbf{v}) \leq -1$ and will show that $\Delta\Phi \leq -1$, by showing that $\Delta M \leq 0$, for all possible jumps of agent i .

- $u_i(\mathbf{v}) = 0$ and $u_i(\mathbf{v}') \geq 1$: If $u_i(\mathbf{v}) = 0$, then either s has two red neighbors and is not adjacent to d , or s has one red neighbor and is adjacent to d , which is an empty node in $\mathbf{v}$. In both cases s has at least one red

neighbor. As d must have at least 1 non-red neighbor for agent i to be motivated to jump, d can have at most 1 red neighbor. When agent i jumps, at least 1 monochromatic red edge is destroyed and at most 1 monochromatic red edge is created. So $\Delta M \leq 0$.

- $u_i(\mathbf{v}) = 1$ and $u_i(\mathbf{v}') = 2$: In this case, d must have 2 non-red neighbors of two different types, and since d is adjacent to the empty node o , no new monochromatic red edges are created, when agent i jumps. This implies that $\Delta M \leq 0$. Notice that $u_i(\mathbf{v}') = 3$ is not possible because the empty node o is a neighbor of d .

The proof is now complete. $\square$

Next we show that that even when there is an arbitrary number of empty nodes, an equilibrium assignment is guaranteed to exist for some families of *connected* graphs, in particular, for trees, cylinder graphs, and tori. For these cases, we explicitly construct an equilibrium rather than showing that the game is potential. The assignment we give will have one of the following two properties:

- ($\mathbf{P}_0$) All empty nodes are adjacent only to red agents or other empty nodes, and $u_i(\mathbf{v}) \geq 1$ for every non-red agent i .
- ($\mathbf{P}_1$) $u_i(\mathbf{v}) \geq 1$ for every agent i , and $\tau_u(\mathbf{v}) \leq 1$ for any empty node u .

Clearly, an assignment $\mathbf{v}$ satisfying ($\mathbf{P}_1$) is an equilibrium assignment as no agent is motivated to jump. Suppose $\mathbf{v}$ satisfies ($\mathbf{P}_0$). Notice that there may be red agents with utility 0, but they cannot increase their utility by jumping to an empty node. Any non-red agent can only get utility 1 by jumping to an empty node, since it already has utility 1, does not have incentive to jump.

THEOREM 3.7. *There exists an equilibrium assignment when the graph is a tree.*

PROOF. We will use an approach that is similar to the one used by [Narayanan, Sabbagh, et al. \(2025\)](#) to derive an equilibrium assignment for tree graphs, but for a different utility function. First, we fix a root node r of degree 1, and then repeatedly remove leaf nodes until we have a tree with exactly $n + 1$ nodes. Call this tree $G' = (V', E')$. Now we use the algorithm of [Narayanan, Sabbagh, et al. \(2025\)](#) for a tree with one empty node, to find an assignment in G' , such that (a) the root node r is empty, (b) every non-red agent i has at least one neighbor of a different type, and hence $u_i(\mathbf{v}) \geq 1$, and (c) a red agent occupies the node v that is the unique neighbor to the empty root r .

For completeness, we describe the algorithm of [Narayanan, Sabbagh, et al. \(2025\)](#) here. Let v be the unique neighbor of the root r (recall that r has degree 1). Consider any ordering of the agents according to their type. The algorithm proceeds in two phases. In the first phase, starting with the deepest odd level of the tree, we place agents in the current level ℓ , moving from left to right. When the current level is completely filled, we skip a level and go to level $\ell - 2$ where we repeat placing agents from left to right, until the unique node v at level 1 is filled, say by a red again. In the second phase, we move downwards from level 2, filling the even levels, again from left to right.

At the end of this procedure, we can observe that since the only empty node is r and it is adjacent to a single red agent, $\tau_r(\mathbf{v}) = 1$ and property ($\mathbf{P}_0$) is satisfied by the assignment. Note that there may be some red agents with utility 0 in this assignment. See Figure 2(a) for an example.

Next we show how to modify the assignment for G' to an assignment for G that satisfies one of the properties ($\mathbf{P}_0$) and ($\mathbf{P}_1$). We start with the same placement of agents $\mathbf{v}$ as in G' . Agents in this assignment on G' may have acquired some new neighboring nodes, but since they are all empty nodes, the utility of agents remains the same in the graph G . However, while every empty node in G is adjacent to at most one agent, some of them may now be adjacent to non-red agents, violating property ($\mathbf{P}_0$). See Figure 2(b) for an example. There may be some red

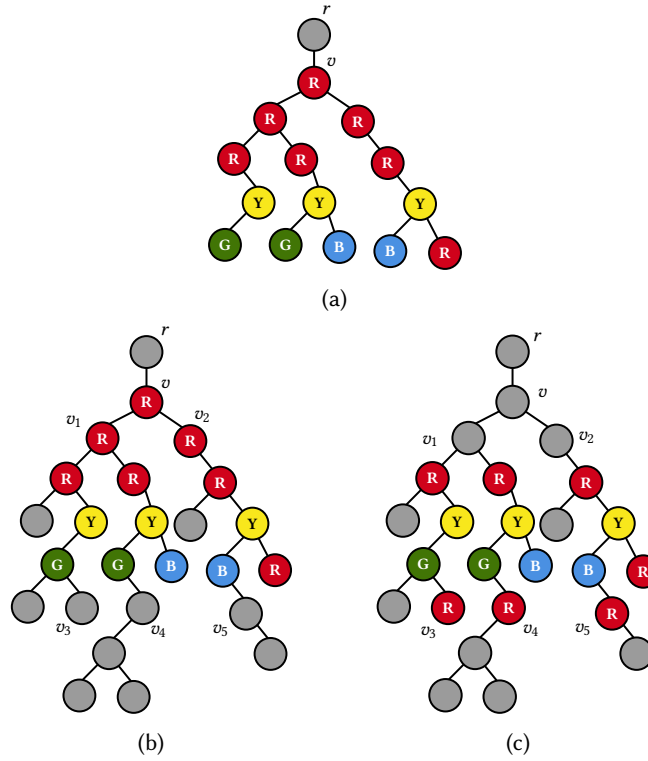


Fig. 2. (a) The assignment $\mathbf{v}$ for the tree G' with a single empty node. Note that all non-red agents have utility at least 1, but some red agents have utility 0. (b) The same assignment for the tree G . Note that the empty nodes v_3 , v_4 , and v_5 are adjacent to non-red agents. (c) The assignment $\mathbf{v}'$: red agents at v , v_1 , and v_2 with utility 0 in $\mathbf{v}$ jump to nodes v_3 , v_4 , and v_5 , respectively, to obtain utility 1 in $\mathbf{v}'$.

agents that have utility 0 that are motivated to jump to such an empty node. Let X be the set of red agents with utility 0 and V_e be the set of empty nodes that are adjacent to non-red agents.

- If $|X| \leq |V_e|$, then let all the agents in X jump to any $|X|$ nodes of V_e . Call the new assignment $\mathbf{v}'$. Consider a red agent i in the set X . Let p be a neighbor of i in $\mathbf{v}$ and let q be a neighbor in $\mathbf{v}'$. First note that since $u_i(\mathbf{v}) = 0$, agent p must be red, and since any node in V_e is adjacent to a single non-red agent, agent q must be non-red. Therefore, the jump does not decrease the utility of p and may increase the utility of q . See Figure 2(c) for an illustration. We also have $u_i(\mathbf{v}') = 1$ for all red agents. Finally, every empty node u is adjacent to at most one agent, so that $\tau_u(\mathbf{v}) \leq 1$. Therefore property (P_1) is satisfied.
- If $|X| > |V_e|$, let an arbitrary set of $|V_e|$ red agents from the set R jump to the empty nodes V_e . As in the previous case, the jumps do not decrease the utility of any agent. Further, all empty nodes are adjacent to red agents. Therefore, property (P_0) is satisfied.

Since $\mathbf{v}'$ satisfies either (P_0) or (P_1) , it is an equilibrium assignment. $\square$

We next consider cylinder graphs. A cylinder graph consists of two cycles of the same length such that each node of a cycle is also connected to a unique node of the other cycle. The graph in the game used to show that

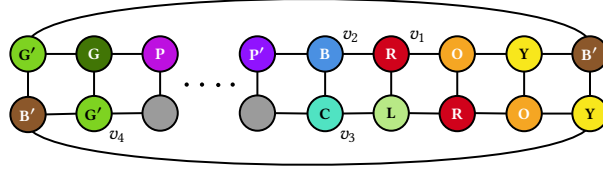


Fig. 3. An example of an equilibrium assignment on a cylinder graph, when $n_k = 2$.

there is an IRC in the proof of Theorem 3.1 is a cylinder (see Figure 1). Hence, constructing a potential function for such graphs is not possible, and an explicit equilibrium assignment is required.

THEOREM 3.8. *There exists an equilibrium assignment when the graph is a cylinder.*

PROOF. Let the dimensions of the cylinder be $2 \times m$, that is, there are $m \geq 3$ columns and 2 rows for a total of $2m$ nodes. Since we know from Theorems 3.3, 3.6, and 3.2 that the game is potential if there are at most 2 empty nodes or only 2 types of agents, we assume that there are at least three empty nodes, and the number of types k is at least 3. Recall that the n agents are partitioned into k types $T_1, T_2, \dots, T_k$ such that $|T_j| = n_j$ for $j \in [k]$. We assume that $n_1 \leq n_2 \leq \dots \leq n_k$. Since it's simple to construct an equilibrium assignment if $n \leq 3$, we assume below that $n \geq 4$. We show explicitly how to construct an equilibrium assignment; there are three main cases, based on the value of n_k the largest number of agents of a single type.

Case 1: $n_k = 1$. This implies $n_1 = n_2 = \dots = n_k = 1$, since $n_k \geq n_j$ for all $j < k$. If $n < m$, place all agents in the top row at consecutive locations. Notice that every agent has utility 2 except the agents at the ends of the row, which have utility 1. When $n < m - 1$, any empty node offers a utility of at most 1, so no agent wants to jump. When $n = m - 1$, the sole empty node in the top row offers utility 2, and other empty nodes have utility 1. However, only the agents adjacent to this empty node have utility 1, and they would not improve their utility by jumping to the empty node. Thus the assignment is an equilibrium. Otherwise, $n \geq m$ and we place m agents in the top row and the remaining $n - m$ agents at consecutive locations in the bottom row. Every agent has utility at least 2 and no empty node offers utility > 2 , so we have an equilibrium assignment.

Case 2: $n_k = 2$. Place the agents in a sequence such that the types of the agents in the sequence are as follows: $T_k, T_{k-1}, \dots, T_1, T_k, T_{k-1}, \dots, T_z$, where z is the smallest index with $|T_z| = 2$.

Case 2 (a): If $n \leq m$, then all agents fit in one row of the cylinder. Place the sequence of agents in the top row of the cylinder. Notice that every agent has a utility of 2 except the agents at both the ends of the sequence. All the empty nodes in the bottom row can offer utility of at most 1 to any agent. If there is only one empty node in the top row, it offers utility at most 2, but only utility 1 to the agents adjacent to it. Therefore no agent wants to jump. If there is more than one empty node in the top row, then all such nodes offer utility at most 1 to any agent, and therefore, no agent is motivated to jump. In both cases, we have an equilibrium.

Case 2 (b): Otherwise, we have $m < n$. Place the first m agents of the sequence in the top row of the cylinder in anticlockwise direction starting from any node, say v_1 . Let v_2 be the last node of the top row to which an agent is assigned and it is adjacent to v_1 . Let v_3 be the node in the bottom row that is adjacent to v_2 . Start assigning agents to the bottom row from v_3 and assign the remaining agents of the sequence in anticlockwise direction, with the last agent being assigned to node v_4 (see Figure 3). This leaves the consecutive nodes from v_4 going counterclockwise to v_3 empty. Since there are at least three empty nodes, every empty node is adjacent to at least

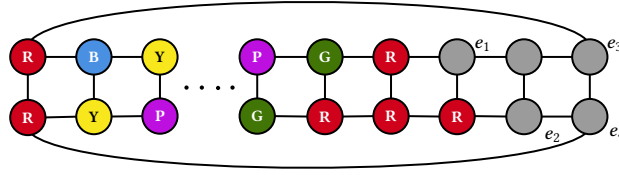


Fig. 4. An example of an equilibrium assignment on a cylinder graph, when $n_k \geq 3$, with 5 empty nodes.

one empty node, and offers utility at most 2. We will show that all agents have utility at least 2; in some cases we need to adjust the above assignment of agents.

- If $k > m + 1$, then every agent in the top row, as well as every agent in the bottom row except those at v_3 and v_4 has utility of at least 2 since it has agents of 2 different types adjacent to it in the same row. Also the agents at v_3 and v_4 have utility 2 as they have one neighbor of a different type in the bottom row, and one in the top row.
- If $k = m + 1$, then every agent starting from the one that is next to v_3 , in the bottom row, will have a neighbor of the same type in the top row. So, move every agent in the bottom row by one node in clockwise direction. Now notice that every agent in the bottom row will also have a utility of at least 2.
- If $k = m$, then v_2 will be occupied by an agent of type T_1 and v_3 will be occupied by an agent of type T_k . Observe that the agent in v_4 has a utility of just 1. Move every agent in the bottom row by one node in the clockwise direction. Now notice that every agent in the bottom row will also have a utility of at least 2.
- If $k < m$, every agent in the top row has utility at least 2 because of neighbors within the top row or neighbors in both the top and bottom rows, and every agent in the bottom row has utility at least 2.

Case 3: $n_k \geq 3$. First we select $\lfloor (|V| - n)/2 \rfloor$ consecutive columns of the graph; if n is odd, we select an additional node in the top row of an adjacent column: these nodes are denoted by V_e and will be empty in our assignment (see the gray nodes in Figure 4). Notice that at most four nodes in V_e are adjacent to nodes outside V_e . Let e_1, e_2, e_3 , and e_4 be the empty nodes of V_e that are adjacent to the nodes of $V - V_e$, where, e_1, e_2 are in the top and bottom rows on one side of V_e (not in the same column if n is odd), and e_3, e_4 are in the top and bottom rows on the other side. Note that when $|V_e| = 3$, $e_2 = e_4$.

Create a sequence of ordered pairs of agents as follows: Add to the sequence n_1 pairs from T_1 and T_2 , followed by $n_2 - n_1$ pairs from the remaining agents in T_2 with T_3 , and so on. Stop the process when all unpaired agents are from T_k .

If n is even, then the four empty nodes e_1, e_2, e_3 and e_4 will have exactly one non-empty node as a neighbor in any assignment, and therefore can only offer utility at most 1 to any agent. Since n is even, the number of unpaired agents is also even. Create new ordered pairs from the unpaired agents of type T_k and add these pairs to the end of the sequence of ordered pairs. Starting with the second next column from one end of V_e , place $n/2 - 1$ ordered pairs in the sequence until you reach the other end of V_e , and then place the last ordered pair in the last available column of $V - V_e$. Notice that this is guaranteed to be an equilibrium assignment, because either every agent has a utility at least 1, or every agent other than type T_k has a utility at least 1 and all empty nodes are adjacent only to the agents of type T_k .

Therefore, we assume that n is odd and we are left with an odd number of unpaired agents. When $|V_e| > 3$, e_1 is adjacent to 2 nodes of $V - V_e$, and e_2, e_3 and e_4 are adjacent to 1 node outside V_e . When $|V_e| = 3$, e_1 and $e_2 = e_4$

have 2 neighbors outside V_e , and e_3 has 1 neighbor outside V_e . Our assignments, depending on the cases below, will ensure that e_1 , e_2 and e_4 are adjacent only to agents of type T_k . Now, we describe how to place the unpaired agents of type T_k as well as the agents in the ordered pairs, depending on the cases below:

Case 3 (a): 1 unpaired agents of type T_k . Place the unpaired agent at the node below e_1 . Since $n_k \geq 3$, the last 2 ordered pairs must contain an agent of type T_k ; remove them from the sequence, and place each pair in the columns beside e_1 and e_3 such there e_1 and e_4 are adjacent only to agents of type T_k . Then, place all other ordered pairs of the sequence in the remaining columns of $V - V_e$. Notice that every agent has utility at least 1 and every node in V_e can offer utility at most 1.

Case 3 (b): 3 unpaired agents of type T_k . Place 2 unpaired agents of type T_k as the two neighbors of e_1 ; place the remaining unpaired agent beside e_4 . Remove the first ordered pair from the sequence (both agents of this ordered pair are not of type T_k , since $k \geq 3$), split the agents in the ordered pair, and place them in the columns of $V - V_e$ that are half filled. Place all other ordered pairs in the remaining columns. Similar to the previous case, every agent has utility at least 1 and every node in V_e can offer utility at most 1.

Case 3 (c): 5 or more unpaired agents of type T_k . Place 5 unpaired agents of type T_k such that one is in e_1 's column and the other four in the two columns adjacent to V_e . Notice that all empty nodes are adjacent only to agents of type T_k . Now, place the agents of each ordered pair in different columns, and finally any remaining unpaired agents of type T_k in any remaining positions in an arbitrary manner. Notice that every agent of type other than T_k has utility at least 1 and none of the agents wants to jump to nodes of V_e . There may be agents of type T_k that have utility 0, but they are not motivated to jump to empty nodes either.

In every case, we exhibited an equilibrium assignment, completing the proof. $\square$

Next we show that, under mild assumptions, games with tori graphs admit equilibrium assignments. A torus graph is a connected, grid-like graph consisting of rows and columns, such that each row and each column of nodes forms a cycle. See Figure 5 for an example. One can think of such graphs as a generalization of cylinders and grids, and can be used to represent, for example, cities with highways connecting its boundaries. We will show that a game in which the graph is a torus with at least 9 rows and largest type has at least 8 agents always admits an equilibrium.

THEOREM 3.9. *There exists an equilibrium assignment when the graph is an $m_1 \times m_2$ torus, where $m_1 \geq m_2 \geq 9$, and there are $n < m_1 m_2$ agents of $k \geq 3$ types, with the largest type containing $n_k \geq 8$ agents.*

PROOF. Consider an $m_1 \times m_2$ torus with $m_1 \geq m_2 \geq 9$. There are k types, $T_1, T_2, \dots, T_k$. Recall that $n_j = |T_j|$ for $1 \leq j \leq k$, and we assume that $n_1 \leq n_2 \leq \dots \leq n_k$. For simplicity, we also assume that $n_k \geq 8$. We start by creating a sequence L of agents of length n as follows: Starting with $j = 1$, alternate an agent from T_j and one from T_{j+1} until the agents in T_j are finished; next alternate agents from T_{j+2} and T_{j+1} , and so on (for example: $T_1, T_2, T_1, T_2, T_3, T_2, T_3, T_2, T_3, T_4, \dots$), without using any agents of type T_k . Once the sequence is created, some agents of type T_{k-1} might be remaining: we denote this set of agents as X . The sequence L has the property that every agent is adjacent to agents of a different type; thus if the sequence is placed on the torus in such a way that every agent is a neighbor on the torus to at least one of its adjacent agents in the sequence, then every agent has utility at least 1.

It remains to describe how to complete the sequence of agents with agents in X , as well as the agents of type T_k , and describe the assignment of agents to locations on the torus; this depends on the number of agents n as described below. The assignments we describe will satisfy either property (P_1) or (P_0) , which were also used in Theorem 3.7.

Case 1: $n \leq 5(m_2 - 2)$. Here, because the number of agents can fit in five rows of the torus, there are a lot of empty nodes. To complete the sequence L , we add agents in X to the sequence L , and then add agents of type T_k between consecutive agents, starting from the end of the sequence. Since $n_k \geq n_{k-1}$, the neighbors of every agent in the sequence are of a type different from it, and the last agent in L is of type T_k . Notice that some agents of type T_k may be remaining.

We now partition the sequence L as follows: Let $i = \lfloor |L|/(m_2 - 2) \rfloor$. We divide L into a subsequence $\hat{L}$ of length $i(m_2 - 2)$, and let L' be the remaining subsequence of length $|L| \bmod (m_2 - 2)$. We now describe the placement of agents in $\hat{L}$, L' , and any remaining agents of type T_k that were not in L .

Place the agents in $\hat{L}$ in the same $m_2 - 2$ columns in consecutive rows, say $[r \dots (r + i - 1)]$, in a snake-like order, and place the agents in L' in row $r + i + 2$ consecutively within the same columns as $\hat{L}$. Since there are at least 9 rows, if L' is empty, there are at least 4 empty rows between the last row of agents in $\hat{L}$ and the first row. If L' is not empty, since agents in $\hat{L}$ occupy at most 4 rows and there are at least 9 rows, there are at least two empty rows between agents in $\hat{L}$ and agents in L' .

- First we consider the case when $n_k \leq n_{k-1} + n_{k-2} + \dots + n_1 + 1$, that is, there are no extra agents of type T_k , and all agents are in the sequence L . If L' has at least two agents, every agent on the torus will have utility 1 in the assignment of agents in L . If L' has a single agent of type T_k , then it is isolated, and has utility 0; to fix this, we move it to place it adjacent to an agent of another type in row $r + i$; once again all agents have utility 1. Observe that property **(P₁)** is satisfied as none of the empty nodes offer utility 2.
- Otherwise $n_k > n_{k-1} + n_{k-2} + \dots + n_1 + 1$, that is, there are some remaining agents of type T_k that were not in the sequence L . If L' has a single agent, remove it from the torus as it is of type T_k and has utility 0 and consider it along with other remaining agents, so that there are no agents in row $r + i + 2$.

It is easy to see that in the assignment so far, no empty node is adjacent to more than one agent. We partition the empty nodes into two sets: E_0 and E_1 are the subsets of empty nodes that offer utility 0 and 1 respectively to agents of type T_k . We need to ensure that as many agents of type T_k as possible are placed in E_1 ; in other words, an agent of type T_k is placed in a node in E_0 only after all candidates in E_1 are already filled. At the same time, we want to ensure that none of the empty nodes will be adjacent to a newly placed agent of type T_k and an agent of a different type, as such an empty node would offer utility 2 to agents of other types, and provide incentive for such agents having utility 1 to jump and improve their utility to 2.

Consider the empty nodes in rows $[r \dots (r + i - 1)]$. They occur in the same columns, in pairs in the rows, between the first and last agents in the rows; we call these *horizontal* runs of length 2 of empty nodes. Similarly there may be *vertical* runs of length 2 between agents of row $r + i - 1$ and $r + i + 2$ that are in the same column or between agents of row r and $r + i + 2$ that are in the same column. Consider the following three cases regarding the two agents adjacent to the 2 empty nodes of runs of length 2:

- both are agents of type T_k , and the two empty nodes between them are both in E_0 .
- one of them is an agent of type T_k and the other is an agent of a different type: the empty node next to the agent of type T_k is in E_0 and the other is in E_1 . By placing an agent in the node in E_1 , the remaining empty node will be between two agents of type T_k and will only offer utility 1 to agents of other types.
- both are agents of types different from T_k ; in this case, both empty nodes in the run are in E_1 . However, if we place an agent of type T_k in one of these empty nodes, then it creates an empty node that is adjacent to 2 different types, so we should either leave both of these empty, or place agents of type T_k in both of them. Observe that this is the only way in which an empty node adjacent to agents of 2 different types, and therefore offering utility of 2 to agents of other types can be created.

We are now ready to describe the algorithm to assign the remaining agents of type T_k . We further partition the empty nodes in E_1 into two sets: E_{11} is the set of pairs of empty nodes corresponding to case (iii) above and E_{12} consists of all other nodes in E_1 . We start placing pairs of agents of type T_k in the pairs of empty nodes in E_{11} , until there are 0 or 1 agent remaining or we run out of nodes in E_{11} .

- If there are 0 agents of type T_k remaining, we are done, and property (P_1) holds.
- If there is just one agent of type T_k remaining, then place it in any empty node in E_{12} if such an empty node is available. If no such empty node is available, then remove the last agent of L' (as it is of type T_k) and place the 2 agents in a pair in E_{11} . Once again, property (P_1) holds.
- If there are agents of type T_k remaining and no more nodes in E_{11} available, then place the remaining agents in E_{12} . Either we run out of agents, in which case, (P_1) holds, or we run out of empty nodes in E_{12} , in which case we place the remaining agents into nodes in E_0 ; these agents will have utility 0, but since there are no empty nodes that offer them utility 1, (P_0) is satisfied.

Case 2: $5(m_2-2) < n \leq m_1m_2-2m_2$. We create a quasi-rectangular area A of empty nodes, with $\lfloor (m_1m_2-n)/m_2 \rfloor$ complete rows and a row that is not complete with $(m_1m_2-n) \bmod m_2$ consecutive nodes. No agent will be assigned to the nodes of A . We call the nodes in A that are adjacent to at least 2 nodes outside A as *corners* (see Figure 5). Notice that in A there can be at most 2 *corners* and they will be adjacent to at most 4 agents.

Case 2 (a): If $n_k \geq 2m_2$, then first use $2m_2$ agents of type T_k to enclose A . Next we describe how to complete the sequence L . Append to the sequence L an alternating sequence of agents of type T_k and T_{k-1} until one of the types is exhausted. Denote the remaining agents by the set Y ; these agents are either all of type T_{k-1} or all of type T_k .

- If Y has agents of type T_{k-1} , then we will first assign these agents. Notice that in this case $|Y| \leq 2m_2$, since $n_{k-1} \leq n_k$. We will assign all these agents *consecutively* adjacent to the agents of type T_k already assigned to the torus, and bordering the empty nodes. Thus all these agents in Y will have a utility of at least 1. Then we assign the agents of L to the available empty nodes of the torus in a snake-like pattern, starting with a row that has the least number of available empty nodes. Notice that property (P_1) is satisfied.
- If instead Y has agents of type T_k , we first assign the agents of L to the available empty nodes in a snake-like pattern, and then place all the agents of type T_k that are not in L in the remaining unassigned nodes. Here, property (P_0) is satisfied.

Case 2 (b): Otherwise, $n_k < 2m_2$, and we cannot ensure that empty nodes are adjacent to a single type T_k . So, we first place 4 agents of type T_k adjacent to the two corners (recall that $n_k \geq 8$). Then we put other agents of type T_k consecutively on the border. Next we place agents in the set X (of type T_{k-1}) adjacent to the agents of type T_k that were just placed on the torus, ensuring that they all obtain utility 1. Finally, we place the sequence L on the torus in snake-like order. See Figure 5 for an example assignment. Notice that property (P_1) is satisfied in this case.

Case 3: $n > m_1m_2 - 2m_2$. In this case, when the number of empty nodes are relatively lower, create a quasi-rectangular area A that is located within 2 columns and the length of A in the two columns differs by at most one.

If agents of type T_k can enclose the area A , then use them to enclose A . After that, follow the approach used in Case 2 (a) to complete the sequence L , and assign the remaining agents.

Otherwise, we first place agents of type T_k next to the 4 corners; this is possible, since at most 8 agents are required, and $n_k \geq 8$. Next, follow the approach used in Case 2 (b) to assign the remaining agents.

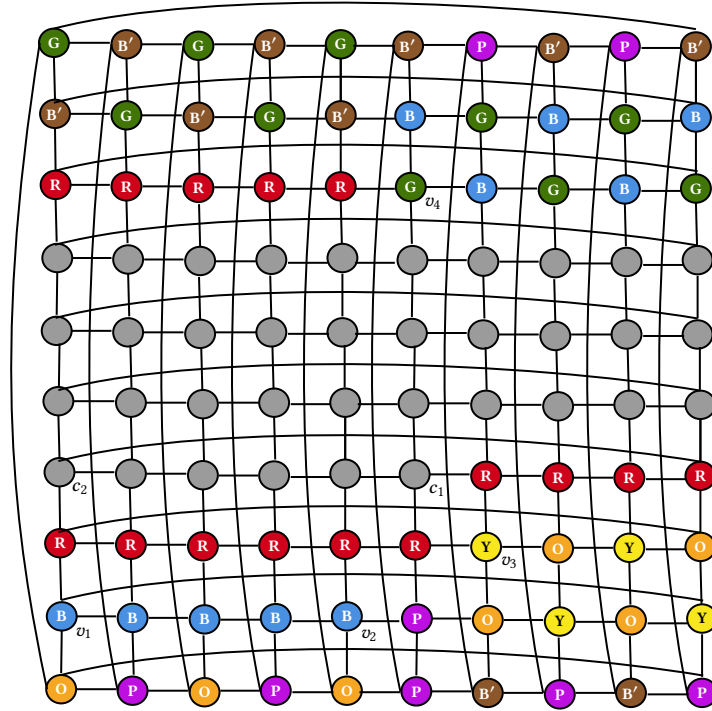


Fig. 5. An example of an equilibrium assignment for Case 2 (b). First, the corners c_1 and c_2 have red neighbors, and the boundary is partially covered with the remaining red agents. Some blue agents are remaining after creating L , so they are placed adjacent to the red agents on the boundary, from nodes v_1 to v_2 . Then, the agents of L are placed in a snake-like pattern starting at v_3 and ending at v_4 .

In every case, we showed that there is an equilibrium assignment, completing the proof. $\square$

4 Quality of Equilibria

In this section, we focus on the diversity quality of equilibria as measured by the social welfare (the total utility of all agents) and the number of colorful edges between agents of different types. We first argue that the price of anarchy is unbounded if the graph is disconnected, for any of the two objectives.

THEOREM 4.1. *If the graph is disconnected, the price of anarchy is unbounded with respect to the social welfare and the number of colorful edges, even when there are just two types with two agents each.*

PROOF. Consider a game with two types of agents, say red and blue, each consisting of two agents. The graph consists of three disconnected components: one node v , one pair of connected nodes, and another pair of connected nodes. The assignment according to which agents of the same type occupy connected nodes is an equilibrium such that all agents have utility 0 and there are 0 colorful edges. On the hand, there is an assignment (which is also an equilibrium) such that agents of different types occupy connected nodes, such that all agents have utility 1 and there are two colorful edges. Hence, the price of anarchy is unbounded with respect to both objectives. $\square$

Given this impossibility, we only consider connected graphs for the remainder of this section. We show tight bounds on the price of anarchy with respect to both of these objective functions, and a lower bound on the price of stability. For general graphs, the price of anarchy depends linearly on the number of agents n or the number of types k . On the other hand, for structured graphs, such as lines, cycles and regular graphs, the price of anarchy is much smaller. These results indicate that an arbitrary equilibrium does not really provide any guarantee about the diversity of the society without a highly structured underlying topology. Our strictly larger than 1 lower bound on the price of stability shows that full diversity cannot be achieved even in the best case.

4.1 Price of Anarchy: Social Welfare

We start by showing an upper bound on the price of anarchy which holds for any class of connected graphs and is a function of the number of agents n and the maximum cardinality over all types.

THEOREM 4.2. *The price of anarchy with respect to the social welfare is exactly $\frac{n(k-1)}{n - \max_T n_T + 1}$.*

PROOF. For the upper bound, consider an arbitrary equilibrium assignment $\mathbf{v}$ and let v be an empty node that is adjacent to an agent of some type R . All agents of type different than R must have utility at least 1 to not have incentive to jump to v . In addition, at least one agent of type R must be adjacent to an agent of a different type; otherwise, since the graph is connected, there would be an empty node (possibly v) that is adjacent to an agent of type different than R where agents of type R would have incentive to jump. Therefore,

$$\text{SW}(\mathbf{v}) \geq 1 + \sum_{T \neq R} n_T = n - n_R + 1 \geq n - \max_T n_T + 1.$$

Since the optimal social welfare is at most $n(k-1)$, the price of anarchy is at most $\frac{n(k-1)}{n - \max_T n_T + 1}$.

For the lower bound, consider a game with $k \geq 2$ types $\{T_1, \dots, T_k\}$. Let $R := T_1$ be the type with the most agents. The graph consists of the following components:

- A clique K_n of size n ;
- A line ℓ_T of size n_T for each type T .

The components are connected as follows: A node of K_n is connected to a node of ℓ_R . For each $t \in \{2, \dots, k-1\}$, all but the last node of ℓ_{T_t} are connected to the last node of $\ell_{T_{t-1}}$. All the nodes of ℓ_{T_k} are connected to the last node of $\ell_{T_{k-1}}$; See Figure 6 for an example for $k = 3$.

The optimal assignment is for the agents to occupy all the nodes of the K_n component, which leads to a utility of $k-1$ for each of them, and a social welfare of $n(k-1)$. A different equilibrium assignment is the following: For each type T , the agents of type T occupy the nodes of line ℓ_T . Since all agents of type $T \neq R$ have utility 1, none of them wants to jump to any of the empty nodes; in particular, there is only one node that is adjacent to an agent occupied by a single agent of type R . The social welfare of this equilibrium is $n - n_R + 1$ (note that the agent of type R that occupies the last node of ℓ_R also has utility 1), and thus the price of anarchy is at least $\frac{n(k-1)}{n - n_R + 1}$. $\square$

Using Theorem 4.2, we can derive many asymptotically tight bounds on the price of anarchy with respect to the social welfare, for example by considering games in which the types might be unbalanced or balanced.

COROLLARY 4.3. *The price of anarchy with respect to the social welfare is $\Theta(n)$ in general, and $\Theta(k)$ for balanced types.*

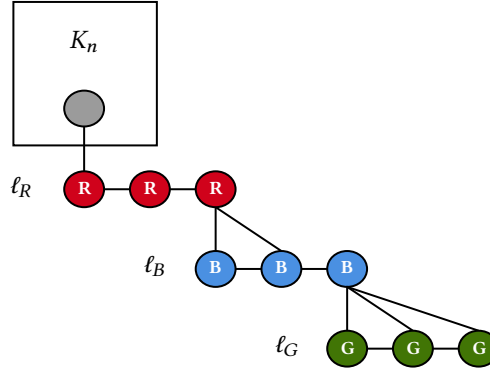


Fig. 6. The graph of the game considered in the proof of the lower bound in Theorem 4.2 for $k = 3$ types: red (R), blue (B), and green (G). The depicted assignment (according to which the agents occupy the nodes of the lines ℓ_R , ℓ_B and ℓ_G) is the equilibrium with minimum social welfare that leads to the desired price of anarchy lower bound.

PROOF. For unbalanced types, since each of the k types has cardinality at least 1, $\max_T n_T \leq n - k + 1$, and thus the price of anarchy is at most

$$\frac{n(k-1)}{n - \max_T n_T + 1} \leq n \cdot \frac{k-1}{k} \leq n.$$

By setting $n_R = n - k + 1$ and $n_T = 1$ for $T \neq R$ in the lower bound construction of Theorem 4.2, we get an asymptotically tight lower bound that is linear in the number of agents.

For balanced types, $n_T = n/k$ for each type T , and thus the price of anarchy is at most

$$\frac{n(k-1)}{n(1-1/k) + 1} = k \cdot \frac{n(k-1)}{n(k-1) + k} \leq k.$$

By setting $n_T = n/k$ for each type T in the lower bound construction of Theorem 4.2, we get an asymptotically tight lower bound that is linear in the number of types. $\square$

The previous statements show that the price of anarchy can be high in general connected graphs, even when the types are balanced. However, in simple graphs, such as lines and cycles, the price of anarchy is much smaller.

THEOREM 4.4. *When the types are balanced and the graph is of degree at most 2, the price of anarchy with respect to the social welfare is $4/3$ for $k = 2$ and $\frac{2k}{k-1}$ for $k \geq 3$.*

PROOF. We start with the case of $k = 2$ types, red and blue. For the upper bound, consider an arbitrary equilibrium assignment $\mathbf{v}$ and let v be an empty node that is adjacent to a red agent. Then, all blue agents must have utility 1 to not have incentive to jump to v . Since each node has degree at most 2, this further means that for every two blue agents, there is at least one red agent with utility 1 as well. Hence, the social welfare of $\mathbf{v}$ is at least $n/2 + n/4 = 3n/4$. Since the maximum possible social welfare is n , the price of anarchy is at most $4/3$.

For the lower bound, consider a cycle or a line with $n + 1$ nodes. Clearly, the assignment according to which agents of alternating types occupy adjacent nodes is the one with optimal social welfare of n . A different equilibrium assignment has the following pattern: There are $n/4$ triplets consisting of a red and two blue agents (in this order) that occupy consecutive nodes. The remaining red agents occupy the nodes after the ones occupied by all the agents in these triplets. See Figure 7 for an example of this assignment with a few agents. Hence, there

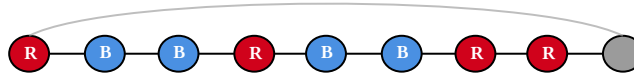


Fig. 7. An example of the assignment that gives the price of anarchy lower bound of $4/3$ with respect to the social welfare for $k = 2$ types (red - R, blue - B) on lines and cycles.

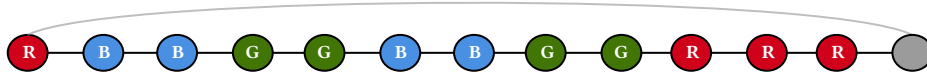


Fig. 8. An example of the assignment that gives the price of anarchy lower bound of $2k/(k - 1)$ with respect to the social welfare for $k \geq 3$ on lines and cycles; here the types are red (R), green (G), and blue (B).

are exactly $3n/4 + 1$ agents with utility 1, leading to a price of anarchy lower bound of approximately $4/3$ for a sufficiently large n .

We now consider the case $k \geq 3$. For the upper bound, if at equilibrium there is an empty node v that is adjacent to a red agent, then all non-red agents must have utility 1. Hence, the social welfare at equilibrium is at least $n - n/k = n(k - 1)/k$. Since the nodes of the graph have degree (at most) 2, the maximum social welfare is $2n$, leading to a price of anarchy of at most $2k/(k - 1)$.

For the lower bound, again consider a cycle or a line with $n + 1$ nodes. The assignment according to which agents of alternating types (in a round-robin manner) occupy adjacent nodes is the one with optimal social welfare of $2n - 2$. A different equilibrium assignment has the following pattern: Consider any ordering of the non-red types. The first node is occupied by a red agent. Then, two agents of the same type are followed by two agents of the type that comes next according to the ordering repeatedly. The remaining red agents occupy the nodes after all the non-red agents. See Figure 8 for an example of this assignment with a few agents. Hence, there are $n - n/k + 2 = n(k - 1)/k + 2$ agents with utility 1, leading to a price of anarchy lower bound that approaches $2k/(k - 1)$ for a sufficiently large n . $\square$

4.2 Price of Anarchy: Colorful Edges

We now turn our attention to the second objective, the number of colorful edges; recall that we consider only connected graphs. We first show a lower bound on the number of colorful edges achieved in any equilibrium; this lemma will be used in several proofs in the following for showing upper bounds on the price of anarchy.

LEMMA 4.5. *The number of colorful edges in any equilibrium is at least $(n - \max_T n_T)/2$.*

PROOF. Consider an arbitrary equilibrium assignment $\mathbf{v}$ and let v be an empty node that is adjacent to an agent of some type R . All agents of type different than R must have at least one neighbor of different type than their own to not have incentive to jump to v . Since a colorful edge connects two agents of different type, the number of colorful edges in $\mathbf{v}$ is at least $(n - n_R)/2 \geq (n - \max_T n_T)/2$. $\square$

Now we are ready to show that the price of anarchy with respect to the number of colorful edges is linear in the number of agents, even when the types have the same exact cardinality.

THEOREM 4.6. *The price of anarchy with respect to the number of colorful edges is $\Theta(n)$, even when the types are balanced.*

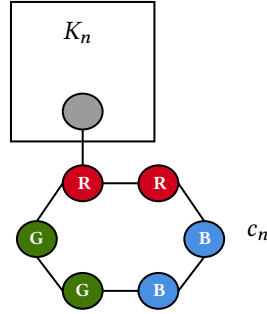


Fig. 9. The graph of the game considered in the proof of the lower bound in Theorem 4.6 for $k = 3$ types (red - R, green - G, and blue - B) consisting of 2 agents each.

PROOF. For the upper bound, observe that the maximum number of colorful edges would be achieved if all agents were to be connected to all agents of different type than their own. Hence, the optimal number of colorful edges is at most the number of edges between all agents minus the number of edges between all agents of the same type. Since $n = \sum_T n_T > \max_T n_T$, we have the following upper bound:

$$\begin{aligned} \frac{1}{2} \left(n(n-1) - \sum_T n_T(n_T-1) \right) &= \frac{1}{2} \left(n^2 - \sum_T n_T^2 \right) \\ &\leq \frac{1}{2} (n^2 - (\max_T n_T)^2) \\ &= \frac{1}{2} (n - \max_T n_T)(n + \max_T n_T). \end{aligned}$$

So, by Lemma 4.5, the price of anarchy is at most $n + \max_T n_T \leq 2n$.

For the lower bound, consider a game with $k \geq 2$ types $\{T_1, \dots, T_k\}$ consisting of n/k agents each, and assume that n/k is an even number. The graph consists of the following components:

- A clique K_n of size n ;
- A cycle c_n of size n .

The components are connected as follows: A node of K_n is connected to a node of c_n . See Figure 9 for an example for $k = 3$.

The optimal assignment is for the agents to occupy all the nodes of the K_n component, which leads to the maximum possible number of colorful edges equal to

$$\frac{1}{2} \left(n(n-1) - k \cdot \frac{n}{k} \left(\frac{n}{k} - 1 \right) \right) = \frac{n^2}{2} \cdot \frac{k-1}{k} \geq \frac{n^2}{4}.$$

Now consider the following equilibrium assignment according to which the agents occupy the nodes of c_n such that each agent has a neighbor of the same type and a neighbor of different type. As all agents have utility 1, no agent has incentive to jump to the empty node of K_n that is adjacent to a node of c_n . The number of colorful edges of this equilibrium is exactly equal to $n/2$, leading to a price of anarchy lower bound of $n/2$. $\square$

We next focus again on the case of balanced types and specific classes of connected graphs including lines, cycles, and also regular graphs.

THEOREM 4.7. *When the types are balanced and the graph is of degree at most 2, the price of anarchy with respect to the number of colorful edges is 2 when $k = 2$ and $\frac{2k}{k-1}$ when $k \geq 3$.*

PROOF. We start with the case $k = 2$. By Lemma 4.5, since $\max_T n_T = n/2$, the number of colorful edges in any equilibrium is at least $n/2$. Since the nodes of the graph have degree (at most) 2, each agent can at most participate in two colorful edges, and thus the maximum number of colorful edges is n . So, the price of anarchy is at most 2. For the lower bound, consider again the game of Theorem 4.4 for $k = 2$ (see also Figure 7). It is not hard to observe that the equilibrium has $n/2$ colorful edges, whereas the optimal number of colorful edges is n , and thus the price of anarchy is at least 2.

We now turn to the case $k \geq 3$. By Lemma 4.5, since $\max_T n_T = n/k$, there are at least $(n - n/k)/2 = n\frac{k-1}{2k}$ colorful edges at any equilibrium. Since the maximum number of colorful edges is n , the price of anarchy is then at most $\frac{2k}{k-1}$. For the lower bound, consider again the game of Theorem 4.4 for $k \geq 3$ (see also Figure 8). Each non-red agent participates in exactly one colorful edge, which implies that there are $n\frac{k-1}{2k}$ colorful edges at equilibrium, and the price of anarchy is at least $2k/(k-1)$. $\square$

THEOREM 4.8. *When the types are balanced and the graph is δ -regular, the price of anarchy with respect to the number of colorful edges is $\Theta(\delta)$.*

PROOF. For the upper bound, observe that the maximum number of colorful edges is $n\delta/2$ since each agent can have at most δ neighbors and can thus participate in at most δ colorful edges. Also, by Lemma 4.5, since $\max_T n_T = n/k$, there are at least $\frac{n}{2} \frac{k-1}{k} \geq n/4$ colorful edges in an equilibrium, thus leading to an upper bound of 2δ on the price of anarchy.

For the lower bound, consider a game with $k \geq 2$ balanced types $\{T_1, \dots, T_k\}$, and $\delta = n/k + 1$. The graph consists of the following components:

- A cycle c of size k consisting of nodes $\{v_1, \dots, v_k\}$;
- k cliques $(K_\ell)_{\ell \in [k]}$ of size $n/k + 1$ each.

The components are connected to each other as follows: For each $\ell \in [k]$, node v_ℓ is connected to all but two nodes of clique $K_{(\ell+1) \bmod k}$. Also, the nodes in the cliques that are not connected to nodes of the cycle c form another cycle (so that each of these nodes has one more neighbor in some other clique). Observe that this is indeed a δ -regular graph: Each node v_ℓ of the cycle c has degree $2 + n/k - 1 = \delta$, and each node of a clique K_ℓ has degree $1 + n/k = \delta$.

An equilibrium is the following: For each $\ell \in [k]$, node v_ℓ is occupied by an agent of type T_ℓ and the $n/k - 1$ nodes of clique $K_{(\ell+1) \bmod k}$ that are adjacent to v_ℓ are occupied by agents of type $T_{(\ell+1) \bmod k}$; the remaining two nodes of each clique are left empty. See also Figure 10 for an example with $k = 3$. This is an equilibrium assignment as all agents have utility at least 1 and each empty node is adjacent to only agents of one type. The only colorful edges in this assignment are those connecting the nodes of c to each other and to nodes in the cliques. So, there are $k + k(\delta - 2) = k(\delta - 1)$ colorful edges in this equilibrium. On the other hand, in the optimal assignment, we can assign to each clique, one agent of every type, so that each agent has exactly $k - 1$ neighbors, all of different type, leading to $k\frac{\delta(\delta-1)}{2}$ colorful edges. So, the price of anarchy is $\Omega(\delta)$. $\square$

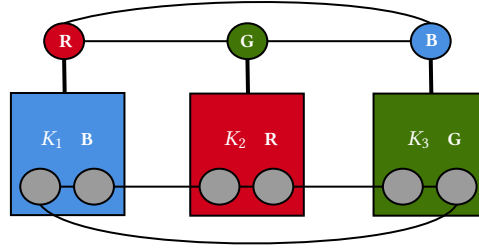


Fig. 10. The δ -regular graph considered in the proof of the lower bound in Theorem 4.8 for $k = 3$ types (red - R, green - G, and blue - B). The depicted assignment is the equilibrium minimizing the number of colorful edges.

4.3 Price of Stability

We conclude with a lower bound of approximately $3/2$ on the price of stability for both objectives. This implies that there are games in which the optimal assignment is not an equilibrium (in contrast to the games considered in the lower bounds on the price of anarchy).

THEOREM 4.9. *The price of stability with respect to the social welfare and the number of colorful edges is at least $3/2 - \varepsilon$, for any $\varepsilon > 0$.*

PROOF. We consider the following game with $k = 4$ types. Let $x \geq 1$ be an integer. There are x red agents, one blue agent, one green agent, and one yellow agent; so, in total, there are $n = x + 3$ agents. The graph is depicted in Figure 11, where each node $p_i, i \in [x]$, is connected to nodes q, s and t .

First consider the assignment according to which the x red agents occupy the nodes $p_1, \dots, p_x$, the blue agent is at node q , the green agent is at node s , the yellow agent is at node t , and thus node r is left empty. Observe that this assignment is not equilibrium since the blue agent has utility 1 and can achieve a utility of 2 by jumping to r . The optimal social welfare is at least the social welfare of this assignment, which is $3x + 5$. Similarly, the optimal number of colorful edges is at least the number of colorful edges of this assignment, which is $3x + 3$.

We now argue about the social welfare and the number of colorful edges that can be achieved in any equilibrium. The possible assignments can be classified into one of the following three cases:

- Case 1: All x red agents occupy nodes $p_1, p_2, \dots, p_x$. The only possible equilibrium is when the blue, green, and yellow agents are in the triangle formed by r, s and t . The social welfare of this equilibrium assignment is $2x + 8$, and the number of colorful edges is $2x + 3$.
- Case 2: There is a red agent at r and no red agents at q, s and t . Then, all remaining $x - 1$ red agents have to occupy $x - 1$ nodes among $p_1, \dots, p_x$. For an equilibrium, the three non-red agents must be at r, s and the remaining node among $p_1, \dots, p_x$. Hence, the social welfare of such an equilibrium is $2x + 8$, and the number of colorful edges is $2x + 3$.
- Case 3: There is a red agent in at least one of q, s and t . Then, there must be at least $x - 4$ red agents at $p_1, \dots, p_x$. Since one of their three neighbors is occupied by a red agent, the utility of these agent is at most 2, and thus the social welfare of any such equilibrium is at most $2(x - 4) + 7 \cdot 3 = 2x + 13$. In addition, the number of colorful edges is at most $2(x - 4) + 4 \cdot 3 + 4 = 2x + 8$.

Overall, the maximum possible social welfare of any equilibrium is at most $2x + 13$ and the maximum possible number of colorful edges is $2x + 8$. Consequently, the price of stability with respect to the social welfare is at

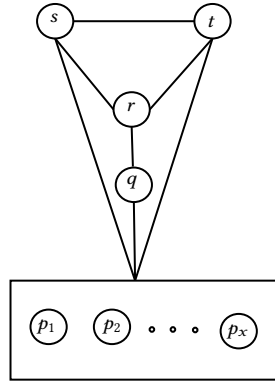


Fig. 11. The graph of the game used to show a lower bound on the price of stability for both the social welfare and the number of colorful edges. Nodes s , t and q are connected to each of the nodes $p_1, \dots, p_x$.

least $\frac{3x+5}{2x+13}$ and the price of stability with respect to the number of colorful edges is at least $\frac{3x+8}{2x+8}$. Both quantities approach $3/2$ as x approaches infinity. $\square$

5 Conclusion and Open Problems

In this paper, we considered a variety-seeking jump game in which agents occupy the nodes of a graph and aim to maximize the number of different types in their neighborhood. We showed the existence of equilibrium assignments for various restrictions of the game depending on the number of agents and their types, as well as the structure of the graph. We also showed tight bounds on the price of anarchy with respect to the social welfare and the number of colorful edges, both of which measure in different ways the achieved diversity of equilibrium assignments, as well as a lower bound on the price of stability for both objectives.

Our work leaves open several interesting, yet quite challenging questions. While we have shown that equilibrium assignments exist for several cases, it remains unknown whether such assignments always exist or if there are games that do not admit any equilibrium. We conjecture that there is always at least one equilibrium and the game is potential when there are only two empty nodes (without any further constraints on the structure of the underlying graph). Preliminary experiments using random graphs strongly support the latter claim; improving response cycles were found only when there were at least three empty nodes in the graph. A formal proof of this, however, remains elusive.

In terms of measuring the diversity of equilibria, while we showed tight bounds on the price of anarchy in terms of the social welfare and the number of colorful edges for connected graphs, we were not able to show tight bounds on the price of stability. We conjecture that the price of stability is constant (i.e., independent of the number of agents or the number of types). One could also consider many other objective functions to measure diversity, such as the degree of integration or variations of it (Agarwal et al. 2021).

References

- A. Agarwal, E. Elkind, J. Gan, A. Igarashi, W. Suksompong, and A. A. Voudouris. 2021. “Schelling games on graphs.” *Artificial Intelligence*, 301, 103576.
- H. Aziz, F. Brandl, F. Brandt, P. Harrenstein, M. Olsen, and D. Peters. 2019. “Fractional Hedonic Games.” *ACM Transactions on Economics and Computation*, 7, 2, 6:1–6:29.

- H. Aziz and R. Savani. 2016. "Hedonic Games." In: *Handbook of Computational Social Choice*. Ed. by F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia. Cambridge University Press, 356–376.
- G. Barmpalias, R. Elwes, and A. Lewis-Pye. 2014. "Digital morphogenesis via Schelling segregation." In: *Proceedings of the 55th IEEE Annual Symposium on Foundations of Computer Science (FOCS)*, 156–165.
- P. Bhakta, S. Miracle, and D. Randall. 2014. "Clustering and mixing times for segregation models on $\mathbb{Z}^2$." In: *Proceedings of the 25th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 327–340.
- D. Bilò, V. Bilò, P. Lenzner, and L. Molitor. 2022a. "Tolerance is Necessary for Stability: Single-Peaked Swap Schelling Games." In: *Proceedings of the 31st International Joint Conference on Artificial Intelligence (IJCAI)*, 81–87.
- D. Bilò, V. Bilò, P. Lenzner, and L. Molitor. 2022b. "Topological influence and locality in swap Schelling games." *Autonomous Agents and Multi-Agent Systems*, 36, 2, 1–60.
- T. Bläsius, T. Friedrich, M. S. Krejca, and L. Molitor. 2023. "The impact of geometry on monochrome regions in the flip Schelling process." *Computational Geometry*, 108, 101902.
- N. Boehmer and E. Elkind. 2020a. "Individual-Based Stability in Hedonic Diversity Games." In: *Proceedings of the 44th AAAI Conference on Artificial Intelligence (AAAI)*, 1822–1829.
- N. Boehmer and E. Elkind. 2020b. "Stable Roommate Problem with Diversity Preferences." In: *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI)*, 96–102.
- C. Brandt, N. Immorlica, G. Kamath, and R. Kleinberg. 2012. "An analysis of one-dimensional Schelling segregation." In: *Proceedings of the 44th Symposium on Theory of Computing (STOC)*, 789–804.
- R. Brederick, E. Elkind, and A. Igarashi. 2019. "Hedonic Diversity Games." In: *Proceedings of the 18th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS)*, 565–573.
- M. Bullinger, P. Lenzner, and A. Melnichenko. 2025. "Network creation with homophilic agents." *Social Choice and Welfare*, 64, 4, 915–959.
- M. Bullinger, W. Suksompong, and A. A. Voudouris. 2021. "Welfare guarantees in Schelling segregation." *Journal of Artificial Intelligence Research*, 71, 143–174.
- H. Chan, M. T. Irfan, and C. V. Than. 2020. "Schelling models with localized social influence: a game-theoretic framework." In: *Proceedings of the 19th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 240–248.
- A. Chauhan, P. Lenzner, and L. Molitor. 2018. "Schelling segregation with strategic agents." In: *Proceedings of the 11th International Symposium on Algorithmic Game Theory (SAGT)*, 137–149.
- W. Clark and M. Fossett. 2008. "Understanding the social context of the Schelling segregation model." *Proceedings of the National Academy of Sciences*, 105, 11, 4109–4114.
- A. Darmann. 2023. "Stability and Welfare in (Dichotomous) Hedonic Diversity Games." *Theory of Computing Systems*, 67, 6, 1133–1155.
- A. Deligkas, E. Eiben, and T. Goldsmith. 2024. "The parameterized complexity of welfare guarantees in Schelling segregation." *Theoretical Computer Science*, 1017, 114783.
- H. Ehzell, T. Friedrich, P. Lenzner, L. Molitor, M. Pappik, F. Schöne, F. Sommer, and D. Stangl. 2019. "Convergence and hardness of strategic Schelling segregation." In: *Proceedings of the 15th International Conference on Web and Internet Economics (WINE)*, 156–170.
- T. Friedrich, P. Lenzner, L. Molitor, and L. Seifert. 2023. "Single-Peaked Jump Schelling Games." In: *Proceedings of the 16th International Symposium on Algorithmic Game Theory (SAGT)*, 111–126.
- R. Ganian, T. Hamm, D. Knop, S. Schierreich, and O. Suchý. 2023. "Hedonic diversity games: A complexity picture with more than two colors." *Artificial Intelligence*, 325, 104017.
- S. Ge and T. Itoh. 2025. "Popularity on the roommate diversity problem." *Theoretical Computer Science*, 1023, 114903.
- P. Kanellopoulos, M. Kyropoulou, and A. A. Voudouris. 2021. "Modified Schelling games." *Theoretical Computer Science*, 880, 1–19.
- P. Kanellopoulos, M. Kyropoulou, and A. A. Voudouris. 2023. "Not all strangers are the same: The impact of tolerance in Schelling games." *Theoretical Computer Science*, 971, 114065.
- D. Knop and S. Schierreich. 2023. "Host Community Respecting Refugee Housing." In: *Proceedings of the 2023 International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 966–975.
- L. Kreisel, N. Boehmer, V. Froese, and R. Niedermeier. 2024. "Equilibria in Schelling games: computational hardness and robustness." *Autonomous Agents and Multi Agent Systems*, 38, 1, 9.
- S. Krogmann, P. Lenzner, and A. Skopalik. 2025. "The Bakers and Millers Game with Restricted Locations." In: *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 1209–1217.
- Y. Li, L. Narayanan, J. Opatrný, and Y. T. Xu. 2025. "Diversity-seeking Swap Games in Networks." In: *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, 2621–2623.
- R. Lorenzo and M. Reeves. 2018. "How and where diversity drives financial performance." *Harvard Business Review*, 30, Jan, 1–5.
- L. Narayanan, J. Opatrný, S. Tummala, and A. A. Voudouris. 2025. "Variety-Seeking Jump Games on Graphs." In: *Proceedings of the 34th International Joint Conference on Artificial Intelligence (IJCAI)*, 4005–4013.
- L. Narayanan, Y. Sabbagh, and A. A. Voudouris. 2025. "Diversity-seeking jump games in networks." *Autonomous Agents and Multi Agent Systems*, 39, 2, 32.

- C. Ntantamis and J. Zhou. 2025. "Firm size and workplace diversity and inclusion." *Economics Letters*, 248, 112234.
- R. Pans and N. J. Vriend. 2007. "Schelling's spatial proximity model of segregation revisited." *Journal of Public Economics*, 91(1–2), 1–24.
- D. Rock and H. Grant. 2016. "Why diverse teams are smarter." *Harvard Business Review*, 4, 4, 2–5.
- T. C. Schelling. 1971. "Dynamic models of segregation." *Journal of Mathematical Sociology*, 1, 2, 143–186.
- T. C. Schelling. 1969. "Models of segregation." *The American Economic Review*, 59, 2, 488–493.
- S. Schierreich. 2023. "Anonymous Refugee Housing with Upper-Bounds." CoRR, abs/2308.09501.
- T. W. Smith, M. Davern, J. Freese, and S. L. Morgan. 2019. "General social surveys, 1972-2018: Cumulative codebook." *Chicago: NORC*.
- D. Vinković and A. Kirman. 2006. "A physical analogue of the Schelling model." *Proceedings of the National Academy of Sciences*, 103, 51, 19261–19265.
- J. Zhang. 2004. "Residential segregation in an all-integrationist world." *Journal of Economic Behavior and Organization*, 54(4), 533–550.

Received 26 January 2026; accepted 04 June 2026