Gradual Semantics for Weighted Higher-Order Argumentation Frameworks

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This paper investigates complex argumentation settings, called *weighted higher-order argumentation frameworks* (wHO-AFs), where both arguments and attacks carry initial weights and may be subject to attacks from arguments. It focuses on developing gradual semantics capable of rationally evaluating these elements by assigning each argument and attack a numerical value, representing their respective degrees of *acceptance* and *seriousness*.

The contributions of the paper are five-fold: i) It identifies key technical challenges for semantics in wHO-AFs, such as handling the distinction between arguments and attacks, and determining how their values should be combined. To address these challenges, the paper introduces a set of guiding principles that capture various semantic strategies. ii) It introduces a broad family of semantics that treat arguments and attacks uniformly, evaluating them in the same way. This family encompasses both semantics inspired by extension-based approaches and those that are purely gradual. iii) It defines a second extensive family of semantics that differentiate between arguments and attacks. This includes instances where a single function evaluates both types of elements, as well as hybrid variants that apply distinct evaluation functions to arguments and attacks. iv) It offers a formal analysis and comprehensive comparison of the resulting diverse semantics. v) It presents an empirical evaluation of these semantics. The results are promising as they show that the values of all elements can be computed in fewer than 20 iterations, even for large frameworks, while maintaining very low runtime. The paper ultimately delivers a catalogue of semantics, each offering distinct formal properties and behaviors, contributing a flexible toolkit for reasoning in rich argumentative contexts.

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1 Introduction

Argumentation is a prominent reasoning paradigm that has gained increasing attention in artificial intelligence over the past two decades (Simari, Giacomin, et al. 2021). It has been applied to a wide range of tasks, including decision making (e.g., (Amgoud and Prade 2009; Bonet and Geffner 1996; Zhong et al. 2019)), reasoning with inconsistent information (e.g., (Besnard and Hunter 2001; Elvang-Gøransson et al. 1993; Simari and Loui 1992)), handling defeasible information (e.g., (García and Simari 2004; Pollock 1994)), and more recently, providing explanations for machine learning models (e.g., (Amgoud 2023; Prentzas et al. 2023; Rago, Baroni, et al. 2022)).

At its core, the argumentation approach involves justifying claims through interacting arguments and evaluating argument *strength* and/or *acceptance status* using a formal mechanism, referred to as a *semantics*. The strength of

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an argument reflects the degree of support it offers for its claim, while its acceptance status determines whether the argument—and consequently its claim—should be accepted or believed (Pollock 2010).

Three main families of semantics can be distinguished in the literature, based on the type of output they provide: *extension semantics*, introduced in (Dung 1995); *gradual semantics*, proposed in (Cayrol and Lagasquie-Schiex 2005); and *ranking semantics* (Amgoud and Ben-Naim 2013). The first family produces sets of jointly acceptable arguments, known as *extensions*. These can be computed in a declarative manner, as proposed in (Dung 1995), or alternatively through a labeling approach (Caminada 2006), or even via equational methods (e.g., (D. M. Gabbay 2012)). The second family focuses on evaluating individual arguments by assigning to each a numerical or qualitative value, reflecting its strength. The third ranks arguments from strongest to weakest. A detailed comparison of the three families can be found in (Amgoud 2019). In (Grossi and Modgil 2019; Skiba et al. 2021), extension semantics that rank their outcomes are also defined. The choice of semantics depends on the nature of the problem and the type of arguments involved. For example, in reasoning with defeasible information, extension semantics have been shown to yield reasonable non-monotonic logics (Dung 1995). However, in decision-making contexts—where fine-grained comparisons between arguments are essential—gradual and ranking semantics are more appropriate. They allow for nuanced evaluation, a finer comparison of arguments, and a more gradual modeling of attack impacts (e.g., (Amgoud and Prade 2009; Baroni, Romano, et al. 2015)). As a result, the past decade has seen the development of numerous gradual semantics (e.g., (Amgoud, Doder, and Vesic 2022; Costa Pereira, Tettamanzi, et al. 2011; Leite and Martins 2011; Potyka 2019; Rago, Toni, et al. 2016)) and ranking semantics (e.g., (Amgoud and Ben-Naim 2013; Amgoud, Ben-Naim, Doder, et al. 2016; Bonzon et al. 2016b; Dondio 2018)). In parallel, a rich body of work has emerged on the formal properties—referred to as *principles*—that extension-based semantics (e.g., (Baroni and Giacomin 2007; Torre and Vesic 2017)) or gradual semantics (e.g., (Amgoud and Ben-Naim 2016a; Amgoud and Doder 2019; Amgoud, Doder, and Vesic 2022; Baroni, Rago, et al. 2019, 2018; Bonzon et al. 2016a, 2017; Potyka 2019)) are expected to satisfy. These properties have been instrumental in analyzing and comparing the behavior of nearly all existing semantics (e.g., (Amgoud and Ben-Naim 2016a; Amgoud, Doder, and Vesic 2022; Baroni, Rago, et al. 2018; Bonzon et al. 2016a)). It is worth noting that all of these semantics—gradual and ranking alike—have so far focused exclusively on basic frameworks, where arguments can be attacked but attacks themselves are not subject to attack.

In (Barringer et al. 2005), the authors advocate for allowing attacks on attacks—a concept known as *higher-order* or *recursive* attacks. This idea has since spurred extensive research into its formal foundations and practical applications (see (Cayrol, Cohen, et al. 2021) for a survey). Higher-order attacks have been shown to be pervasive in various domains, including natural language dialogues, reasoning about preferences (Modgil 2009b), coalition formation (Boella et al. 2008), and deductive mathematical proofs (Boudjani et al. 2018). For example, the tool CLEAR (Constructing and evaLuating dEductive mAthematical pRoofs) was developed to support students in mathematics and logic courses. It enables them to collaboratively construct deductive proofs through structured argumentative debate, while also providing instructors with a means to assess the quality of those proofs. Higher-order attacks have also been investigated in the context of explainable AI (XAI), notably in (Dauphin and Cramer 2017), where the authors introduced the *Extended Explanatory Argumentation Framework*. This framework builds upon the model proposed in (Seselja and Straßer 2013) by incorporating higher-order attacks, joint attacks, and a support relation. It is particularly well-suited to capturing the interplay between explanation and argumentation in scientific debates. The following example illustrates the concept of higher-order interactions in the context of classifier explainability.

Example 1. *Consider a model that generates explanations for the predictions made by a classifier κ. The model takes as input a dataset $\mathbf{D}$—a set of tuples, each consisting of feature-value pairs (f_i, v_i)—as well as a specific input $x \in \mathbf{D}$. It outputs* sufficient reasons *for the decision $\kappa(x) = c$, which explain why the class c is assigned to x by κ. A sufficient reason for $\kappa(x) = c$ is a set $\mathcal{E}$ of feature-value pairs such that $\mathcal{E} \subseteq x$ and for all $y \in \mathbf{D}$, if $\mathcal{E} \subseteq y$, then*

$\kappa(y) = c$. In other words, any input that contains all the elements of $\mathcal{E}$ is assigned class c by the classifier. It has been argued in (Amgoud and Ben-Naim 2022) that explanation models should satisfy certain desirable properties, one of which is the ***global coherence*** *of the set of all explanations they may generate. This property ensures the correctness of explanations, as it forbids two compatible sets of features-values from explaining distinct classes. Formally, two explanations $\mathcal{E}$ and $\mathcal{E}'$ for predictions $\kappa(x)$ and $\kappa(x')$, respectively, are said to be coherent if $\kappa(x) \neq \kappa(x')$ implies that $\mathcal{E} \cup \mathcal{E}'$ is logically inconsistent.*

Assume that $\mathbf{D}$ *contains information about employees of a company, and that the model generates reason $\mathcal{E}$ for Paul (input x) and reason $\mathcal{E}'$ for John (input y), for whom the classifier κ predicts "no hiking" and "hiking," respectively.*

- $\mathcal{E} = \{(V, 0)\}$, *which reads as* κ predicts *no hiking* because Paul is not on vacation.
- $\mathcal{E}' = \{(M, 0)\}$, *which reads as* κ predicts *hiking* because John has no meeting.

The explanations $\mathcal{E}$ and $\mathcal{E}'$ are clearly incoherent, since the classifier assigns different outcomes ($\kappa(x) \neq \kappa(y)$) while $\mathcal{E} \cup \mathcal{E}'$ are logically consistent. Indeed the condition of not being on vacation is not incompatible with the condition of not having a meeting; for instance both conditions could simultaneously hold for an employee, say Peter (input z), who is neither on vacation $(V, 0)$ nor scheduled for a meeting $(M, 0)$. If $\kappa(z) = hiking$, then $\mathcal{E}$ is incorrect (as z contains $\mathcal{E}$ while $\kappa(z) \neq \kappa(x)$) and if $\kappa(z) = no\ hiking$, then $\mathcal{E}'$ is incorrect (as z contains $\mathcal{E}'$ while $\kappa(z) \neq \kappa(y)$). This situation is represented in (Amgoud and Ben-Naim 2022) as an argumentation framework, where an argument is a pair $\langle \mathcal{E}, \kappa(x) \rangle$ with $\mathcal{E}$ being a sufficient reason for $\kappa(x)$. An argument attacks another argument if their reasons are incompatible. Thus, in our example, we have two arguments with a symmetric conflict:

- $A = \langle \mathcal{E}, no\ hiking \rangle$,
- $B = \langle \mathcal{E}', hiking \rangle$.

Now assume that a constraint X on the data is available, as discussed in (Cooper and Amgoud 2023), and that this constraint states that employers who are not on vacation have meetings ($X = (V, 0) \to (M, 1)$). This information gives birth to the following argument C, whose conclusion states that the reasons $\mathcal{E}$ and $\mathcal{E}'$ are incompatible in presence of X:

- $C = \langle \{(V, 0) \to (M, 1)\}, \mathcal{E} \cup \mathcal{E}' \cup X \vdash \bot \rangle$

The argument C does not attack the individual arguments A and B directly, but rather targets the conflicts between them (see Figure 1). As a result, both $\mathcal{E}$ and $\mathcal{E}'$ can be regarded as reasonable explanations.

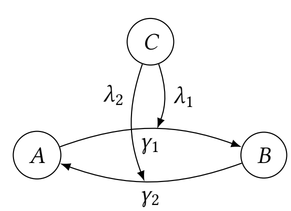


Fig. 1. Framework representing Example 1

Higher-order attacks naturally allow for the formalization of such conflicts, making argumentation a powerful and intuitive tool for addressing problems in explainable AI.

Although argumentation frameworks allowing attacks on attacks were first introduced in 2005 (Barringer et al. 2005), the development of semantics capable of handling such structures emerged much later. To date, all proposed semantics are extension-based; no gradual or ranking semantics have been defined in the literature for these frameworks. Existing approaches have been specifically designed for three unweighted settings, where neither arguments nor attacks carry initial weights: the *Extended Argumentation Framework* (EAF) (Modgil

2009b), the *Argumentation Framework with Recursive Attacks* (AFRA) (Baroni, Cerutti, et al. 2011), and the *Recursive Argumentation Framework* (RAF) (Cayrol, Fandinno, et al. 2020). While examples clearly demonstrate that the semantics defined in these frameworks may produce different outcomes, the precise sources of these divergences remain unexamined. More generally, there is still no formal analysis of the underlying assumptions that differentiate these semantics.

To address the gaps identified above, this paper focuses on *weighted higher-order argumentation frameworks* (wHO-AFs), in which both arguments and attacks are assigned weights and may themselves be subject to attack by arguments. The weight of an argument can reflect various notions, such as the degree of certainty with which its claim follows from its premises (Benferhat et al. 1993), or the reliability of its source (Costa Pereira, Tettamanzi, et al. 2011). Similarly, the weight of an attack may capture different interpretations, including its relevance (Amgoud 2020; Dunne et al. 2011) or the proportion of supporting votes it receives (Egilmez et al. 2013; Leite and Martins 2011). The objective of the paper is to investigate *gradual semantics* that can evaluate the elements of wHO-AFs—both arguments and attacks—in a rational and principled way. These semantics assign a numerical value to each element, representing the argument's *strength* and the attack's *seriousness*. The contributions of the paper are five-fold:

(1) The paper starts by highlighting the technical challenges that semantics face in the context of wHO-AFs—particularly, how to treat attacks in relation to arguments and how to combine their respective strengths in a principled way. To address these challenges, the paper introduces sixteen principles that delineate the range of possible strategies for defining semantics. Several of these principles extend existing properties from (Amgoud and Doder 2019) to the weighted higher-order setting, thereby illustrating how the strength of attacks can be integrated into the evaluation of arguments. Four additional principles are entirely novel and apply specifically to wHO-AFs.
(2) The paper proposes a broad family of gradual semantics that treat arguments and attacks as comparable entities, allowing for a consistent treatment of both types of elements within wHO-AFs. The family is sufficiently large to encompass semantics inspired by extension-based approaches from the literature—such as those defined in (Cayrol, Fandinno, et al. 2020)—as well as semantics that are purely gradual in nature.
(3) The paper also introduces a second broad family of gradual semantics that treat arguments and attacks differently. This family includes instances where a single function is used to evaluate both arguments and attacks, as well as *hybrid* approaches that combine two distinct functions—one for each element type. Moreover, the family encompasses semantics inspired by extension-based approaches, such as those from (Baroni, Cerutti, et al. 2011), as well as semantics that are purely gradual in nature.
(4) The paper conducts a formal analysis and comprehensive comparison of the wide range of novel semantics introduced. In doing so, it offers the first systematic analysis of the extension semantics defined for RAF and AFRA, highlighting their key differences—most notably, their divergent treatment of attacks.
(5) Finally, the paper reports an empirical evaluation of some pure gradual semantics on a benchmark of weighted higher-order argumentation frameworks. The results show that the proposed semantics converge rapidly in practice: the strength values of all elements are obtained in fewer than 20 iterations, even for large frameworks, while the overall runtime remains low.

This paper significantly extends the work presented in (Amgoud, Doder, and Lagasquie-Schiex 2024). It provides detailed proofs for all the results and generalizes the family of pure gradual semantics that treat arguments and attacks equally, demonstrating that this family encompasses the two semantics discussed in (Amgoud, Doder, and Lagasquie-Schiex 2024). Additionally, it broadens the scope of the hybrid semantics family and proves the conjecture regarding the well-definedness of these semantics. Finally, the paper offers an experimental evaluation of the new pure gradual semantics.

Table 1. Kinds of wHO-AFs $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$.

Framework	Definition
Flat	$\forall x \in \mathcal{A} \cup \mathcal{R}, \mathbf{w}(x) = 1$
Semi-Weighted	$\exists x \in \mathcal{A}$ s.t. $\mathbf{w}(x) \neq 1$ and $\forall x \in \mathcal{R}, \mathbf{w}(x) = 1$
Weighted	$\exists x \in \mathcal{A}$ s.t. $\mathbf{w}(x) \neq 1$ and $\exists x \in \mathcal{R}$ s.t. $\mathbf{w}(x) \neq 1$
Basic	$\forall \alpha \in \mathcal{R}, \mathbf{t}(\alpha) \in \mathcal{A}$
Two-order	$\forall \langle \alpha_1, \ldots, \alpha_n \rangle$ s.t. $\alpha_i \in \mathcal{R}$ and $\forall 1 \leq i < n, \mathbf{t}(\alpha_i) = \alpha_{i+1}$, it holds that $n \leq 2$
Higher-order	$\exists \langle \alpha_1, \ldots, \alpha_n \rangle$ s.t. $\alpha_i \in \mathcal{R}, \forall 1 \leq i < n, \mathbf{t}(\alpha_i) = \alpha_{i+1}$, and $n \geq 2$

The paper is structured as follows: Section 2 introduces weighted higher-order argumentation frameworks. Section 3 reviews existing semantics for these frameworks, as well as gradual semantics developed for frameworks in which attacks are restricted to targeting arguments. Section 4 defines a set of principles that gradual semantics would satisfy. Sections 5 and 6 introduce gradual semantics that treat arguments and attacks equally and differently, respectively. Preliminary experimental results are presented in Section 7, followed by a discussion of related work in Section 8. The final section concludes the paper. The proofs of all results are provided in an appendix.

2 Weighted Higher-order Argumentation Frameworks

In this paper we focus on weighted higher-order argumentation frameworks (wHO-AFs) where both arguments and attacks can be attacked by arguments, and have *initial weights*. The term *higher-order* refers to the possibility of chaining more than two attacks in a framework. Put differently, there may exist a sequence of $n > 2$ attacks $\langle \alpha_1, \ldots, \alpha_n \rangle$ such that for any $i < n$, α_i targets α_{i+1}. Throughout the paper, we assume a set $\mathsf{P_a}$ containing names of arguments (denoted by $a, b, c, \ldots$) and a set $\mathsf{P_r}$ (with $\mathsf{P_a} \cap \mathsf{P_r} = \emptyset$) containing names of attacks (denoted by $\pi, \mu, \delta, \ldots$). We refer to an arbitrary *element*, which may be either argument or attack, by variables $x, y, z, \ldots$.

Definition 1. *A* weighted higher-order argumentation framework *(wHO-AF)*[1] *is a tuple* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$ *such that:*

- $\mathcal{A} \subseteq \mathsf{P_a}$ *with* $\mathcal{A} \neq \emptyset$ *and* $\mathcal{A}$ *is finite (the set of argument names),*
- $\mathcal{R} \subseteq \mathsf{P_r}$ *(the set of attack names),*
- $\mathbf{s} : \mathcal{R} \rightarrow \mathcal{A}$ *(the source function assigning each attack name its source),*
- $\mathbf{t} : \mathcal{R} \rightarrow \mathcal{A} \cup \mathcal{R}$ *(the target function assigning each attack name its target),*
- $\mathbf{w} : \mathcal{A} \cup \mathcal{R} \rightarrow [0, 1]$ *(the weighting function assigning each element its initial weight).*

Moreover, only one attack can exist between two same elements: $\forall \alpha, \beta \in \mathcal{R}$, $(\mathbf{s}(\alpha), \mathbf{t}(\alpha)) \neq (\mathbf{s}(\beta), \mathbf{t}(\beta))$ *if* $\alpha \neq \beta$. *Table 1 distinguishes various kinds of wHO-AFs. Let* AF *denote the set of all possible wHO-AFs.*

The functions $\mathbf{s}$ and $\mathbf{t}$ return the source and target of an attack, respectively. By definition, the source must always be an argument; that is, an attack cannot serve as the source of another attack. Moreover, there is at most one attack between any two elements. The function $\mathbf{w}$ assigns an initial weight to each element, the greater the weight, the more important the element is considered to be. A basic wHO-AF corresponds to a classical argumentation framework in which only arguments are subject to attack. In a flat framework, all arguments and attacks are assigned an initial weight of 1, thereby modeling scenarios in which weight information is either unavailable or all elements are treated as equally important. In semi-weighted frameworks, only arguments are weighted, while attacks are not.

[1]In (Amgoud, Doder, and Lagasquie-Schiex 2024), a different terminology is adopted: the acronym HO-AF is used instead of wHO-AF. In the present work, we introduce the prefix "w" to explicitly distinguish between higher-order argumentation frameworks with weights and those without.

We now illustrate the definition with a running example that will be used throughout the paper.

Example 2. *Consider the following exchange of five arguments between two individuals discussing a washing machine:*

- c**:** *The washing machine is no longer working, so we need to buy a new one.*
- d**:** *The ecological transition policy prohibits purchasing new appliances and requires that all existing equipment be repaired.*
- a**:** *There is an interesting website, Do It Yourself, that offers inexpensive and easy-to-implement repair solutions.*
- b**:** *Following the instructions on Do It Yourself, the machine works perfectly. However, in the event of a breakdown, the repair solutions for this type of machine are actually costly and complex, as spare parts are hard to find.*
- g**:** *That website is not reliable.*

The following attacks are present in the above exchange:

- *Arguments c and d attack each other, as they support contradictory conclusions regarding whether a new washing machine should be purchased.*
- *Argument b attacks c, as it contradicts the claim that the machine is no longer working.*
- *Argument b also attacks a, challenging the claim that the proposed repair solutions are inexpensive and easy to implement.*
- *Argument g does not directly attack any individual argument, but rather challenges the relevance of b's attack on c, by undermining the reliability of the source (*Do It Yourself*) on which b is based.*
- *Arguments a and d both attack the attack from g. Indeed, they support the idea that even a non-reliable website can be useful—a by emphasizing the availability of repair solutions, and d by insisting that repairs must be performed regardless of the source.*

The above information is represented by the weighted higher-order argumentation framework $\mathbf{H}_1 = \langle \mathcal{A}_1, \mathcal{R}_1, \mathbf{s}_1, \mathbf{t}_1, \mathbf{w}_1 \rangle$*, which is depicted in Figure 2.*

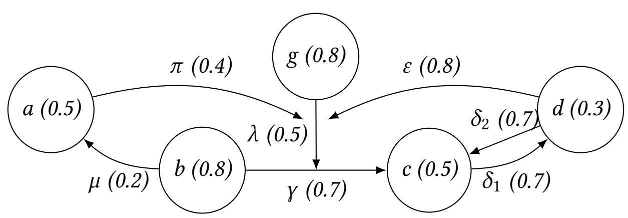


Fig. 2. Representation of $\mathbf{H}_1$

For instance, $\mathbf{s}_1(\pi) = a$*,* $\mathbf{t}_1(\pi) = \lambda$*,* $\mathbf{w}_1(\pi) = 0.4$ *and* $\mathbf{w}_1(a) = \mathbf{w}_1(\lambda) = 0.5$*. The origin of these weights is left unspecified in this example. Note that* $\mathbf{H}_1$ *is not two-order since there exists at least one sequence* $\langle \epsilon, \lambda, \gamma \rangle$ *containing more than 2 consecutive attacks.*

Example 3. *Consider the wHO-AF* $\mathbf{H}_2 = \langle \mathcal{A}_2, \mathcal{R}_2, \mathbf{s}_2, \mathbf{t}_2, \mathbf{w}_2 \rangle$ *depicted in Figure 3. It is basic (see the formal definition in Table 1), meaning that none of its attacks are themselves attacked.*

Although a wHO-AF is represented as a graphical object, it does not qualify as a graph in the strict sense of Graph Theory (Trudeau 1994). Therefore, some basic notions—specifically, *path* and *cycle*—must be explicitly defined. A *path* in a wHO-AF is a sequence of arguments and attacks such that each attack (respectively, argument) is immediately followed by its target (respectively, the attack originating from it). A *cycle* is a path whose first and last elements coincide. We recall below the definitions of these notions as given in (Doutre et al. 2022; Lafages 2021).

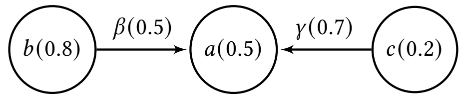


Fig. 3. Representation of $\mathbf{H}_2$

Definition 2. *Let* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathtt{AF}$ *and* $x, y \in \mathcal{A} \cup \mathcal{R}$*. A* path *from* x *to* y *is a sequence of* n *elements* $(e_1, \ldots, e_n)$ *with* $n \in \mathbb{N}^*$ *such that:*[2]

- $e_i \in \mathcal{A} \cup \mathcal{R}$*, all the* $e_{i=1,\ldots,n}$ *are distinct except possibly* e_1 *and* e_n*,*
- $x = e_1$ *and* $y = e_n$*,*
- *if* $n > 1$*,* $\forall i = 1, \ldots, n-1$*,* $e_i \in \mathcal{A}$ *implies that* $e_{i+1} \in \mathcal{R}$ *and* $e_i = \mathbf{s}(e_{i+1})$*,*
- *if* $n > 1$*,* $\forall i = 1, \ldots, n-1$*,* $e_i \in \mathcal{R}$ *implies* $\mathbf{t}(e_i) = e_{i+1}$*.*

A cycle *is a path* $(e_1, \ldots, e_n)$ *with* $n \geq 2$ *and* $e_1 = e_n$*.*

Example 2 (Cont) *The wHO-AF* $\mathbf{H}_1$ *contains two cycles:* $(\lambda, \gamma, c, \delta_1, d, \varepsilon, \lambda)$ *and* $(c, \delta_1, d, \delta_2, c)$*.*

Let us now define the notion of *isomorphism* between weighted higher-order argumentation frameworks.

Definition 3. *Let* $\mathbf{H}, \mathbf{H}' \in \mathtt{AF}$ *such that* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$ *and* $\mathbf{H}' = \langle \mathcal{A}', \mathcal{R}', \mathbf{s}', \mathbf{t}', \mathbf{w}' \rangle$*. An* isomorphism *from* $\mathbf{H}$ *to* $\mathbf{H}'$ *is a bijective function* $F : \mathcal{A} \cup \mathcal{R} \rightarrow \mathcal{A}' \cup \mathcal{R}'$ *such that the following holds:*

- $F : \mathcal{A} \rightarrow \mathcal{A}'$*,* $F : \mathcal{R} \rightarrow \mathcal{R}'$*,*
- $\forall\, x \in \mathcal{A} \cup \mathcal{R}$*,* $\mathbf{w}(x) = \mathbf{w}'(F(x))$*,*
- $\forall\, x \in \mathcal{A}$*,* $\forall y \in \mathcal{A} \cup \mathcal{R}$*,* $\forall\, z \in \mathcal{R}$*,* $\mathbf{s}(z) = x$ *and* $\mathbf{t}(z) = y$ *iff* $\mathbf{s}'(F(z)) = F(x)$ *and* $\mathbf{t}'(F(z)) = F(y)$*.*

We introduce below some notations that will be used throughout the paper.
Notations: Let $\mathbf{H}, \mathbf{H}' \in \mathtt{AF}$ such that $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle, \mathbf{H}' = \langle \mathcal{A}', \mathcal{R}', \mathbf{s}', \mathbf{t}', \mathbf{w}' \rangle$, $\mathcal{A} \cap \mathcal{A}' = \emptyset$, $\mathcal{R} \cap \mathcal{R}' = \emptyset$, $x \in \mathcal{A} \cup \mathcal{R}$ and $y \in \mathcal{A}$.

- $\mathtt{Att}_{\mathbf{H}}(x) = \{\pi \in \mathcal{R} \mid \mathbf{t}(\pi) = x\}$ (i.e., the set of all attacks targeting x in $\mathbf{H}$).
- $\mathtt{Sr}_{\mathbf{H}}(x) = \{a \in \mathcal{A} \mid \exists \pi \in \mathcal{R}$ such that $\mathbf{s}(\pi) = a$ and $\mathbf{t}(\pi) = x\}$ (i.e., the set of arguments attacking x in $\mathbf{H}$).
- $\mathtt{SrF}_{\mathbf{H}}(x) = \{a \in \mathtt{Sr}_{\mathbf{H}}(x) \mid \mathbf{w}(a) > 0\}$ (i.e., the set of attackers of x having a positive initial weight).
- Assume that y attacks x. The function $\mathbf{n}(y, x)$ returns the name of this attack (i.e., $\mathbf{n}(y, x) \in \mathcal{R}$).
- $\mathbf{H} \oplus \mathbf{H}' = \langle \mathcal{A}'', \mathcal{R}'', \mathbf{s}'', \mathbf{t}'', \mathbf{w}'' \rangle$ such that $\mathbf{H} \oplus \mathbf{H}' \in \mathtt{AF}$ and:
  - $\mathcal{A}'' = \mathcal{A} \cup \mathcal{A}'$,
  - $\mathcal{R}'' = \mathcal{R} \cup \mathcal{R}'$,
  - $\forall x \in \mathcal{R}$ (resp. $\forall x \in \mathcal{R}'$), $\mathbf{s}''(x) = \mathbf{s}(x)$ (resp. $\mathbf{s}''(x) = \mathbf{s}'(x)$),
  - $\forall x \in \mathcal{R}$ (resp. $\forall x \in \mathcal{R}'$), $\mathbf{t}''(x) = \mathbf{t}(x)$ (resp. $\mathbf{t}''(x) = \mathbf{t}'(x)$),
  - $\forall x \in \mathcal{A} \cup \mathcal{R}$ (resp. $x \in \mathcal{A}' \cup \mathcal{R}'$), $\mathbf{w}''(x) = \mathbf{w}(x)$ (resp. $\mathbf{w}''(x) = \mathbf{w}'(x)$).

Example 2 (Cont) *Note that* $\mathtt{Att}_{\mathbf{H}_1}(a) = \{\mu\}$*,* $\mathtt{Att}_{\mathbf{H}_1}(\lambda) = \{\pi, \varepsilon\}$*,* $\mathtt{Sr}_{\mathbf{H}_1}(a) = \{b\}$*,* $\mathtt{Sr}_{\mathbf{H}_1}(\lambda) = \{a, d\}$*,* $\mathbf{n}(b, a) = \mu$*, and* $\mathbf{n}(a, \lambda) = \pi$*. In this example, for any element* x *(argument or attack), we have* $\mathtt{SrF}_{\mathbf{H}_1}(x) = \mathtt{Sr}_{\mathbf{H}_1}(x)$*.*

3 State of the Art

A wide range of semantics have been proposed for different argumentation frameworks. In this section, we highlight two important works on extension-based semantics for flat wHO-AFs (Baroni, Cerutti, et al. 2011; Cayrol, Fandinno, et al. 2020), alongside a notable contribution on gradual semantics tailored to basic, weighted

[2]By definition, a path may consist of a single element. Note that for any $e \in \mathcal{A} \cup \mathcal{R}$, both (e) and (e, e) qualify as paths; however, the former is a path that is not a cycle, while the latter constitutes a cycle.

frameworks (Amgoud and Doder 2019). We will explore their respective extensions and formal properties in subsequent sections.

3.1 Extension Semantics for Flat wHO-AFs

Several frameworks in the literature address the notion of attacks on attacks, including the Extended Argumentation Framework (EAF) (Modgil 2009b), the Argumentation Framework with Recursive Attacks (AFRA) (Baroni, Cerutti, et al. 2011), and the Recursive Argumentation Framework (RAF) (Cayrol, Fandinno, et al. 2020). All three frameworks are **flat**; EAF supports only second-order attacks, whereas AFRA and RAF allow for higher-order attacks. All three frameworks adopt extension-based semantics. However, there is a key difference in their outputs: both AFRA and RAF semantics yield sets of arguments and attacks, while EAF semantics yield sets of arguments only. For the sake of generality, the discussion in the remainder of this subsection will focus on RAF and AFRA. Their relationships to other frameworks will be explored in Section 8. Throughout this subsection, we assume a fixed but arbitrary flat wHO-AF $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \equiv 1 \rangle$.

3.1.1 RAF Semantics. RAF semantics are grounded in the notion of a *structure*, along with adapted versions of *acceptability* and *conflict-freeness* from Dung's seminal work (Dung 1995). A structure, which generalizes the concept of an extension, is defined as a set of arguments and attacks. The notions of acceptability and conflict-freeness are based on a *defeat* relation: a structure defeats an element (either an argument or an attack) if it contains an attack targeting that element, and the source of the attack also belongs to the structure. In Example 2, the structures $\{\pi\}$ and $\{a, d\}$ do not defeat the attack λ, since the former does not contain π's source a, and the latter does not include any attack targeting λ. A structure is said to be *conflict-free* if it does not contain any attack whose source and target are both elements of the structure. For instance, the structure $\{a, \pi, \lambda\}$ is not conflict-free, because $\mathbf{s}_1(\pi) = a$ and $\mathbf{t}_1(\pi) = \lambda$, both of which are included in the structure. Finally, an element is said to be *acceptable* with respect to a structure if the structure defeats every attack (or its source) that targets the element.

Definition 4. *Let $x \in \mathcal{A} \cup \mathcal{R}$, $S \subseteq \mathcal{A}$, $\Gamma \subseteq \mathcal{R}$.*

- *$U = S \cup \Gamma$ is called* structure.[3]
- *U* defeats *x iff $\exists \pi \in \Gamma$ such that $\mathbf{s}(\pi) \in S$ and $\mathbf{t}(\pi) = x$.*
- *x is* acceptable *wrt U iff $\forall \pi \in \mathcal{R}$, if $\mathbf{t}(\pi) = x$, then $\exists \mu \in \Gamma$ such that $\mathbf{s}(\mu) \in S$ and either $\mathbf{t}(\mu) = \pi$ or $\mathbf{t}(\mu) = \mathbf{s}(\pi)$. $\mathtt{Acc}(U)$ is the set of acceptable elements wrt U.*
- *U is* conflict-free *iff $\nexists \pi \in \Gamma$ such that $\mathbf{s}(\pi) \in U$ and $\mathbf{t}(\pi) \in U$.*

We recall below the structure-based semantics as defined in (Cayrol, Fandinno, et al. 2020).

Definition 5. *Let $U = S \cup \Gamma$ be a structure.*

- *U is* complete *iff U is conflict-free and $\mathtt{Acc}(U) = U$.*
- *U is* preferred *iff U is a $\subseteq$-maximal conflict-free structure such that $U \subseteq \mathtt{Acc}(U)$.*
- *U is* grounded *iff U is the $\subseteq$-minimal complete structure.*
- *U is* stable *iff U is conflict-free and defeats any element $x \in (\mathcal{A} \cup \mathcal{R}) \setminus U$.*

Let $\mathtt{Struc}_\bullet(\mathbf{H})$ be the set of structures of $\mathbf{H}$ under semantics $\bullet$ where $\bullet \in \{c, p, s, g\}$ and c, p, s, g stand respectively for complete, preferred, stable and grounded.

Example 2 (Cont) *The set of complete structures of $\mathbf{H}_1$ is $\mathtt{Struc}_c(\mathbf{H}_1) = \{U_1, U_2, U_3\}$ such that:*

- *$U_1 = \{b, g, \pi, \mu, \delta_1, \delta_2, \varepsilon\}$,*
- *$U_2 = \{b, c, g, \pi, \mu, \delta_1, \delta_2, \varepsilon, \lambda\}$,*

[3]Note that in (Cayrol, Fandinno, et al. 2020), U is a pair (S, Γ).

- $U_3 = \{b, d, g, \pi, \mu, \delta_1, \delta_2, \varepsilon, \gamma\}$.

Note that U_1 is the grounded structure while U_2, U_3 are preferred and stable structures.

It is worth noting that when attacks occur only between arguments, the semantic notions defined in RAF coincide with the classical ones introduced in the seminal work of Dung (Dung 1995).

3.1.2 AFRA Semantics. Like other extension-based approaches, the AFRA semantics introduced in (Baroni, Cerutti, et al. 2011) rely on the notions of conflict-freeness and acceptability. However, they are distinguished by a specific form of the defeat relation, which sets them apart from RAF semantics. First, in AFRA, defeats can only originate from attacks—arguments cannot be sources of defeat. Second, the framework introduces **new conflicts** between attacks: specifically, one attack may defeat another if it targets the source of the latter. In Example 2, the attack π is not directly attacked. However, since μ attacks the source of π, it is said that "μ e-defeats π". The following definition recalls these fundamental notions as introduced in (Baroni, Cerutti, et al. 2011).

Definition 6. *Let $x \in \mathcal{A} \cup \mathcal{R}$, $\pi \in \mathcal{R}$ and $S \subseteq \mathcal{A} \cup \mathcal{R}$.*

- π e-defeats[4] x *if* $\mathbf{t}(\pi) = x$, *or* $x \in \mathcal{R}$ *and* $\mathbf{t}(\pi) = \mathbf{s}(x)$.
- S *is* e-conflict-free *iff* $\nexists x, y \in S$ *such that* $x \in \mathcal{R}$ *and* x *e-defeats* y.
- x *is* e-acceptable *wrt* S *iff* $\forall \mu \in \mathcal{R}$ *such that* μ *e-defeats* x, $\exists \delta \in S \cap \mathcal{R}$ *such that* δ *e-defeats* μ. $\texttt{Acc}(S)$ *denotes the set of* e-acceptable *elements wrt* S.

AFRA semantics provide *extensions* which are defined as in Definition 5. It is sufficient to consider the above definitions of *e-defeat, e-conflict-freeness and e-acceptability.*

Definition 7. *Let $S \subseteq \mathcal{A} \cup \mathcal{R}$.*

- S *is* complete *iff* S *is e-conflict-free and* $\texttt{Acc}(S) = S$.
- S *is* preferred *iff* S *is* $\subseteq$*-maximal e-conflict-free such that* $S \subseteq \texttt{Acc}(S)$.
- S *is* grounded *iff* S *is the* $\subseteq$*-minimal complete extension.*
- S *is* stable *iff* S *is e-conflict-free and e-defeats any element* $x \in (\mathcal{A} \cup \mathcal{R}) \setminus S$.

Let $\texttt{Ext}_{\blacksquare}(\mathbf{H})$ *is the set of extensions of* $\mathbf{H}$ *under semantics* $\blacksquare$ *where* $\blacksquare \in \{c, p, s, g\}$.

Example 2 (Cont) *The set of complete extensions of* $\mathbf{H}_1$ *under AFRA is* $\texttt{Ext}_c(\mathbf{H}_1) = \{S_1, S_2, S_3\}$ *such that:*

- $S_1 = \{b, g, \mu\}$,
- $S_2 = \{b, c, g, \mu, \delta_1, \lambda\}$,
- $S_3 = \{b, d, g, \mu, \delta_2, \varepsilon, \gamma\}$.

The set S_1 is the grounded extension and the two others are preferred and stable extensions.

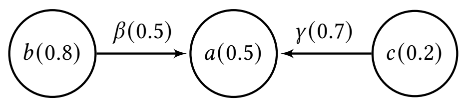
Note that AFRA extensions and RAF structures contain exactly the same arguments but the former contain less attacks. Indeed, AFRA semantics introduce additional conflicts, thereby modifying the topology of the argumentation graph. As a result, even attacks that are not explicitly targeted—such as π—may be rejected, since they can be e-defeated by newly introduced conflicts (i.e., attacks on their sources). The running example illustrates how AFRA semantics can yield outcomes that differ from those produced by RAF semantics. For instance, the attack π does not appear in any AFRA extension, whereas it is included in all RAF structures. In Subsections 5.1 and 6.1, we will clarify the formal properties that account for these differences.

[4]We use the prefix "e-" to distinguish the basic notions of AFRA from those of RAF.

3.2 Gradual Semantics for Basic wHO-AFs

Several gradual semantics have been proposed in the literature to handle different classes of basic argumentation frameworks. These include flat frameworks, which consist solely of arguments and attacks (Amgoud and Ben-Naim 2016a; Besnard and Hunter 2001; Pu et al. 2014); semi-weighted frameworks, where arguments are weighted and may be attacked (Amgoud, Ben-Naim, Doder, et al. 2017; Costa Pereira, Tettamanzi, et al. 2011; Leite and Martins 2011) or supported (Amgoud and Ben-Naim 2016b); weighted frameworks, in which both arguments and their relations carry initial weights (Amgoud and Doder 2019; Egilmez et al. 2013); and bipolar semi-weighted settings (Amgoud and Ben-Naim 2018; Mossakowski and Neuhaus 2016; Rago, Toni, et al. 2016). However, to date, no gradual semantics have been defined for wHO-AFs. In what follows, we recall the semantics introduced in (Amgoud and Doder 2019), as the weighted setting presented there is the closest to wHO-AFs.

The authors of (Amgoud and Doder 2019) proposed a family of gradual semantics designed to evaluate arguments within **basic frameworks** $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$, where for every $\pi \in \mathcal{R}$, the target $\mathbf{t}(\pi)$ belongs to $\mathcal{A}$—that is, only arguments can be attacked. To illustrate the core ideas underlying these semantics, we recall the graph $\mathbf{H}_2$ from Example 3 and focus on the evaluation of argument a:



To evaluate the strength of argument a, a gradual semantics proposed in (Amgoud and Doder 2019) proceeds as follows:

- It begins by computing the strength of each *attacker* of a, namely b and c. Let $\mathbf{v}(b)$ and $\mathbf{v}(c)$ denote the computed strength values, each belonging to the unit interval $[0, 1]$.
- Next, it evaluates the strength of each individual *attack*. This is done by aggregating the initial weight of the attack (i.e., the arrow) with the strength of its source using an *aggregation function* h. In the case of a, this yields two values: $v_1 = \mathsf{h}(0.5, \mathbf{v}(b))$ and $v_2 = \mathsf{h}(0.7, \mathbf{v}(c))$, both in $[0, 1]$.
- Then, it computes the strength of the entire *group of attacks* against a by aggregating the individual attack strengths using a second aggregation function g. The result is $v_3 = \mathsf{g}(v_1, v_2)$, which belongs to $[0, +\infty)$.
- Finally, an *influence function* f is used to combine the initial weight of a with the aggregated attack strength v_3. The result, $\mathsf{f}(\mathbf{w}(a), v_3)$, represents the strength degree assigned to a.

To summarize, the strength of a is given by:

$$\mathsf{f}\Big(\mathbf{w}(a), \mathsf{g}\big(\mathsf{h}(0.5, \mathbf{v}(b)), \mathsf{h}(0.7, \mathbf{v}(c))\big)\Big).$$

Hence, formally a gradual semantics uses a tuple of three functions $\langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, called *evaluation method*.

Definition 8. *An* evaluation method *(EM) is a triple* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$ *such that:*

- $\mathsf{h} : [0, 1] \times [0, 1] \longrightarrow [0, 1]$
- $\mathsf{g} : \bigcup_{n=0}^{+\infty} [0, 1]^n \longrightarrow [0, +\infty)$
- $\mathsf{f} : [0, 1] \times [0, +\infty) \longrightarrow [0, 1]$

In general, a gradual semantics is a function, determined by an evaluation method, that assigns each argument a value in $[0, 1]$, where larger values correspond to stronger arguments.

Definition 9. *A* gradual semantics *is a function* $\mathbf{S}$ *assigning to any basic framework* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$ *a weighting* $\mathbf{Deg}^{\mathbf{S}}_{\mathbf{H}}$ *on* $\mathcal{A}$*, i.e.,* $\mathbf{Deg}^{\mathbf{S}}_{\mathbf{H}} : \mathcal{A} \longrightarrow [0, 1]$ *such that* $\forall a \in \mathcal{A}$,

$$\mathbf{Deg}^{\mathbf{S}}_{\mathbf{H}}(a) = \mathsf{f}\Bigg(\mathbf{w}(a), \mathsf{g}\Bigg(\mathsf{h}\Bigg(\mathbf{w}(\alpha_1), \mathbf{Deg}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_1))\Bigg), \ldots, \mathsf{h}\Bigg(\mathbf{w}(\alpha_n), \mathbf{Deg}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_n))\Bigg)\Bigg)\Bigg), \tag{1}$$

where $\{\alpha_1, \ldots, \alpha_n\} = \mathtt{Att}_{\mathbf{H}}(a)$ *and* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$ *is an evaluation method. We say that* $\mathbf{S}$ is based *on* $\mathbf{M}$. $\mathbf{Deg}^{\mathbf{S}}_{\mathbf{H}}(a)$ *represents the* strength *(sometimes also called the* degree*) of* a *in the framework* $\mathbf{H}$ *under the semantics* $\mathbf{S}$.

Remark: Throughout the paper, an element (argument or attack) whose strength equals zero is referred to as *worthless*; otherwise, it is considered *alive*.

The scope of definable gradual semantics is constrained by specific requirements imposed on each of the three functions that constitute the evaluation method.

(1) a) $\mathsf{f}(x_1, y) > \mathsf{f}(x_2, y)$ whenever $x_1 > x_2$,
b) $\mathsf{f}(y, x_1) > \mathsf{f}(y, x_2)$ whenever $x_1 < x_2$ and $y \neq 0$,
c) $\mathsf{f}(x, 0) = x$,
d) $\mathsf{f}(0, x) = 0$,

(2) a) $\mathsf{g}() = 0$,
b) $\mathsf{g}(x) = x$,
c) $\mathsf{g}(x_1, \ldots, x_n) = \mathsf{g}(x_1, \ldots, x_n, 0)$,
d) $\mathsf{g}(x_1, \ldots, x_n, y) \leq \mathsf{g}(x_1, \ldots, x_n, z)$ if $y \leq z$,
e) g is commutative, i.e., $\mathsf{g}(x_1, \ldots, x_n) = \mathsf{g}(x_{\rho(1)}, \ldots, x_{\rho(n)})$, for any permutation ρ of the set $\{1, \ldots, n\}$;

(3) a) $\mathsf{h}(0, x) = 0$,
b) $\mathsf{h}(1, x) = x$,
c) $\mathsf{h}(x, y) > 0$ whenever $xy > 0$,
d) h is non-decreasing in both components.

The conditions set on the function f state the following: 1a) ensures that the greater the initial weight of an argument, the stronger the argument; 1b) states that the stronger the group of attacks on an argument, the weaker its strength; 1c) guarantees that an argument retains its entire initial weight if the group of attacks targeting it is worthless; 1d) ensures that an argument's strength is zero if its initial weight is zero.

Regarding the constraints on the aggregation function g: 2a) ensures that g returns zero for non-attacked arguments; 2b) states that in the case of a single attack, g returns the full strength of that attack's source; 2c) states that adding a worthless attack does not affect the strength of the group, and hence the target's strength; 2d) ensures that g is non-decreasing; 2e) (called *symmetry* in (Amgoud and Doder 2019)) ensures that the order of attacks does not affect the aggregation result.

The condition set on the function h state the following: 3a) ensures that an attack whose initial weight equals zero has no impact regardless of the strength of its source; 3b) states that the strength of an attack equals the strength of its source when the attack's initial weight is maximal; 3c) guarantees that every attack possesses a certain strength as soon as it, together with its source, is alive.

Table 2 provides examples of functions f and g that satisfy the above conditions. Table 3 presents various well-known aggregation operators (Calvo et al. 2002), some of which—such as h_1 and h_7—meet all four conditions. Note that h_2, for instance, violates condition 3a), and therefore cannot be used for defining gradual semantics within the family presented in (Amgoud and Doder 2019).

Let us define a subset $\mathbb{F}$ of gradual semantics encompassed by this family. The set $\mathbb{F}$ includes semantics that extend the two gradual semantics $\mathtt{Hbs}$ and $\mathtt{Mbs}$, originally introduced in (Amgoud, Ben-Naim, Doder, et al. 2017; Amgoud, Doder, and Vesic 2022) for semi-weighted frameworks (i.e., frameworks where all attacks have an initial weight of 1) and another $\mathtt{Ebs}$ which deals with weighted graphs. These semantics are based on the evaluation

Table 2. Examples of functions f and g.

f	g
$\mathsf{f}_{\mathsf{frac}}(x_1, x_2) = \frac{x_1}{1+x_2}$	$\mathsf{g}_{\mathsf{sum}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i$
$\mathsf{f}_{\mathsf{exp}}(x_1, x_2) = x_1 e^{-x_2}$	$\mathsf{g}_{\mathsf{max}}(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$

Table 3. Aggregation operators.

Operator name	Mathematical definition
min	$\mathsf{h}_1(x, y) = \min(x, y)$
max	$\mathsf{h}_2(x, y) = \max(x, y)$
Average (avg.)	$\mathsf{h}_3(x, y) = \frac{x+y}{2}$
Weighted avg. (w-avg)	$\mathsf{h}_4(x, y) = wx + (1-w)y, w \in [0, 1]$
Modified weighted avg. (mw-avg)	$\mathsf{h}_5(x, y) = \begin{cases} wx + (1-w)y, w \in [0, 1] & \text{if } xy \neq 0 \\ 0 & \text{else.} \end{cases}$
Modified avg. (m-avg)	$\mathsf{h}_6(x, y) = \begin{cases} \frac{x+y}{2} & \text{if } xy \neq 0 \\ 0 & \text{else.} \end{cases}$
Product	$\mathsf{h}_7(x, y) = xy$

methods $\langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h} \rangle$, $\langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle$, and $\langle \mathsf{f}_{\mathsf{exp}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle$, and are denoted by $\mathsf{Hbs}^{\mathsf{h}}$, $\mathsf{Mbs}^{\mathsf{h}}$, and $\mathsf{Ebs}^{\mathsf{h}}$ respectively. Thus, each of Hbs, Mbs and Ebs is extended by varying the aggregation function h. Formally, for any basic argumentation framework $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$ and any argument $a \in \mathcal{A}$,

$$\mathbf{Deg}_{\mathbf{H}}^{\mathsf{Hbs}^{\mathsf{h}}}(a) = \frac{\mathbf{w}(a)}{1 + \sum\limits_{\mathbf{t}(\alpha)=a} \mathsf{h}\Big(\mathbf{w}(\alpha), \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Hbs}^{\mathsf{h}}}(\mathbf{s}(\alpha))\Big)} \tag{2}$$

$$\mathbf{Deg}_{\mathbf{H}}^{\mathsf{Mbs}^{\mathsf{h}}}(a) = \frac{\mathbf{w}(a)}{1 + \max\limits_{\mathbf{t}(\alpha)=a} \mathsf{h}\Big(\mathbf{w}(\alpha), \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Mbs}^{\mathsf{h}}}(\mathbf{s}(\alpha))\Big)} \tag{3}$$

$$\mathbf{Deg}_{\mathbf{H}}^{\mathsf{Ebs}^{\mathsf{h}}}(a) = \mathbf{w}(a) \times e^{-\max\limits_{\mathbf{t}(\alpha)=a} \mathsf{h}\Big(\mathbf{w}(\alpha), \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Ebs}^{\mathsf{h}}}(\mathbf{s}(\alpha))\Big)} \tag{4}$$

The key difference between the three semantics is that $\mathsf{Hbs}^{\mathsf{h}}$ takes into account all attackers of an argument when evaluating its strength, whereas $\mathsf{Mbs}^{\mathsf{h}}$ and $\mathsf{Ebs}^{\mathsf{h}}$ consider only a single attacker—the strongest one. Naturally, both $\mathsf{Hbs}^{\mathsf{h}}$ and $\mathsf{Mbs}^{\mathsf{h}}$ yield the same values in graphs where each argument has at most one attacker.

Example 3 (Cont) *Consider again the framework $\mathbf{H}_2$ and the product function h_7. The three semantics $\mathsf{Hbs}^{\mathsf{h}_7}$, $\mathsf{Mbs}^{\mathsf{h}_7}$ and $\mathsf{Ebs}^{\mathsf{h}_7}$ assign the same values to b and c while $\mathsf{Hbs}^{\mathsf{h}_7}$ is more harmful than the two others regarding a (see the scores in Table 4).*

Every instance of the broad family of gradual semantics introduced in (Amgoud and Doder 2019) is well-defined in the sense that it assigns a unique value to each argument within a basic framework. However, as discussed in Section 5.2, some of these semantics cannot be generalized to weighted higher-order frameworks, since certain aggregation functions—such as h_7—may present challenges in that context.

Table 4. Argument strengths for Example 3

Argument	*a*	*b*	*c*
$\mathtt{Hbs}^{h_7}$	*0.32*	*0.8*	*0.2*
$\mathtt{Mbs}^{h_7}$	*0.35*	*0.8*	*0.2*
$\mathtt{Ebs}^{h_7}$	*0.33*	*0.8*	*0.2*

4 Gradual Semantics for wHO-AFs and Principles

This paper investigates gradual semantics that evaluate the strength of elements — both arguments and attacks — within weighted higher-order argumentation frameworks. The strength of an argument reflects the quality of its components, including its premises, claim, and the link between them, while the strength of an attack represents its degree of seriousness. A gradual semantics thus assigns to each element of a wHO-AF a single value drawn from an ordered scale, where higher values indicate stronger elements. Although different numerical or qualitative scales can be employed (see (Baroni, Rago, et al. 2019)), for simplicity, we restrict our attention to the unit interval $[0, 1]$.

Definition 10. *A* gradual semantics *is a function* S *assigning to any wHO-AF* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathtt{AF}$ *a weighting* $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}} : \mathcal{A} \cup \mathcal{R} \to [0, 1]$. *For* $x \in \mathcal{A} \cup \mathcal{R}$, $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(x)$ *represents the* strength *of* x *in* $\mathbf{H}$ *under* S.

There exists a substantial body of work on formal properties—often referred to as principles or axioms—that semantics are expected to satisfy (e.g., (Amgoud, Doder, and Vesic 2022; Baroni, Rago, et al. 2018; Bonzon et al. 2017)). These principles serve multiple purposes: they deepen our understanding of the foundational assumptions underlying semantics, provide a systematic framework for comparing different semantics, and guide the design of novel ones. Importantly, principles are **semantics-agnostic**—they are defined independently of any specific semantics and do not rely on its internal mechanisms. Instead, they conceptualize a semantics as a black-box function that, given an argumentation framework, assigns strength degrees to its arguments (and attacks). The principles then impose constraints on these degrees under particular configurations of the framework. Some principles are fundamental and should therefore be satisfied by any semantics, whereas others are context-dependent, being appropriate in certain settings but not in others.

In what follows, we introduce a set of sixteen principles tailored to gradual semantics for weighted higher-order argumentation frameworks. These principles are grouped into two categories: *general* and *specific*. General principles apply to both arguments and attacks, while specific principles target attacks only and therefore have no direct counterparts in the existing literature. Furthermore, they apply exclusively to wHO-AFs.

Throughout the paper, we say that a semantics S satisfies a principle P if the conditions of P holds under S.

4.1 General Principles

The first group of principles builds on properties introduced in (Amgoud and Doder 2019) for evaluating arguments in basic frameworks. These properties are extended here in two ways: first, to incorporate the strengths of attacks into the evaluation of arguments; and second, to adapt them for the evaluation of attacks themselves. Consequently, each principle consists of two sub-principles: one concerning arguments and the other concerning attacks. As we will show later, a semantics may satisfy one sub-principle while violating the other. For any principle, say P, we refer to its argument-related version as A-P, and its attack-related version as R-P.

The first principle, called *Anonymity*, states that the evaluation of arguments and attacks should not depend on their identities. This principle is considered fundamental and must be satisfied by any reasonable semantics.

Anonymity: For all $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$, $\mathbf{H}' = \langle \mathcal{A}', \mathcal{R}', \mathbf{s}', \mathbf{t}', \mathbf{w}' \rangle \in \mathsf{AF}$, for any isomorphism F from $\mathbf{H}$ to $\mathbf{H}'$, we have: $\forall\, x \in \mathcal{R}$ (respectively $\forall x \in \mathcal{A}$), $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(F(x))$.

Consider, for instance, the two graphs depicted in Figure 4. They are isomorphic, meaning there exists a bijective function F such that $F(a) = c$, $F(b) = d$, and $F(\alpha) = \beta$. The Anonymity principle ensures that each element receives the same strength value as its image under F (i.e., $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(c)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(d)$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\alpha) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(\beta)$).

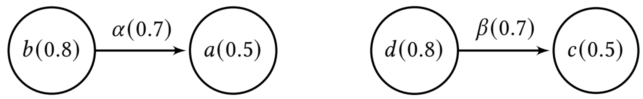


Fig. 4. Illustration of the anonymity principle ($a, b, \alpha \in \mathcal{A} \cup \mathcal{R}$ and $c, d, \beta \in \mathcal{A}' \cup \mathcal{R}'$)

The second principle, *Independence*, ensures that the evaluation of an element x must not depend on arguments and attacks that are not in the same connected components as x. The principle ensures that the strength of the argument a in Figure 4 is independent from those of c, d, and β. The rational behind is that the latter are unrelated to a and thus may not concern the premises or claim of a.

Independence: For all $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle, \mathbf{H}' = \langle \mathcal{A}', \mathcal{R}', \mathbf{s}', \mathbf{t}', \mathbf{w}' \rangle \in \mathsf{AF}$ such that $\mathcal{A} \cap \mathcal{A}' = \emptyset$ and $\mathcal{R} \cap \mathcal{R}' = \emptyset$, it holds that: $\forall\, x \in \mathcal{R}$ (respectively $\forall x \in \mathcal{A}$), $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H} \oplus \mathbf{H}'}(x)$.

The above principle does not prohibit the strength of a from being taken into account when evaluating the strength of b. Rather, it is the principle of *Directionality* that imposes such a restriction: it ensures that the strength of an element is independent of its outgoing attacks. The underlying idea is that the mere fact that an argument attacks other elements should have no influence on its own strength.

Directionality: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall a \in \mathcal{A}$, $\forall x \in \mathcal{A}$ (respectively $\forall x \in \mathcal{R}$), for any $\mathbf{H}' = \langle \mathcal{A}', \mathcal{R}', \mathbf{s}', \mathbf{t}', \mathbf{w}' \rangle \in \mathsf{AF}$ such that $\mathcal{A}' = \mathcal{A}$, $\mathcal{R}' = \mathcal{R} \cup \{\pi\}$ with $\pi \notin \mathcal{R}$, $\pi = \mathbf{n}(a, x)$, $\forall y \in \mathcal{A} \cup \mathcal{R}$, $\mathbf{w}'(y) = \mathbf{w}(y)$, it holds that: $\forall z \in \mathcal{A} \cup \mathcal{R}$, if there is no path from x to z, then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(z) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(z)$.[5]

The previous two principles (Independence and Directionality) identify factors that should not influence the evaluation of elements in a weighted higher-order argumentation framework. This naturally raises the question: *which factors should be taken into account*? The next principle, *Equivalence*, addresses this by asserting that the strength of an element should depend only on three factors: its *initial weight*, the *strength of the attacks targeting it*, and the *strength of the sources of those attacks*. In the graph depicted in Figure 4, the strength of a depends on its weight (0.5), the strength of b and the strength of α. While all existing semantics—whether extension-based or gradual—respect the previously discussed principles, almost all extension-based semantics violate Equivalence, even in the context of basic frameworks (Amgoud, Ben-Naim, Doder, et al. 2017; Amgoud and Doder 2019). Instead, these semantics often take into account the entire *topology* of the graph.

Equivalence: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x, y \in \mathcal{R}$ (respectively $\forall x, y \in \mathcal{A}$), if

- $\mathbf{w}(x) = \mathbf{w}(y)$,
- there exists a bijective function F from $\mathtt{Att}_{\mathbf{H}}(x)$ to $\mathtt{Att}_{\mathbf{H}}(y)$ such that $\forall \pi \in \mathtt{Att}_{\mathbf{H}}(x)$,
  - $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\pi) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(F(\pi))$ and
  - $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{s}(\pi)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{s}(F(\pi)))$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Maximality states that every element keeps its initial weight if it is not attacked. For instance, any semantics that satisfies this principle would assign the value 0.8 to b and d, and the value 0.7 to α and β in the graphs depicted in Figure 4.

Maximality: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x \in \mathcal{R}$ (resp. $\forall x \in \mathcal{A}$), if $\mathtt{Att}_{\mathbf{H}}(x) = \emptyset$, then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{w}(x)$.

[5]If $x = z$, then by definition there exists a path from x to z, and thus the property does not apply in this case.

The following principle clarifies the functional role of attacks—namely, to weaken their targets. The Weakening principle states that any element loses part of its initial weight if it is targeted by at least one alive attack originating from an alive argument. Consider again the graph depicted in Figure 4. A semantics that satisfies Maximality assigns the value 0.8 to the argument b and 0.7 to the attack α. The strength of a must be strictly less than 0.5 if the semantics also satisfies the Weakening principle.

Weakening: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x \in \mathcal{A}$ (resp. $\forall x \in \mathcal{R}$), if

- $\mathbf{w}(x) > 0$ and
- $\exists \pi \in \mathtt{Att}_{\mathbf{H}}(x)$ such that $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\pi) > 0$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{s}(\pi)) > 0$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) < \mathbf{w}(x)$.

The next principle restricts the scope of attacks that can influence their targets. Specifically, the *Neutrality* principle ensures that worthless attacks—as well as attacks originating from worthless arguments—have no impact on the strength of their targets. This property is particularly important in many application domains, including argumentative dialogues, where fallacious or irrelevant arguments are common. Neutrality ensures that such arguments do not undermine the strength of reasonable ones. Suppose that the initial weight of argument b is 0 in the graph of Figure 4. Under a semantics satisfying Maximality, this implies that b is worthless (i.e., has strength 0). Consequently, by the Neutrality principle, the attack α originating from b has no influence on the strength of a. It is also worth noting that α would have no effect on the strength of a if its own initial weight were 0, regardless of the strength of its source.

Neutrality: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x, y \in \mathcal{R}$ (resp. $\forall x, y \in \mathcal{A}$), if

- $\mathbf{w}(x) = \mathbf{w}(y)$,
- $\mathtt{Sr}_{\mathbf{H}}(y) = \mathtt{Sr}_{\mathbf{H}}(x) \cup \{a\}$ such that $a \in \mathcal{A} \setminus \mathtt{Sr}_{\mathbf{H}}(x)$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = 0$ or $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, y)) = 0$,
- $\forall z \in \mathtt{Sr}_{\mathbf{H}}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Let us now turn to the intensity of attacks—specifically, the extent to which an attack can diminish the strength of its target. *Can an attack cause an element to lose all of its initial weight*? The principle of Resilience states that this is not possible, as long as the element has a strictly positive initial weight. A semantics that satisfies Resilience guarantees that all elements—both arguments and attacks—in the graph depicted in Figure 4 receive a strictly positive strength value. In the context of basic frameworks, it has been shown in (Amgoud, Ben-Naim, Doder, et al. 2017; Amgoud and Doder 2019) that Resilience is violated by Dung's extension semantics—specifically, grounded, stable, preferred, and complete semantics (Dung 1995)—while it is satisfied by most existing gradual semantics. The applicability of Resilience depends on the nature of the argument under consideration. For instance, an epistemic argument based on beliefs may be entirely defeated by an attack that demonstrates one of its premises is false. In contrast, a practical argument whose claim expresses an option (e.g., a candidate for a job) is generally not fully defeated by a single attack. Such an attack typically highlights drawbacks, such as negative consequences or violated criteria, thereby lowering the perceived utility of the option without rendering the argument worthless.

Resilience: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x \in \mathcal{R}$ (respectively $\forall x \in \mathcal{A}$), if $\mathbf{w}(x) > 0$ then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$.

The next three properties address the *quality* and *quantity* of attacks, respectively. They are particularly relevant in decision-making contexts, where distinguishing between arguments is essential, as it ultimately leads to a differentiation between the alternatives those arguments support.

The *Reinforcement* principle captures the idea that the stronger the attackers, the more harmful their impact on their targets. Consider the wHO-AF shown in Figure 5. A semantics satisfying Maximality assigns the value 0.8 to b, 0.5 to d, and 0.7 to both α and β. As a result, a is attacked by a stronger source than c. Reinforcement guarantees that a receives a lower value than c.

Reinforcement: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x, y \in \mathcal{R}$ (resp. $\forall x, y \in \mathcal{A}$), if

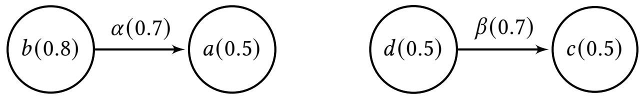


Fig. 5. Illustration of the reinforcement principle

- $\mathbf{w}(x) = \mathbf{w}(y)$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$,
- $\mathsf{Sr}_{\mathbf{H}}(x) \setminus \mathsf{Sr}_{\mathbf{H}}(y) = \{a\}$, $\mathsf{Sr}_{\mathbf{H}}(y) \setminus \mathsf{Sr}_{\mathbf{H}}(x) = \{b\}$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a)$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)) > 0$ and $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x) \cap \mathsf{Sr}_{\mathbf{H}}(y)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,y))$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

The next principle, *Attack-sensitivity*, is similar to the previous one but focuses on the attacks themselves rather than their sources. It states that the stronger the attacks on an element, the weaker that element becomes. In the graph depicted in Figure 6, the initial weight of β is lower than that of α, while their respective sources both have the same value, 0.8. Attack-sensitivity ensures that a receives a lower value than c.

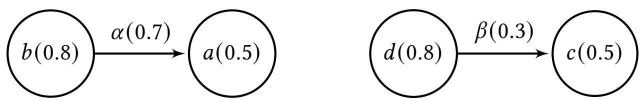


Fig. 6. Illustration of the attack-sensitivity principle

Attack-sensitivity: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x, y \in \mathcal{R}$ (resp. $\forall x, y \in \mathcal{A}$), if

- $\mathbf{w}(x) = \mathbf{w}(y)$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$,
- $\mathsf{Sr}_{\mathbf{H}}(x) \setminus \mathsf{Sr}_{\mathbf{H}}(y) = \{a\}$, $\mathsf{Sr}_{\mathbf{H}}(y) \setminus \mathsf{Sr}_{\mathbf{H}}(x) = \{b\}$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b) > 0$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x))$ and $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x) \cap \mathsf{Sr}_{\mathbf{H}}(y)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,y))$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

The *Counting* principle states that every alive attack further weakens its target. Put differently, every attack has an effect on the strength of the element it targets. In the wHO-AF depicted in Figure 7, argument c is attacked by two elements, one of which—namely β—is identical to the attack targeting a. Counting ensures that c receives a lower value than a, since the additional alive attack from e further reduces c's strength.

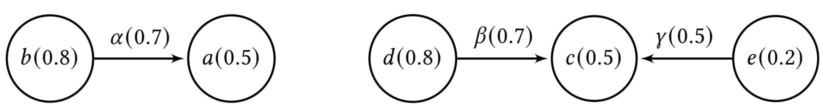


Fig. 7. Illustration of the counting principle

Counting: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x, y \in \mathcal{R}$ (resp. $\forall x, y \in \mathcal{A}$), if

- $\mathbf{w}(x) = \mathbf{w}(y)$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$,

- $\mathsf{Sr}_{\mathbf{H}}(y) = \mathsf{Sr}_{\mathbf{H}}(x) \cup \{a\}$ with $a \in \mathcal{A} \setminus \mathsf{Sr}_{\mathbf{H}}(x)$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) > 0$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, y)) > 0$,
- $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

The last general principle, *Proportionality*, concerns the influence of initial weights on the strength of elements. It states that the higher an element's initial weight, the greater its overall strength. In Figure 8, c has a higher initial weight than a, and both are attacked by equally strong attacks and attackers. Proportionality ensures that c receives a higher value, as it starts from a stronger initial position.

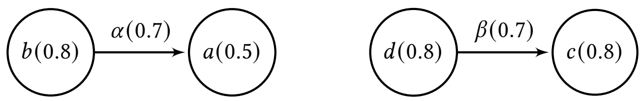


Fig. 8. Illustration of the proportionality principle

Proportionality: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x, y \in \mathcal{R}$ (resp. $\forall x, y \in \mathcal{A}$), if

- $\mathbf{w}(x) > \mathbf{w}(y)$,
- $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y)$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$,
- $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

4.2 Specific Principles

We now introduce a second set of principles, specific to weighted higher-order argumentation frameworks. These principles are novel, as they have no direct counterparts in the existing literature. They address distinctive challenges posed by wHO-AFs that require dedicated treatment, namely:

(i) whether the semantics penalizes attacks more strongly than arguments, or vice versa, or treats both as equivalent entities;
(ii) how a semantics determines the overall strength of an attack—particularly, how it combines the strength of the attack itself with that of its source.

The first three principles address the first challenge. Depending on the types of arguments involved (e.g., deductive, inductive, or abductive) and on the nature of the attacks, different semantics strategies may be appropriate. The Equality principle stipulates that arguments and attacks should be evaluated in the same way. This principle is particularly suitable in contexts where the impact of an attack—whether directed at an argument or another attack—is essentially identical. This occurs, for instance, when an argument is defeated by showing that one of its premises is false, and analogously when an attack is invalidated by demonstrating that the conditions for its existence are not met. Consider Example 1 presented in the introduction, now extended by adding a new argument D:

A: *No hiking* because Paul is not on vacation.
B: *Hiking* because John has no meeting.
C: Employers who are not on vacation have meetings.
D: Paul is on vacation.

The corresponding flat wHO-AF is depicted in Figure 9.

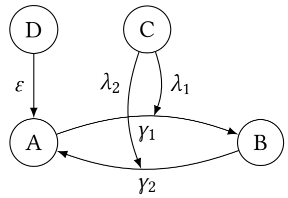


Fig. 9. Complete representation of Example 1

Let us compare the effect of the three attacks $\varepsilon, \lambda_1, \lambda_2$ on their respective targets. The attack ε is fatal to A, as D undermines one of A's premises. As a result, the level of justification that A provides for its claim is reduced to zero. The conflict between A and B reflects the incompatibility of their respective explanations—one supporting hiking and the other opposing it. Indeed, both explanations cannot simultaneously hold, as there could be employees who are neither on holiday nor attending a meeting and who should therefore both hike and not hike. The attacks λ_1 and λ_2 are also fatal to γ_1, γ_2, respectively. This is because C offers evidence that invalidates the assumption underlying the original conflict between A and B, thereby rendering the attacks γ_1, γ_2 unjustified.
Equality: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x \in \mathcal{A}$, $\forall y \in \mathcal{R}$, if

- $\mathbf{w}(x) = \mathbf{w}(y)$,
- $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y)$,
- $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Remark: Note that if a semantics satisfies Equality, then the two sub-principles of each of the general principles can be combined into a single principle that applies to arbitrary elements (it is sufficient to consider $x, y \in \mathcal{A} \cup \mathcal{R}$).

The Equality principle is, however, not appropriate when arguments are inductive, since attacks can only weaken such arguments but do not eliminate them entirely. This asymmetry does not necessarily apply to attacks on attacks, which can be fully defeated. Consider the following dialogue between two friends about a novel, along with its representation as a flat wHO-AF depicted in Figure 10:

*D***:** This novel has a similar plot to the one we read before. That one was boring, so this one is likely boring too.
*E***:** But the covers of the two books are very different.
*F***:** The cover has nothing to do with the book's content.

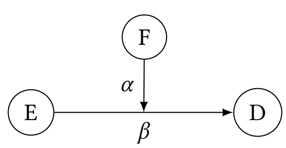


Fig. 10. Dialog about a novel

Argument D is analogical, as it draws on accepted similarities between two items (novels) to support the conclusion that they share an additional feature. One common way to challenge an analogical argument is to point out one or more dissimilarities between the items being compared (Amgoud 2020); in this sense, E attacks D. Countering such an attack involves showing that the cited dissimilarity is irrelevant to the comparison. In the example, argument F attacks β by arguing that the cover of a book is not a valid criterion for comparing the content of two novels. This attack is lethal to β and effectively neutralizes it. The situation is different for arguments. Suppose we consider a graph consisting solely of D, E, and β. In this case, D is weakened by β but not eliminated—it remains alive (Juthe 2005; Waller 2001). This illustrates that, under similar conditions, a semantics should give precedence to arguments, penalizing them less severely than attacks. The next principle formalizes this behavior.

A-Precedence: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x \in \mathcal{A}$ and $\forall y \in \mathcal{R}$, if

- $\mathbf{w}(x) = \mathbf{w}(y)$,
- $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y)$,
- $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) \geq \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

To motivate the third strategy where precedence is given to attacks over arguments, consider the following dialogue between two friends (see Figure 11):

G: Daisy is white since it is a swan and swans are white.
H: Dante is a black-necked swan.
I: But that information was given by someone on the *Arguman* debate platform.

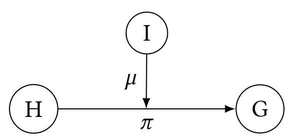


Fig. 11. Dialog about swans

Note that I does not refute H, but highlights the fact that it comes from an uncertain source (the Arguman platform), which casts doubt on the relevance of the attack π. Semantics should take this uncertainty into account but, instead of effectively inhibiting π, reduce its strength and therefore its effect on G. Now consider the simple graph with only H, G, π. The argument H undermines a premise of G preventing the latter from providing sufficient support for its claim. Therefore, a reasonable semantics would declare G worthless. In this example, as the following principle promotes, semantics can be more harmful to arguments than to attacks.

R-Precedence ensures that a semantics gives precedence to attacks over arguments.

R-Precedence: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x \in \mathcal{A}$ and $\forall y \in \mathcal{R}$, if

- $\mathbf{w}(x) = \mathbf{w}(y)$,
- $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y)$,
- $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) \leq \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

The last principle concerns the second challenge (ii), namely how a semantics evaluates the overall strength of an attack—specifically, how it combines the strength of the attack itself with that of its source. In the previous example, π is attacked by I via μ. To determine the strength of π, a gradual semantics should consider both the strength of I and that of μ. It may either assign equal importance to both elements or give priority to one over the other. The next principle endorses the former approach, treating the source and the attack as equally important in the evaluation.

Local Equality: For any $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $\forall x, y \in \mathcal{R}$ (resp. $\forall x, y \in \mathcal{A}$), if

- $\mathbf{w}(x) = \mathbf{w}(y)$,

- $\mathsf{Sr}_{\mathbf{H}}(x) \setminus \mathsf{Sr}_{\mathbf{H}}(y) = \{a\}$, $\mathsf{Sr}_{\mathbf{H}}(y) \setminus \mathsf{Sr}_{\mathbf{H}}(x) = \{b\}$,
- $\forall c \in \mathsf{Sr}_{\mathbf{H}}(x) \setminus \{a\}$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(c,x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(c,y))$,
- $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x))$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y))$,

then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

All the principles presented so far are compatible, i.e., can be satisfied all together by a semantics.

Theorem 1. *The following properties hold.*

- *Equality* $\Longleftrightarrow$ *A-precedence and R-precedence.*
- *The set of all above-defined principles is compatible.*

In summary, we have proposed sixteen principles that form a formal foundation for analyzing and comparing gradual semantics. These principles capture the different strategies a semantics may adopt to evaluate both arguments and attacks. The first twelve are adaptations of existing properties to the wHO-AF setting—they extend well-known principles originally defined for arguments so as to also encompass attacks. The remaining four principles are novel, specifically designed to address the distinctive features of weighted higher-order argumentation frameworks. These new principles motivate the introduction of new semantics and structure their presentation across Sections 5 and 6.

5 Treating Arguments and Attacks Equally

This section introduces the first family of gradual semantics designed for weighted higher-order frameworks. The semantics in this family treat arguments and attacks equally, thereby satisfying the *Equality* principle. We distinguish two subfamilies within this approach. The first focuses on flat frameworks and extends the RAF semantics recalled in Section 3.1.1 by simply assigning scores to individual elements. The second addresses weighted frameworks, following the methodology outlined in Section 3.2.

5.1 Structure-based Gradual Semantics

As discussed in Section 3.1.1, four structure-based semantics were introduced in (Cayrol, Fandinno, et al. 2020) to address flat higher-order frameworks $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, where both arguments and attacks are unweighted—that is, $\mathbf{w} \equiv 1$. These semantics generate sets of jointly acceptable elements. To align with Definition 10 of gradual semantics, we slightly extend them by assigning scores to individual elements. In the literature (Baroni and Giacomin 2007; Cayrol and Lagasquie-Schiex 2005), arguments are usually assigned a qualitative status (*sceptically* accepted, *credulously* accepted, *undecided*, or *rejected*) according to their membership across the generated sets. For the sake of homogeneity, we instead adopt a fixed numerical four-valued scale $\mathbf{D} = \{0, \alpha, \beta, 1\}$, where α and β are fixed but arbitrary values satisfying $0 < \alpha < \beta < 1$.

Definition 11. *Let* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ *and* $\mathbf{X}$ *be a set of subsets of* $\mathcal{A} \cup \mathcal{R}$. *An* aggregator *is the mapping* $\mathcal{V}^{\mathbf{X}}_{\mathbf{H}} : \mathcal{A} \cup \mathcal{R} \rightarrow \mathbf{D}$ *such that* $\forall x \in \mathcal{A} \cup \mathcal{R}$, $\mathcal{V}^{\mathbf{X}}_{\mathbf{H}}(x) = \alpha$ *if* $\mathbf{X} = \emptyset$. *Else,*

- $\mathcal{V}^{\mathbf{X}}_{\mathbf{H}}(x) = 1$ *if* $x \in \bigcap_{E \in \mathbf{X}} E$.
- $\mathcal{V}^{\mathbf{X}}_{\mathbf{H}}(x) = \beta$ *if* $\exists E, E' \in \mathbf{X}$ *such that* $x \in E$ *and* $x \notin E'$.
- $\mathcal{V}^{\mathbf{X}}_{\mathbf{H}}(x) = \alpha$ *if* $x \notin \bigcup_{E \in \mathbf{X}} E$ *and* $\nexists E \in \mathbf{X}$ *such that* E *defeats* x *(in the sense of Definition 4).*
- $\mathcal{V}^{\mathbf{X}}_{\mathbf{H}}(x) = 0$ *if* $x \notin \bigcup_{E \in \mathbf{X}} E$ *and* $\exists E \in \mathbf{X}$ *such that* E *defeats* x *(in the sense of Definition 4).*

Property 1. *For fixed* $\mathbf{H}$ *and* $\mathbf{X}$*, there is a unique aggregator on* $\mathbf{X}$.

Let us now define the first family of gradual semantics. Its instances assign to every flat wHO-AF the aggregator on its set of structures.

Definition 12. *We define* $\mathbf{S}_\bullet$ *as a function assigning to any* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ *the aggregator* $\mathcal{V}_{\mathbf{H}}^{\mathsf{Struc}_\bullet(\mathbf{H})}$ *where* $\bullet \in \{c, p, s, g\}$*, i.e., for any* $x \in \mathcal{A} \cup \mathcal{R}$, $\mathbf{V}_{\mathbf{H}}^{\mathbf{S}_\bullet}(x) = \mathcal{V}_{\mathbf{H}}^{\mathsf{Struc}_\bullet(\mathbf{H})}(x)$.

Example 2 (Cont) *Let* $\bullet = p$*. Ignoring* $\mathbf{w}_1$*, we get the following scores under preferred semantics.*

- $\mathbf{V}_{\mathbf{H}_1}^{\mathbf{S}_p}(x) = \mathcal{V}_{\mathbf{H}_1}^{\mathsf{Struc}_p(\mathbf{H}_1)}(x) = 1$ *for* $x \in \{b, g, \pi, \mu, \delta_1, \delta_2, \varepsilon\}$
- $\mathbf{V}_{\mathbf{H}_1}^{\mathbf{S}_p}(x) = \mathcal{V}_{\mathbf{H}_1}^{\mathsf{Struc}_p(\mathbf{H}_1)}(x) = \beta$ *for* $x \in \{c, d, \gamma, \lambda\}$
- $\mathbf{V}_{\mathbf{H}_1}^{\mathbf{S}_p}(x) = \mathcal{V}_{\mathbf{H}_1}^{\mathsf{Struc}_p(\mathbf{H}_1)}(x) = 0$ *for* $x \in \{a\}$

From the uniqueness of the aggregator, it follows that every $\mathbf{S}_\bullet$ is a gradual semantics. The latter ignores the initial weights of elements since it is grounded on RAF semantics which deal only with flat graphs.

Theorem 2. *For any* $\bullet \in \{c, p, g, s\}$, $\mathbf{S}_\bullet$ *is a gradual semantics.*

The next result shows the behaviour of the four gradual semantics with respect to the sixteen principles.

Theorem 3. *Let* $\mathbf{S}_\bullet$ *be such that* $\bullet \in \{c, p, g, s\}$*. Table 5 gives the principles satisfied/violated by* $\mathbf{S}_\bullet$*.*

The table shows that all four gradual semantics satisfy Anonymity, Weakening, Proportionality, and Equality, indicating that they treat arguments and attacks in a uniform manner. However, only $\mathbf{S}_g$, which is based on the grounded semantics, satisfies Local Equality and Equivalence. None of the semantics satisfy Resilience, meaning that elements may lose all their initial weight. Moreover, they are insensitive to both the quality and quantity of their attackers or incoming attacks. These limitations make the semantics less suitable for contexts like decision making, where such distinctions are often critical. Of the four semantics, stable uniquely violates both Independence and Directionality, as will be demonstrated later.

5.2 Pure Gradual Semantics

Unlike the first subfamily of gradual semantics—which disregard the initial weights of elements and are grounded in extension-based semantics—the second subfamily operates in weighted settings and is purely gradual, in the sense of (Cayrol and Lagasquie-Schiex 2005), as it directly assigns values to elements. These semantics are defined by evaluation methods, each represented as a tuple of three numerical functions: f, g, and h. The function h computes the strength of each individual attack on an element by aggregating the strength of the attack itself with that of its source. The function g evaluates the strength of the group of attacks on the element by aggregating the values returned by h for each member of the group. Finally, f determines the strength of the element by combining its initial weight with the aggregate value computed by g. In existing literature, such semantics have only been defined for evaluating arguments, as the underlying graphs typically allow attacks only on arguments. In contrast, wHO-AFs permit attacks on attacks, which makes it necessary to evaluate the strength of attacks themselves in order to assess arguments accurately. In what follows, we revise Definition 9 of gradual semantics by extending its functionality and overhauling the conditions that the three functions of an evaluation method must satisfy. Specifically, we relax some constraints on h to broaden the class of admissible functions, while introducing new critical conditions—such as continuity—which are essential for ensuring the existence of well-defined semantics.

Definition 13. *An evaluation method* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$ *is* well-behaved[6] *if the following conditions hold.*

(1) *a)* $\mathsf{f}(x_1, y) > \mathsf{f}(x_2, y)$ *whenever* $x_1 > x_2$*,*
b) $\mathsf{f}(y, x_1) > \mathsf{f}(y, x_2)$ *whenever* $x_1 < x_2$ *and* $y \neq 0$*,*
c) $\mathsf{f}(x, 0) = x$*,*

[6]The term well-behaved is used in (Amgoud and Doder 2019; Leite and Martins 2011) for evaluation methods that satisfy other sets of constraints.

Table 5. The symbols ✓ and × stand for satisfied and violated respectively. If A-P and R-P have the same status, this status is given once. The colored line concerns the gradual semantics that satisfies all the principles.

		Principles (A-P/R-P)															
		Anonymity	Independence	Directionality	Equivalence	Maximality	Neutrality	Weakening	Proportionality	Resilience	Reinforcement	Counting	Attack-S.	Equality	A-precedence	R-precedence	Local Equality
Gradual semantics $\mathbf{S}_\bullet$ (case of RAF)																	
$\bullet =$	g	✓	✓	✓	✓	✓	✓	✓	✓	×	×	×	×	✓	✓	✓	✓
	s	✓	×	×	×	×	✓	✓	✓	×	×	×	×	✓	✓	✓	×
	p	✓	✓	✓	×	✓	×	✓	✓	×	×	×	×	✓	✓	✓	×
	c	✓	✓	✓	×	✓	×	✓	✓	×	×	×	×	✓	✓	✓	×
Gradual semantics $\mathbf{E}_\blacksquare$ (case of AFRA)																	
$\blacksquare =$	g	✓	✓	✓	✓/×	✓/×	✓	✓	✓	×	×	×	×	×	✓	×	✓/×
	s	✓	×	×	×	×	✓	✓	✓	×	×	×	×	×	✓	×	×
	p	✓	✓	✓	×	✓/×	×	✓	✓	×	×	×	×	×	✓	×	×
	c	✓	✓	✓	×	✓/×	×	✓	✓	×	×	×	×	×	✓	×	×
Gradual Semantics $\mathsf{eHbs}^{\mathsf{h}}$																	
$\mathsf{h} =$	h_1	✓	✓	✓	✓	✓	✓	✓	✓	✓	×	✓	×	✓	✓	✓	✓
	h_5	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	×
	h_6	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Gradual Semantics $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$																	
$\mathsf{h} =$	h_1	✓	✓	✓	✓	✓	✓	✓	✓	✓	×	×	×	✓	✓	✓	✓
	h_5	✓	✓	✓	✓	✓	✓	✓	✓	✓	×	×	×	✓	✓	✓	×
	h_6	✓	✓	✓	✓	✓	✓	✓	✓	✓	×	×	×	✓	✓	✓	✓
Hybrid Semantics Hyb																	
Hyb_1		✓	✓	✓	✓	✓	✓	✓	✓	✓	✓/×	✓/×	✓/×	×	×	✓	✓
Hyb_2		✓	✓	✓	✓	✓	✓	✓	✓	✓	×/✓	✓	×/✓	×	✓	×	✓

d) $\mathsf{f}(0, x) = 0$,
e) f *is continuous on the second variable,*

(2) *a)* $\mathsf{g}() = 0$,
b) $\mathsf{g}(x) = x$,
c) $\mathsf{g}(x_1, \ldots, x_n) = \mathsf{g}(x_1, \ldots, x_n, 0)$,
d) $\mathsf{g}(x_1, \ldots, x_n, y) \leq \mathsf{g}(x_1, \ldots, x_n, z)$ *whenever* $y \leq z$,
e) g *is commutative, i.e.,* $\mathsf{g}(x_1, \ldots, x_n) = \mathsf{g}(x_{\rho(1)}, \ldots, x_{\rho(n)})$, *for any permutation* ρ *of the set* $\{1, \ldots, n\}$;
f) g *is continuous,*

(3) *a)* $\mathsf{h}(0, x) = 0$,
b) $\mathsf{h}(x, 0) = 0$,
c) $\mathsf{h}(x, y) > 0$ *whenever* $xy > 0$,
d) h *is non-decreasing in both variables,*
e) h *is continuous on both variables.*

Let us now define a gradual semantics, based on a well-behaved evaluation method, that assesses strengths of elements of any weighted higher-order argumentation framework.

Definition 14. *An* extended gradual semantics *is a function* S *assigning to any* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathtt{AF}$ *a function* $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}} : \mathcal{A} \cup \mathcal{R} \to [0,1]$ *such that* $\forall x \in \mathcal{A} \cup \mathcal{R}$,

$$\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(x) = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\Bigg(\mathsf{h}\Big(\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha_1), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha_n), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_n))\Big)\Bigg)\Bigg) \tag{5}$$

where $\{\alpha_1, \ldots, \alpha_n\} = \mathtt{Att}_{\mathbf{H}}(x)$ *and* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$ *is a well-behaved evaluation method.*

The above definition generalizes Equation 1 from Definition 9. Specifically, when the wHO-AF is basic—that is, it contains no higher-order attacks—then Equations 1 and 5 yield the same values for both arguments and attacks, with the latter simply retaining their initial weights.

Theorem 4. *Let* S *be an extended gradual semantics and* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathtt{AF}$ *such that* $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$. *The following properties hold:*

- $\forall \alpha \in \mathcal{R}$, $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha) = \mathbf{w}(\alpha)$.
- $\forall a \in \mathcal{A}$, $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(a) = \mathsf{f}\Bigg(\mathbf{w}(a), \mathsf{g}\Bigg(\mathsf{h}\Big(\mathbf{w}(\alpha_1), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{w}(\alpha_n), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_n))\Big)\Bigg)\Bigg)$, *where* $\mathtt{Att}_{\mathbf{H}}(a) = \{\alpha_1, \ldots, \alpha_n\}$.

It is worth noting that evaluating arguments and attacks in a wHO-AF amounts to solving a system of equations—one equation per element. A natural question that arises is *whether this system admits a solution.* The existence of a solution directly determines whether an extended gradual semantics exists. The following result establishes that the answer is indeed positive.

Theorem 5. *For any well-behaved evaluation method* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, *there exists an extended gradual semantics that is based on* $\mathbf{M}$.

A second natural question is *whether the system of equations admits a unique solution or possibly multiple ones.* Equivalently, does a given evaluation method $\mathbf{M}$ define a single extended gradual semantics or several? From a technical perspective, the existence of multiple solutions—say n—implies the existence of n distinct extended gradual semantics S_i, all based on the same evaluation method $\mathbf{M}$, yet indistinguishable in practice, as none can be uniquely characterized. We show that the system of equations admits a unique solution whenever the evaluation method satisfies a specific set of additional properties.

Definition 15. *We define* $\mathbb{M}$ *as the set of all well-behaved evaluation methods* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$ *such that:*

(1) $\lambda \mathsf{f}(x_1, \lambda x_2) < \mathsf{f}(x_1, x_2)$, $\forall \lambda < 1$, $x_1 \neq 0$.
(2) $\mathsf{g}(\mathsf{h}(\lambda y_1, \lambda x_1), \ldots, \mathsf{h}(\lambda y_n, \lambda x_n)) \geq \lambda \mathsf{g}(\mathsf{h}(y_1, x_1), \ldots, \mathsf{h}(y_n, x_n))$, $\forall \lambda \in [0,1]$.

The first condition has been used by the authors in (Amgoud and Doder 2019) to ensure a semantics is well-defined, while the second is new. It rules out certain functions such as $\mathsf{h}_7(x, y) = x \times y$, which can be used by the semantics of (Amgoud and Doder 2019) but not in the context of wHO-AFs. This shows that the definition of a gradual semantics that deals with weighted higher-order attacks needs to be done with great care.

The following result guarantees that, for any evaluation method $\mathbf{M}$ in the set $\mathbb{M}$, there exists a unique extended gradual semantics based on $\mathbf{M}$. Equivalently, the system of equations (5) admits a single solution.

Theorem 6. *For any evaluation method* $\mathbf{M} \in \mathbb{M}$, *there exists a single extended gradual semantics that is based on* $\mathbf{M}$.

The set $\mathbb{M}$ defines a new family of extended gradual semantics that can handle higher-order attacks and evaluate the strength of both arguments and attacks.

Definition 16. *We define* $\mathbf{ES}$ *as the set of all extended gradual semantics that are based on an evaluation method* $\mathbf{M} \in \mathbb{M}$.

Each semantics of the new family assigns to each element x of a wHO-AF a value from the interval $[0, \mathbf{w}(x)]$. Thus, the initial weight is the maximum strength an element can have.

Theorem 7. *Let* $\mathrm{S} \in \mathbf{ES}$ *and* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$. *For every* $x \in \mathcal{A} \cup \mathcal{R}$, $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(x) \in [0, \mathbf{w}(x)]$.

Example 2 (Cont) *For any* $\mathrm{S} \in \mathbf{ES}$, *the following hold:* $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}_1}(a) \leq 0.5$, $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}_1}(\pi) \leq 0.4$, *and* $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}_1}(\gamma) \leq 0.7$.

The definition of extended gradual semantics is declarative in nature and inherently recursive. In what follows, we present its procedural counterpart: an algorithm that computes the strengths of arguments and attacks according to the new semantics. We show in Section 7 that this algorithm is both efficient and scalable.

Theorem 8. *Let* $\mathrm{S} \in \mathbf{ES}$ *be based on the evaluation method* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, *and* $x \in \mathcal{A} \cup \mathcal{R}$ *such that* $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots \alpha_k\}$. *We define the sequence* $\{\sigma(x)^{(n)}\}_{n=1}^{+\infty}$ *in the following way:*

- $\sigma(x)^{(1)} = \mathbf{w}(x)$,
- $\sigma(x)^{(n+1)} = \mathsf{f}\Big(\mathbf{w}(x), \mathsf{g}\big(\mathsf{h}(\sigma(\alpha_1)^{(n)}, \sigma(\mathbf{s}(\alpha_1))^{(n)}), \ldots, \mathsf{h}(\sigma(\alpha_k)^{(n)}, \sigma(\mathbf{s}(\alpha_k))^{(n)})\big)\Big)$.

For every $x \in \mathcal{A} \cup \mathcal{R}$, *the following hold:*

(1) $\{\sigma(x)^{(n)}\}_{n=1}^{+\infty}$ *converges,*
(2) $\lim\limits_{n \to +\infty} \sigma(x)^{(n)} = \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(x)$.

The following result summarizes the conditions under which the principles introduced in Section 4 are satisfied by any instance of the new family of semantics. Some results hold under additional properties, as specified in Table 6, for the three functions of an evaluation method $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$.

[h]

Table 6. Properties of f, g, h.

Property	Definition
Positivity	$\mathsf{f}(x_1, x_2) > 0$ whenever $x_1 > 0$
Strict Monotony	$\mathsf{g}(x_1, \ldots, x_n, y) < \mathsf{g}(x_1, \ldots, x_n, z)$ when $y < z$
Strict Monotony (F1)	$\mathsf{h}(x_1, y) > \mathsf{h}(x_2, y)$ whenever $x_1 > x_2$, $y \neq 0$
Strict Monotony (F2)	$\mathsf{h}(y, x_1) > \mathsf{h}(y, x_2)$ whenever $x_1 > x_2$, $y \neq 0$
Symmetry	$\mathsf{h}(x, y) = \mathsf{h}(y, x)$

Theorem 9. *Let* S *be a semantics based on a well-behaved evaluation method* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$.

- S *satisfies Equivalence, Maximality, Neutrality, Weakening, Proportionality, Equality, A-Precedence, and R-Precedence.*
- *If* $\mathbf{M} \in \mathbb{M}$, *then* S *satisfies Anonymity, Independence, and Directionality.*
- *If* f *is positive, then* S *satisfies Resilience.*
- *If* g *is strictly monotonic, then:*
  - S *satisfies Counting.*

  - *If* h *is strictly monotonic on the first variable (F1), then* $\mathbf{S}$ *satisfies Attack-sensitivity.*
  - *If* h *is strictly monotonic on the second variable (F2), then* $\mathbf{S}$ *satisfies Reinforcement.*
- *If* h *is symmetric,* $\mathsf{h}(x, y) = \mathsf{h}(y, x)$*, then* $\mathbf{S}$ *satisfies Local-Equality.*

Below we show that if Local Equality is satisfied by a semantics, then the function h is necessarily symmetric, provided that g is strictly monotonic.

Property 2. *Let* $\mathbf{S} \in \mathbf{ES}$ *be based on* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$*. If* $\mathbf{S}$ *satisfies Local Equality and* g *is strictly monotonic, then* h *is symmetric.*

It is worth noting that any semantics within the family **ES** satisfies all the mandatory principles, including Anonymity, Independence, Directionality, Neutrality, Weakening, and Maximality. Additionally, some non-mandatory principles—such as Counting—are also satisfied under certain additional constraints on the evaluation methods. More precisely, Theorem 9 establishes formal connections between the three functions defining an evaluation method and the principles themselves. For instance, the satisfaction of Resilience depends on whether the influence function f is positive; Counting is ensured if the aggregation function g is strictly monotonic; Local Equality holds when h is symmetric; and Attack-Sensitivity and Reinforcement rely on the strict monotonicity of both g and h. Moreover, instances of **ES** satisfy Equivalence, meaning their evaluation of each element depends solely on its initial weight and the strengths of its direct attackers and their sources. In contrast, the extension-based gradual semantics from the previous subsection may also consider other factors, such as the global topology of the argumentation graph.

Instantiating the general family **ES** of semantics amounts to identifying well-behaved evaluation methods that satisfy the additional properties outlined in Definition 15. In the following, we discuss specific instances based on selected functions from Tables 2 and 3. We show that some of these function tuples indeed belong to the set $\mathbb{M}$.

Definition 17. *Let* $\langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$ *be an evaluation method. We say that* f *(respectively* g, h*) is* well-behaved *iff it satisfies the conditions (1.x), with* $x = a, \ldots, e$ *(respectively (2.y) with* $y = a, \ldots, f$*, and (3.z) with* $z = a, \ldots, e$*) in Definition 13.* f *is* rational *iff it satisfies the first condition in Definition 15. The pair* $\langle \mathsf{g}, \mathsf{h} \rangle$ *is rational if it satisfies the second condition in Definition 15.*

The following result summarizes the properties of the functions given in Tables 2 and 3.

Property 3. *The following properties hold.*

(1) *The function* $\mathsf{f}_{\mathtt{frac}}$ *is well-behaved, positive, and rational.*
(2) *The function* $\mathsf{f}_{\mathtt{exp}}$ *is well-behaved and positive.*
(3) $\lambda \mathsf{f}_{\mathtt{exp}}(x, \lambda y) < \mathsf{f}_{\mathtt{exp}}(x, y)$*,* $\forall \lambda < 1$*,* $x \neq 0$ *and* $y \in [0, 1]$*.*
(4) *The functions* $\mathsf{g}_{\mathsf{sum}}$ *and* $\mathsf{g}_{\mathsf{max}}$ *are well-behaved. In addition,* $\mathsf{g}_{\mathsf{sum}}$ *is strictly monotonic.*
(5) *For any* $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$*, the following holds:*
   - h *is well-behaved,*
   - *for any* $\mathsf{g} \in \{\mathsf{g}_{\mathsf{sum}}, \mathsf{g}_{\mathsf{max}}\}$*, the pair* $\langle \mathsf{g}, \mathsf{h} \rangle$ *is rational.*
(6) *The functions* $\mathsf{h}_2, \mathsf{h}_3, \mathsf{h}_4$ *are not well-behaved.*
(7) *The function* h_7 *is well-behaved; for any* $\mathsf{g} \in \{\mathsf{g}_{\mathsf{sum}}, \mathsf{g}_{\mathsf{max}}\}$*, the pair* $\langle \mathsf{g}, \mathsf{h}_7 \rangle$ *is not rational.*
(8) *The functions* $\mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3, \mathsf{h}_6, \mathsf{h}_7$ *are symmetric. The weighted functions* $\mathsf{h}_4, \mathsf{h}_5$ *are not.*

The above result shows that the function $\mathsf{f}_{\mathtt{frac}}$ can be combined with either $\mathsf{g}_{\mathsf{sum}}$ or $\mathsf{g}_{\mathsf{max}}$, whereas $\mathsf{f}_{\mathtt{exp}}$ can only be paired with $\mathsf{g}_{\mathsf{max}}$. This restriction arises because the combination $(\mathsf{f}_{\mathtt{exp}}, \mathsf{g}_{\mathsf{sum}})$ violates the first condition of Definition 15, which is essential for the proofs of the key convergence Theorems 6 and 8.

The result also shows that both functions $\mathsf{g}_{\mathsf{max}}$ and $\mathsf{g}_{\mathsf{sum}}$ can be combined with h_1, h_5, and h_6. Notably, h_1 and h_6 are symmetric, which ensures that the resulting semantics satisfy the Local Equality principle, whereas h_5 is not symmetric, and its corresponding semantics violate this principle.

It is also worth noting that the product function h_7, used in the semantics from (Amgoud and Doder 2019), is excluded from **ES** because it violates the second condition in Definition 15 that characterizes $\mathbb{M}$. Conversely, h_5 and h_6 are discarded in (Amgoud and Doder 2019) for violating the property $\mathsf{h}(1, x) = x$, which, in contrast, is not required in our framework, allowing for a broader range of aggregation functions. This illustrates that the family **ES** neither fully generalizes all semantics proposed in (Amgoud and Doder 2019) nor is entirely subsumed by it; rather, it excludes some semantics from (Amgoud and Doder 2019) while encompassing novel ones. The following result provides tuples that belong to the set $\mathbb{M}$.

Theorem 10. *For any* $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$*, the following properties hold:*

- $\langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h} \rangle \in \mathbb{M}$,
- $\langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle \in \mathbb{M}$.
- $\langle \mathsf{f}_{\mathtt{exp}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle \in \mathbb{M}$.

The above evaluation methods give birth to the semantics $\mathsf{eHbs}^{\mathsf{h}}$, $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$, which are based respectively on $\langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h} \rangle$, $\langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle$ and $\langle \mathsf{f}_{\mathtt{exp}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle$, where $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$.

Definition 18. *Let* $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$*. We define* $\mathsf{eHbs}^{\mathsf{h}}$ *(resp.* $\mathsf{eMbs}^{\mathsf{h}}$*,* $\mathsf{eEbs}^{\mathsf{h}}$*) as a function which assigns to each* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ *a weighting* $\mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}}}$ *(respectively* $\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}$*,* $\mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}$*) on* $\mathcal{A} \cup \mathcal{R}$ *such that* $\forall x \in \mathcal{A} \cup \mathcal{R}$*:*

$$\mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}}}(x) = \frac{\mathbf{w}(x)}{1 + \sum\limits_{\mathbf{t}(\alpha)=x} \left(\mathsf{h}\left(\mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}}}(\alpha), \mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}}}(\mathbf{s}(\alpha)) \right) \right)}$$

$$\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(x) = \frac{\mathbf{w}(x)}{1 + \max\limits_{\mathbf{t}(\alpha)=x} \left(\mathsf{h}\left(\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(\alpha), \mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(\mathbf{s}(\alpha)) \right) \right)}$$

$$\mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(x) = \mathbf{w}(x) \times e^{-\max\limits_{\mathbf{t}(\alpha)=x} \left(\mathsf{h}\left(\mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(\alpha), \mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(\mathbf{s}(\alpha)) \right) \right)}$$

The above semantics are instances of the family **ES**, which also means that **ES** is not empty.

Theorem 11. *For any* $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$*, the inclusion* $\{\mathsf{eHbs}^{\mathsf{h}}, \mathsf{eMbs}^{\mathsf{h}}, \mathsf{eEbs}^{\mathsf{h}}\} \subseteq \mathbf{ES}$ *holds.*

Let us illustrate the semantics with examples.

Example 2 (Cont) *Table 7 below gives the strength degree of each element according to* $\mathsf{eHbs}^{\mathsf{h}}$*,* $\mathsf{eMbs}^{\mathsf{h}}$ *and* $\mathsf{eEbs}^{\mathsf{h}}$*. For the weighted function* h_5*, we fix the weight* w *to 0.5.*

Table 7. Strengths of arguments and attacks in Example 2. Case of $\mathsf{eHbs}^{\mathsf{h}}$, $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$, with $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$.

		a	b	c	d	g	π	μ	δ_1	δ_2	γ	ε	λ
$\mathsf{eHbs}^{\mathsf{h}}$	h_1	*0.416*	*0.8*	*0.282*	*0.233*	*0.8*	*0.4*	*0.2*	*0.7*	*0.7*	*0.535*	*0.8*	*0.306*
with h =	$\mathsf{h}_5, \mathsf{h}_6$	*0.333*	*0.8*	*0.240*	*0.204*	*0.8*	*0.4*	*0.2*	*0.7*	*0.7*	*0.456*	*0.8*	*0.267*
$\mathsf{eMbs}^{\mathsf{h}}$	h_1	*0.416*	*0.8*	*0.329*	*0.225*	*0.8*	*0.4*	*0.2*	*0.7*	*0.7*	*0.515*	*0.8*	*0.357*
with h =	$\mathsf{h}_5, \mathsf{h}_6$	*0.333*	*0.8*	*0.307*	*0.199*	*0.8*	*0.4*	*0.2*	*0.7*	*0.7*	*0.446*	*0.8*	*0.333*
$\mathsf{eEbs}^{\mathsf{h}}$	h_1	*0.409*	*0.8*	*0.303*	*0.221*	*0.8*	*0.4*	*0.2*	*0.7*	*0.7*	*0.500*	*0.8*	*0.335*
with h =	$\mathsf{h}_5, \mathsf{h}_6$	*0.303*	*0.8*	*0.274*	*0.184*	*0.8*	*0.4*	*0.2*	*0.7*	*0.7*	*0.402*	*0.8*	*0.305*

Unattacked elements (arguments or attacks) retain their initial weight, while attacked elements lose weight under the six semantics. Note also that $\mathsf{eHbs}^{\mathsf{h}}$ *can be more harmful than* $\mathsf{eMbs}^{\mathsf{h}}$*, since each attack contributes to further* *weakening its target. This is the case for c and λ, which have two attackers (see the colored cells); for instance,* $\mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}}}(c) \leq \mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(c)$ *whatever the function* h.

The following result shows that the semantics $\mathsf{eHbs}^{\mathsf{h}}$, $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ generalize their counterparts in (Amgoud and Doder 2019), provided that h satisfies the relevant conditions in (Amgoud and Doder 2019) and in Definitions 13 and 15. Note that h_1 is one of the concerned functions. When a wHO-AF does not contain higher-order attacks, each semantics assigns exactly the same values to arguments and attacks as its counterpart. In particular, the value of every attack is its initial weight. Before presenting the formal result, recall that $\mathbb{F}$ denotes the set of all semantics $\mathsf{Hbs}^{\mathsf{h}}, \mathsf{Mbs}^{\mathsf{h}}, \mathsf{Ebs}^{\mathsf{h}}$ defined in (Amgoud and Doder 2019).

Theorem 12. *Let* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ *such that* $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$ *(so a basic wHO-AF). For any semantics* $\mathsf{Hbs}^{\mathsf{h}} \in \mathbb{F}$ *(respectively* $\mathsf{Mbs}^{\mathsf{h}} \in \mathbb{F}$, $\mathsf{Ebs}^{\mathsf{h}} \in \mathbb{F}$*), if* $\langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h} \rangle \in \mathbb{M}$ *(respectively* $\langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle \in \mathbb{M}$, $\langle \mathsf{f}_{\mathsf{exp}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle \in \mathbb{M}$*), then* $\forall a \in \mathcal{A}$:

- $\mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}}}(a) = \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Hbs}^{\mathsf{h}}}(a)$,
- $\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(a) = \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Mbs}^{\mathsf{h}}}(a)$,
- $\mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(a) = \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Ebs}^{\mathsf{h}}}(a)$.

Regarding their behaviour with respect to the principles of Section 4, it is clear that these semantics inherit all the properties described in Theorem 9.

Theorem 13. *Table 5 summarizes the list of principles that are satisfied/violated by* $\mathsf{eHbs}^{\mathsf{h}}$, $\mathsf{eMbs}^{\mathsf{h}}$ *and* $\mathsf{eEbs}^{\mathsf{h}}$ *for any* $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$.

Among the semantics considered, $\mathsf{eHbs}^{\mathsf{h}_6}$ is the only one that satisfies all the principles. The choice of the function h directly impacts several principles—specifically, Reinforcement, Attack-Sensitivity, and Local Equality. For instance, $\mathsf{eHbs}^{\mathsf{h}_1}$, which employs the symmetric function h_1, satisfies Local Equality but violates Reinforcement and Attack-Sensitivity. These violations are not due to the symmetry of h_1 per se, but rather to its behavior: it selects the minimum of its two inputs—the strength of the attack and that of its source—which dampens the influence of stronger attackers. In contrast, $\mathsf{eHbs}^{\mathsf{h}_6}$ satisfies both principles while still relying on a symmetric operator, highlighting that symmetry alone is not problematic. The third semantics, $\mathsf{eHbs}^{\mathsf{h}_5}$, uses the non-symmetric function h_5. It satisfies Reinforcement and Attack-Sensitivity, but violates Local Equality, as expected due to the asymmetry of h_5. The $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ semantics differ from their $\mathsf{eHbs}^{\mathsf{h}}$ counterparts in that they always violate Reinforcement, Attack-Sensitivity, and Counting, regardless of the choice of h. This is due to the use of the aggregation function $\mathsf{g}_{\mathsf{max}}$, which fails to reflect the cumulative strength of multiple attackers and thus undermines these principles.

To summarize, we introduced a broad family of gradual semantics. Its instances are well-defined and constitute the first gradual semantics in the literature capable of handling weighted higher-order attacks. All semantics in this family satisfy both the Equality and Equivalence principles. Unlike the extension-based semantics of the first family, which take into account the global topology of the graph while disregarding initial weights, these purely gradual semantics rely exclusively on initial weights and direct attacks, offering a more localized and weight-sensitive evaluation.

6 Treating Arguments and Attacks Differently

This section introduces two families of semantics that distinguish between arguments and attacks. The first family comprises the four AFRA semantics, which satisfy the A-Precedence principle—meaning they assign lower values to attacks than to arguments, thereby treating attacks more harshly. The second family consists of pure gradual semantics, which follow a hybrid approach by combining two distinct evaluation methods: one for arguments and another for attacks. Depending on how their evaluation functions are configured, some instances of this family satisfy A-Precedence, while others satisfy R-Precedence.

6.1 Extension-based Gradual Semantics

In this section, we introduce four gradual semantics, each derived from one of the AFRA semantics. Given a flat higher-order argumentation framework, an AFRA semantics first computes a set of extensions. Subsequently, a value is assigned to each element of the graph using the aggregation function defined in Definition 11, where **the relation *defeat* is replaced by *e-defeat*** as specified in Definition 6. These semantics can thus be regarded as direct adaptations of the AFRA semantics, enriched with strength degrees that reflect the elements' membership status in the extensions generated by the underlying AFRA semantics.

Definition 19. *We define* $\mathbf{E}_{\blacksquare}$ *as a function assigning to any* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ *the aggregator* $\mathcal{V}_{\mathbf{H}}^{\mathtt{Ext}_{\blacksquare}(\mathbf{H})}$, *where* $\blacksquare \in \{c, p, s, g\}$*, i.e., for any* $x \in \mathcal{A} \cup \mathcal{R}$, $\mathbf{V}_{\mathbf{H}}^{\mathbf{E}_{\blacksquare}}(x) = \mathcal{V}_{\mathbf{H}}^{\mathtt{Ext}_{\blacksquare}(\mathbf{H})}(x)$.

Recall that the AFRA semantics consider flat graphs, thus any $\mathbf{E}_{\blacksquare}$ ignores the initial weights of arguments and attacks.

Example 2 (Cont) *Here, ignoring* $\mathbf{w}_1$ *and with* $\blacksquare = p$*, we have:*

- $\mathbf{V}_{\mathbf{H}_1}^{\mathbf{E}_p}(x) = \mathcal{V}_{\mathbf{H}_1}^{\mathtt{Ext}_p(\mathbf{H}_1)}(x) = 1$ *for* $x \in \{b, g, \mu\}$
- $\mathbf{V}_{\mathbf{H}_1}^{\mathbf{E}_p}(x) = \mathcal{V}_{\mathbf{H}_1}^{\mathtt{Ext}_p(\mathbf{H}_1)}(x) = \beta$ *for* $x \in \{c, d, \lambda, \delta_1, \delta_2, \gamma, \varepsilon\}$
- $\mathbf{V}_{\mathbf{H}_1}^{\mathbf{E}_p}(x) = \mathcal{V}_{\mathbf{H}_1}^{\mathtt{Ext}_p(\mathbf{H}_1)}(x) = 0$ *for* $x \in \{a, \pi\}$

$\mathbf{E}_p$ *treats arguments and attacks differently. Indeed, in the wHO-AF* $\mathbf{H}_1$*, the two arguments* b *and* g *are not attacked and therefore receive the maximal value, whereas the unattacked attack* π *in* $\mathbf{H}_1$ *is assigned a value of* 0*. Recall that* $\mathbf{S}_p$ *assigns the value 1 to* π *(see Section 5.1). Thus, the two semantics provide different evaluations.*

We show next that any function $\mathbf{E}_{\blacksquare}$ is a gradual semantics.

Theorem 14. *For any* $\blacksquare \in \{c, p, g, s\}$, $\mathbf{E}_{\blacksquare}$ *is a gradual semantics.*

The result below sheds light on the principles that are satisfied by the semantics $\mathbf{E}_{\blacksquare}$.

Theorem 15. *Let* $\mathbf{E}_{\blacksquare}$ *be such that* $\blacksquare \in \{c, p, g, s\}$*. Table 5 gives the principles satisfied/violated by* $\mathbf{E}_{\blacksquare}$.

Table 5 shows that the four semantics $\mathbf{E}_{\blacksquare}$ satisfy Anonymity, Weakening, Proportionality, and A-Precedence. The principles Independence, Directionality, and A-Maximality are satisfied by almost all semantics except $\mathbf{E}_s$—which is unsurprising, as these principles are already violated by stable semantics in basic argumentation frameworks. For the same reason, the four semantics also violate Resilience, Reinforcement, Counting, and Attack-Sensitivity. In other words, some of the negative characteristics (i.e., principle violations) of Dung's seminal extension semantics (Dung 1995) are inherited by the AFRA semantics, while certain positive characteristics are lost due to incompatibility with the A-Precedence principle.

The semantics $\mathbf{E}_g$ satisfies A-Equivalence and A-Local-Equality but violates the corresponding principles for attacks. In contrast, $\mathbf{S}_g$, defined in the RAF setting and treating arguments and attacks equally, satisfies both R-Equivalence and R-Local-Equality.

A third source of principle violations by $\mathbf{E}_{\blacksquare}$ arises from the AFRA setting itself. Under grounded, preferred, and complete semantics, Maximality holds for arguments but is violated for attacks. These violations stem primarily from the additional conflicts introduced in AFRA via the e-defeat relation (Definition 6). For instance, in Example 2, the attack π is not attacked in $\mathbf{H}_1$ but is e-defeated by μ. As a result, π does not appear in any extension and is assigned a value of 0 rather than the maximum, whereas all non-attacked arguments are still accepted.

6.2 Hybrid Semantics

So far, we have defined semantics that satisfy either *Equality* or *A-Precedence*, but none that follow *R-Precedence* while violating *Equality*. Moreover, those satisfying *A-Precedence* fail to meet several desirable properties—such as *R-Maximality*, *Neutrality*, *Reinforcement*, and *Attack-Sensitivity*—which are particularly relevant in decision-making contexts. In what follows, we introduce a fourth family of pure gradual semantics whose instances may satisfy different subsets of these principles. The key idea is that, to satisfy *A-* (or *R-*) *Precedence* without enforcing *Equality*, a gradual semantics must evaluate arguments and attacks differently, through distinct functions referred to as *base semantics*. We therefore propose a broad family of such *hybrid semantics*, whose base semantics rely on evaluation methods belonging to the set $\mathbb{M}$ defined in Definition 15.

Definition 20. *Let* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, $\mathbf{M}' = \langle \mathsf{f}', \mathsf{g}', \mathsf{h}' \rangle$ *be such that* $\mathbf{M}, \mathbf{M}' \in \mathbb{M}$ *and* $\mathbf{M} \neq \mathbf{M}'$. *A* hybrid semantics *based on* $(\mathbf{M}, \mathbf{M}')$ *is a function* $\mathtt{Hyb}$ *mapping any* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ *into a function* $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}} : \mathcal{A} \cup \mathcal{R} \to [0, 1]$ *such that* $\forall x \in \mathcal{A} \cup \mathcal{R}$, *with* $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots, \alpha_k\}$*:*

$$\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x) = \begin{cases} \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\bigg(\mathsf{h}\Big(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_n), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_n))\Big)\bigg)\Bigg) & \text{if } x \in \mathcal{A}, \\ \mathsf{f}'\Bigg(\mathbf{w}(x), \mathsf{g}'\bigg(\mathsf{h}'\Big(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}'\Big(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_n), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_n))\Big)\bigg)\Bigg) & \text{if } x \in \mathcal{R}. \end{cases}$$

The above family relies on two distinct evaluation methods, thereby covering the following seven cases:

- $\mathsf{f} \neq \mathsf{f}', \mathsf{g} = \mathsf{g}', \mathsf{h} = \mathsf{h}'$,
- $\mathsf{f} = \mathsf{f}', \mathsf{g} \neq \mathsf{g}', \mathsf{h} = \mathsf{h}'$,
- $\mathsf{f} = \mathsf{f}', \mathsf{g} = \mathsf{g}', \mathsf{h} \neq \mathsf{h}'$,
- $\mathsf{f} \neq \mathsf{f}', \mathsf{g} \neq \mathsf{g}', \mathsf{h} = \mathsf{h}'$,
- $\mathsf{f} \neq \mathsf{f}', \mathsf{g} = \mathsf{g}', \mathsf{h} \neq \mathsf{h}'$,
- $\mathsf{f} = \mathsf{f}', \mathsf{g} \neq \mathsf{g}', \mathsf{h} \neq \mathsf{h}'$,
- $\mathsf{f} \neq \mathsf{f}', \mathsf{g} \neq \mathsf{g}', \mathsf{h} \neq \mathsf{h}'$.

In all these disparate cases, a pair of evaluation methods defines a single hybrid semantics.

Theorem 16. *For every pair of evaluation methods* $\mathbf{M}, \mathbf{M}' \in \mathbb{M}$, *there exists a unique hybrid semantics that is based on* $(\mathbf{M}, \mathbf{M}')$.

We provide below an algorithm, which calculates the strengths of elements in an iterative and efficient way.

Theorem 17. *Let* $\mathtt{Hyb}$ *be a hybrid semantics based on the evaluation methods* $(\mathbf{M}, \mathbf{M}')$, *where* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, $\mathbf{M}' = \langle \mathsf{f}', \mathsf{g}', \mathsf{h}' \rangle$. *Let* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, *and* $x \in \mathcal{A} \cup \mathcal{R}$ *such that* $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots \alpha_k\}$. *We define the sequence* $\{\sigma(x)^{(n)}\}_{n=1}^{+\infty}$ *in the following way:*

- $\sigma(x)^{(1)} = \mathbf{w}(x)$.
- $\sigma(x)^{(n+1)} = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\bigg(\mathsf{h}\Big(\sigma(\alpha_1)^{(n)}, \sigma(\mathbf{s}(\alpha_1))^{(n)}\Big), \ldots, \mathsf{h}\Big(\sigma(\alpha_k)^{(n)}, \sigma(\mathbf{s}(\alpha_k))^{(n)}\Big)\bigg)\Bigg)$, *if* $x \in \mathcal{A}$.
- $\sigma(x)^{(n+1)} = \mathsf{f}'\Bigg(\mathbf{w}(x), \mathsf{g}'\bigg(\mathsf{h}'\Big(\sigma(\alpha_1)^{(n)}, \sigma(\mathbf{s}(\alpha_1))^{(n)}\Big), \ldots, \mathsf{h}'\Big(\sigma(\alpha_k)^{(n)}, \sigma(\mathbf{s}(\alpha_k))^{(n)}\Big)\bigg)\Bigg)$, *if* $x \in \mathcal{R}$.

For every $x \in \mathcal{A} \cup \mathcal{R}$, *the following hold:*

(1) $\{\sigma(x)^{(n)}\}_{n=1}^{+\infty}$ *converges,*

(2) $\lim_{n\to+\infty} \sigma(x)^{(n)} = \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x)$.

If the argumentation graph contains no higher-order attacks, the equations in Definition 20 simplify as follows.

Theorem 18. *Let* Hyb *be a hybrid semantics based on the evaluation methods* $(\mathbf{M}, \mathbf{M}')$*, where* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h}\rangle$*,* $\mathbf{M}' = \langle \mathsf{f}', \mathsf{g}', \mathsf{h}'\rangle$*, and* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w}\rangle \in \mathsf{AF}$ *such that* $\forall \alpha \in \mathcal{R}$*,* $\mathbf{t}(\alpha) \in \mathcal{A}$*. For any* $x \in \mathcal{A} \cup \mathcal{R}$*, the following hold:*

- *If* $x \in \mathcal{R}$*, then* $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x) = \mathbf{w}(x)$.
- *If* $x \in \mathcal{A}$*, then* $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x) = \mathsf{f}\Big(\mathbf{w}(x), \mathsf{g}\Big(\mathsf{h}\Big(\mathbf{w}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{w}(\alpha_k), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_n))\Big)\Big)\Big)$,
  where $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots \alpha_k\}$.

From Theorems 4 and 18, it follows that the extended gradual semantics defined in Section 5.2 and the hybrid semantics coincide in the case of basic argumentation frameworks (i.e., there are no higher-order attacks). This naturally holds when both semantics rely on the same evaluation method for assessing arguments.

Corollary 1. *Let* $\mathbf{S}_1 \in \mathbf{ES}$ *be an extended gradual semantics based on an evaluation method* $\mathbf{M}$*, and let* $\mathbf{S}_2$ *be a hybrid semantics based on* $(\mathbf{M}', \mathbf{M}'')$*. Let* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w}\rangle \in \mathsf{AF}$ *be such that* $\forall \alpha \in \mathcal{R}$*,* $\mathbf{t}(\alpha) \in \mathcal{A}$*.*

$$\text{If } \mathbf{M} = \mathbf{M}', \text{ then } \forall x \in \mathcal{A} \cup \mathcal{R},\ \mathbf{V}_{\mathbf{H}}^{\mathsf{S}_1}(x) = \mathbf{V}_{\mathbf{H}}^{\mathsf{S}_2}(x).$$

Before analysing the behaviour of hybrid semantics with respect to the principles presented in Section 4, let us first introduce a new relation between aggregation operators h.

Definition 21. *Let* $\mathsf{h}_1, \mathsf{h}_2$ *be such that for* $i = 1, 2$, $\mathsf{h}_i : [0,1]\times[0,1] \longrightarrow [0,1]$*. We write* $\mathsf{h}_1 \succeq \mathsf{h}_2$ *iff* $\forall (x,y) \in [0,1]^2$, $\mathsf{h}_1(x,y) \geq \mathsf{h}_2(x,y)$.

The properties of hybrid semantics are summarized in the following result.

Theorem 19. *Let* Hyb *be a hybrid semantics based on the evaluation methods* $(\mathbf{M}, \mathbf{M}')$*, where* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h}\rangle$ *and* $\mathbf{M}' = \langle \mathsf{f}', \mathsf{g}', \mathsf{h}'\rangle$.

- Hyb *satisfies Anonymity, Independence, Directionality, Equivalence, Maximality, Neutrality, Weakening and Proportionality.*
- *If* f *(resp.* f'*) is positive, then* Hyb *satisfies A- (resp. R-) Resilience.*
- *If* g *(resp.* g'*) is strictly monotonic, then:*
  - Hyb *satisfies A- (resp. R-) Counting.*
  - *If* h *(resp.* h'*) is strictly monotonic on the first variable (F1), then* Hyb *satisfies A- (resp. R-) Attack-sensitivity.*
  - *If* h *(resp.* h'*) is strictly monotonic on the second variable (F2), then* Hyb *satisfies A- (resp. R-) Reinforcement.*
- *If* h *(resp.* h'*) is symmetric,* $\mathsf{h}(x,y) = \mathsf{h}(y,x)$*, then* Hyb *satisfies A- (resp. R-) Local-Equality.*
- *If* $\mathsf{f} = \mathsf{f}'$*,* $\mathsf{g} = \mathsf{g}'$*,* $\mathsf{h}' \succeq \mathsf{h}$ *(resp.* $\mathsf{h} \succeq \mathsf{h}'$*), then* Hyb *satisfies A-Precedence (resp. R-Precedence).*

The instances of this new family satisfy all the mandatory properties like Anonymity, Independence, Directionality, Maximality and Weakening. Some of them satisfy *A-Precedence* while others *R-Precedence*. Depending on the application at hand, one chooses the appropriate base semantics.

Let us now discuss two instances of the family. The first semantics, denoted Hyb_1, uses the same functions f and h and distinct functions g.

Definition 22. *We define* Hyb_1 *as a hybrid semantics based on the evaluation methods* $(\mathbf{M}_1, \mathbf{M}_2)$*, where* $\mathbf{M}_1 = \langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h}_6\rangle$ *and* $\mathbf{M}_2 = \langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h}_6\rangle$.

Example 2 (Cont) *Table 8 presents the strengths of all elements in* $\mathbf{H}_1$ *under the hybrid semantics* Hyb_1*. The colored cells indicate the non-trivial cases, namely the elements that are attacked in* $\mathbf{H}_1$*.*

Table 8. Strengths of arguments and attacks in Example 2. Case of Hyb_1.

	a	b	c	d	g	π	μ	δ_1	δ_2	γ	ε	λ
Hyb_1	*0.333*	*0.8*	*0.240*	*0.204*	*0.8*	*0.4*	*0.2*	*0.7*	*0.7*	*0.446*	*0.8*	*0.332*

The next result shows that the semantics Hyb_1 satisfies *R-Precedence* and violates *Equality* and *A-Precedence.* So, Hyb_1 is more harmful with arguments than it is with attacks.

Theorem 20. *The hybrid semantics* Hyb_1 *satisfies R-Precedence and violates Equality and A-Precedence.*

The following example illustrates how Hyb_1 penalizes arguments more strongly than attacks.

Example 4. *Let us consider the wHO-AF* $\mathbf{H}_3$ *depicted in the figure below.*

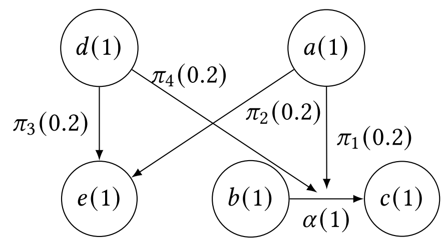


Fig. 12. Representation of $\mathbf{H}_3$

Consider the argument e *and the attack* α*. They have the same initial weights, are attacked by the same arguments, and the attacks on both targets have equal strengths. It is easy to check that:*

- $\mathbf{V}^{\mathsf{Hyb}_1}_{\mathbf{H}_3}(e) = \frac{1}{1+2\mathsf{h}_6(0.2,1)} = 0.454$,
- $\mathbf{V}^{\mathsf{Hyb}_1}_{\mathbf{H}_3}(\alpha) = \frac{1}{1+\mathsf{h}_6(0.2,1)} = 0.625$.

Then, $\mathbf{V}^{\mathsf{Hyb}_1}_{\mathbf{H}_3}(e) < \mathbf{V}^{\mathsf{Hyb}_1}_{\mathbf{H}_3}(\alpha)$.

The second hybrid semantics, denoted by Hyb_2, relies on evaluation methods that differ only in their h function. Specifically, one employs h_1 (the minimum aggregation operator), while the other uses h_6 (the modified average).

Definition 23. *We define* Hyb_2 *as a hybrid semantics based on the evaluation methods* $(\mathbf{M}_1, \mathbf{M}_2)$*, where* $\mathbf{M}_1 = \langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h}_1 \rangle$ *and* $\mathbf{M}_2 = \langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h}_6 \rangle$.

Example 2 (Cont) *The outputs of* Hyb_2 *are summarized in Table 9.*

Table 9. Strengths of arguments and attacks in Example 2. Case of Hyb_2.

	a	b	c	d	g	π	μ	δ_1	δ_2	γ	ε	λ
Hyb_2	*0.416*	*0.8*	*0.296*	*0.231*	*0.8*	*0.4*	*0.2*	*0.7*	*0.7*	*0.457*	*0.8*	*0.259*

We show that, unlike Hyb_1, the hybrid semantics Hyb_2 is more harmful to attacks than to arguments. Indeed, it satisfies A-Precedence while violating Equality and R-Precedence.

Theorem 21. *The hybrid semantics* $\mathtt{Hyb}_2$ *satisfies A-Precedence and violates Equality and R-Precedence.*

Example 4 (Cont) *Consider again the wHO-AF* $\mathbf{H}_3$ *depicted in Figure 12 and let us focus the argument* e *and the attack* α*, which fulfill the conditions of the three principles: A-Precedence, R-Precedence and Equality.*

$$\mathbf{V}^{\mathtt{Hyb}_2}_{\mathbf{H}_3}(e) = \frac{1}{1+2\mathtt{h}_1(0.2,1)} = \frac{1}{1+2\mathtt{min}(0.2,1)} = 0.714,$$

$$\mathbf{V}^{\mathtt{Hyb}_2}_{\mathbf{H}_3}(\alpha) = \frac{1}{1+2\mathtt{h}_6(0.2,1)} = \frac{1}{1+2((0.2+1)/2)} = 0.454.$$

This semantics is clearly less damaging for arguments than for attacks.

Table 5 provides an overview of all the properties satisfied or violated by the two hybrid semantics, $\mathtt{Hyb}_1$ and $\mathtt{Hyb}_2$.

To sum up, the paper shows that the rich structure of wHO-AF graphs introduces novel challenges that require specific treatment. It identifies two main issues: (i) whether a semantics should treat arguments and attacks equally or differently, and (ii) how a semantics evaluates the overall strength of an attack, specifically how it combines the strength of the attack itself with that of its source. These challenges motivated the introduction of four new principles: Equality, A-Precedence, R-Precedence, and Local Equality. Based on these principles, a catalogue of new semantics is defined in Sections 5 and 6, each providing different formal guarantees.

Gradual semantics grounded in RAF semantics and pure gradual semantics satisfy the Equality principle. Semantics based on AFRA semantics, as well as some hybrid semantics such as $\mathtt{Hyb}_2$, satisfy A-Precedence while violating Equality. Other hybrid semantics, including $\mathtt{Hyb}_1$, satisfy R-Precedence and also violate Equality. The Local Equality principle is violated by most semantics derived from extension semantics, but it is satisfied by gradual semantics based on evaluation methods that employ a symmetric aggregation operator $\mathtt{h}$.

7 Empirical Analysis

One of our primary future objectives is to analyze the theoretical complexity of the proposed gradual semantics. In the meantime, to provide insights into the practical computability of the proposed semantics, we conducted an empirical study investigating the behavior of some novel semantics on benchmark graphs. The analysis focuses on two key aspects of iterative semantics: the *number of iterations* required for convergence and the *execution time.*

Semantics. The choice of implemented semantics was guided by the objective of ensuring a fair comparison with existing work. Specifically, we selected semantics based on $\mathtt{Mbs}$ and $\mathtt{Hbs}$—two semantics designed for basic argumentation frameworks and implemented in (Amgoud, Doder, and Vesic 2022). The goal is to see how the introduction of higher-order attacks affects both the number of iterations required for convergence and the computation time. To this end, we implemented eight semantics, namely:

- $\mathtt{eHbs}^{\mathtt{h}}$ with $\mathtt{h} \in \mathtt{h}_1, \mathtt{h}_5, \mathtt{h}_6$;
- $\mathtt{eMbs}^{\mathtt{h}}$ with $\mathtt{h} \in \mathtt{h}_1, \mathtt{h}_5, \mathtt{h}_6$;
- $\mathtt{Hyb1}$, which is based on $\mathbf{M}_1 = \langle \mathtt{f}_{\mathtt{frac}}, \mathtt{g}_{\mathtt{sum}}, \mathtt{h}_6 \rangle$ and $\mathbf{M}_2 = \langle \mathtt{f}_{\mathtt{frac}}, \mathtt{g}_{\mathtt{max}}, \mathtt{h}_6 \rangle$;
- $\mathtt{Hyb2}$, which is based on $\mathbf{M}_1 = \langle \mathtt{f}_{\mathtt{frac}}, \mathtt{g}_{\mathtt{sum}}, \mathtt{h}_1 \rangle$ and $\mathbf{M}_2 = \langle \mathtt{f}_{\mathtt{frac}}, \mathtt{g}_{\mathtt{sum}}, \mathtt{h}_6 \rangle$.

Algorithms for Semantics Computation. Our objective is not to develop the fastest possible solvers, but rather to investigate the average number of iterations required to compute the strengths of elements, as well as the general evolution of the execution time when the number of arguments (respectively attacks and cycles) increases. To this end, we implemented eight naive algorithms following the general scheme described in Theorems 8 and 16. More precisely, these algorithms iteratively compute the values of all elements in a wHO-AF, where the value of an element at a given iteration depends on the values of its attackers at the previous iteration. Initially, each element is assigned its initial weight. The algorithms then iteratively update the values of the elements until the difference between two successive states becomes negligible; more precisely, the process stops when this difference falls below a fixed threshold ϵ for all elements of the wHO-AF. For illustration purpose, we provide below Algorithm 1, which computes the strengths of elements using the gradual semantics $\mathsf{eHbs}^{\mathsf{h}_1}$ (where $\mathsf{h}_1 = \min$). Note that the complete code can be found in a public web repository (see (Amgoud, Doder, and Lagasquie-Schiex 2026)).

Algorithm 1 Naive algorithm for the computation of $\mathsf{eHbs}^{\mathsf{h}_1}$

Require: $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$, a wHO-AF
Require: ϵ, a threshold

Start($timer$)
for every element $e \in \mathcal{A} \cup \mathcal{R}$ **do** ▷ initialization of the *current* strength for e
  $\mathbf{V}(e) \longleftarrow \mathbf{w}(e)$
end for
$ni \longleftarrow 0$ ▷ ni represents the number of iterations
$continue \longleftarrow true$ ▷ *continue* means that the computation of strengths is not finished
while $continue$ **do**
  for every element $e \in \mathcal{A} \cup \mathcal{R}$ **do** ▷ Computation of the *new* strength of e (so $\mathbf{V}'(e)$) using the ▷ *current* strength ($\mathbf{V}$) of the attacks targeting e and of their sources
    $\mathbf{V}'(e) \longleftarrow \frac{\mathbf{w}(e)}{1+\sum_{\alpha_i \in \mathtt{Att}(e)} \min(\mathbf{V}(\alpha_i), \mathbf{V}(\mathbf{s}(\alpha_i)))}$
  end for
  if there exists at least one element $e \in \mathcal{A} \cup \mathcal{R}$ such that $|\mathbf{V}(e) - \mathbf{V}'(e)| > \epsilon$ **then** ▷ the computation of strengths is not finished
    $ni \longleftarrow ni + 1$
    for any $e \in \mathcal{A} \cup \mathcal{R}$ **do**
      $\mathbf{V}(e) \longleftarrow \mathbf{V}'(e)$
    end for
  else ▷ the computation of strengths is finished
    $continue \longleftarrow false$
    Stop($timer$)
  end if
end while
return $\mathbf{V}$ (final strength of the elements), ni (final number of iterations) and $timer$ (final computation time)

Benchmark Generation. While several benchmarks of argumentation graphs have been proposed in the literature (e.g., (Costa Pereira, Dragoni, et al. 2016; Thimm and Villata 2017)) to evaluate the performance of extension-based semantics and certain gradual semantics, none of them take into account higher-order and weighted attacks. Consequently, to empirically evaluate the performance of the eight proposed semantics, we randomly generated **eight types** of dedicated benchmarks, which differ with respect to the five parameters listed below.

- the number of arguments (nb of Arg);
- the minimum number of attacks explicitly included in the wHO-AF (this value is necessarily smaller than the square of the number of arguments) (nb of Att);

- the minimum percentage of higher-order attacks among all attacks (% of HO-att);
- the minimum number of cycles explicitly introduced in the wHO-AF (min nb of cycles);
- the maximum size of these cycles (max size of cycles).

Table 10 specifies, for each type, the values assigned to the corresponding parameters:

Table 10. Benchmark generation parameters

Type of benchmark	nb of Arg	nb of Att	% of HO-att	nb of cycles	size of cycles
0	10	15	0	0	0
1	10	15	20	2	3
2	100	150	20	10	5
3	1000	1500	30	100	10
4	10000	20000	50	1000	20
5	100000	200000	0	0	0
6	100000	200000	60	1000	20
7	100000	200000	60	1000	1000

For each benchmark type, 20 wHO-AFs were generated using Algorithm 2, which invokes Algorithms 3 to 6. The random number generator used in these algorithms produces integer or floating-point values within specified ranges according to a uniform distribution.

Algorithm 2 proceeds as follows:

(1) Generate a set of argument names according to the specified parameters, and assign a random weight to each argument. This step is described in Algorithm 3.
(2) Generate a set of attack names according to the specified parameters, and assign a random weight to each attack. This step is described in the first loop (the “for” loop) of Algorithm 4.
(3) For each attack, randomly select a source and a target while ensuring the absence of duplicates (i.e., no two attacks share both the same source and target), and respecting the required percentage of higher-order attacks. This step is described in the second loop (the “while” loop) of Algorithm 4.
(4) Create cycles by possibly adding new attacks (together with their corresponding weights, sources, and targets), according to the specified parameters controlling the number and size of cycles, while again avoiding duplicates. This step is described in Algorithm 5. Furthermore, the selection of the elements composing each cycle is detailed in Algorithm 6.

The python implementation of these algorithms can be found in a public web repository (see (Amgoud, Doder, and Lagasquie-Schiex 2026)).

Algorithm 2 Benchmark generator (using the five parameters described previously)

Require: *nbArg*, the number of arguments
Require: *nbAtt*, the minimum number of attacks explicitly included in the wHO-AF
Require: *pHOAtt*, the minimum percentage of higher-order attacks among all attacks
Require: *minNbCyc*, the minimum number of cycles explicitly introduced in the wHO-AF
Require: *maxSizeCyc*, the maximum size of these cycles

$\mathcal{A} \leftarrow \emptyset$
$\mathcal{R} \leftarrow \emptyset$
$\mathbf{s} \leftarrow \emptyset$
$\mathbf{t} \leftarrow \emptyset$
$\mathbf{w} \leftarrow \emptyset$
$\mathbf{H} \leftarrow (\mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w})$
CreateSetOfArguments(*nbArg*, **H**) ▷ Cf. Algorithm 3
CreateSetOfAttacks(*nbAtt*, *pHOAtt*, **H**) ▷ Cf. Algorithm 4
CreateCycles(*minNbCyc*, *maxSizeCyc*, **H**) ▷ Cf. Algorithm 5
return $\mathbf{H} = (\mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w})$

Algorithm 3 Creation of the set of arguments (function CreateSetOfArguments)

Require: *nbArg*, the number of arguments
Require: **H**, an empty wHO-AF

for $i \in [1, nbArg]$ **do**
  CreateArgumentName(i) ▷ Creation of a new argument name, a_i
  $\mathcal{A} \leftarrow \mathcal{A} \cup \{a_i\}$ ▷ Addition of this new name to the set
  $\mathbf{w}(a_i) \leftarrow$ RandomFloatInRange$(0, 1)$ ▷ The initial weight of a_i is a random float ▷ chosen uniformly in the range $[0, 1]$
end for
return $\mathbf{H} = (\mathcal{A}, \emptyset, \emptyset, \emptyset, \mathbf{w})$

Algorithm 4 Creation of the set of attacks (function CreateSetOfAttacks)

Require: $nbAtt$, the minimum number of attacks explicitly included in the wHO-AF
Require: $pHOAtt$, the minimum percentage of higher-order attacks among all attacks
Require: $\mathbf{H} = (\mathcal{A}, \emptyset, \emptyset, \emptyset, \mathbf{w})$, a wHO-AF containing arguments but no attack

```
for i ∈ [1, nbAtt] do
    CreateAttackName(i)                     ▷ Creation of a new attack name, α_i
    R ← R ∪ {α_i}                           ▷ Addition of this new name to the set
    w(α_i) ← RandomFloatInRange(0, 1)       ▷ The initial weight of α_i is updated with a random float
                                            ▷ chosen uniformly in the range [0, 1]
end for
i ← 1
while i ≤ nbAtt do                          ▷ Definition of each attack in terms of source and target
    s(α_i) ← RandomSelectionInSet(A)        ▷ selection of an argument as the source of α_i
    if i < (nbAtt * pHOAtt/100) then        ▷ α_i is a higher-order attack, its target is an attack
        t(α_i) ← RandomSelectionInSet(R)
    else                                    ▷ α_i is a first-order attack, its target is an argument
        t(α_i) ← RandomSelectionInSet(A)
    end if
    if (s(α_i), t(α_i)) is not a duplicate in H then  ▷ no another attack having the same source and target as α_i
        i ← i + 1
    end if
end while
return H = (A, R, s, t, w)
```

Algorithm 5 Creation of cycles (function CreateCycles)

Require: $minNbCyc$, the minimum number of cycles explicitly introduced in the wHO-AF
Require: $maxSizeCyc$, the maximum size of these cycles
Require: $\mathbf{H} = (\mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w})$, a wHO-AF already containing arguments and attacks

```
for c ∈ [1, minNbCyc] do
    sizeCyc ← RandomIntInRange(1, maxSizeCyc)        ▷ Choice of the size for the current cycle
    cycle ← SelectElemForCycle(sizeCyc, H)           ▷ Cf Algorithm 6
    for j ∈ [1, Length(cycle)[ do
        if cycle[j] ∈ A and (cycle[j], cycle[j + 1]) is not a duplicate in H then
                                 ▷ This pair corresponds to a new attack that must be created
            nbAtt ← nbAtt + 1
            CreateAttackName(nbAtt)                  ▷ Creation of the attack name
            R ← R ∪ {α_nbAtt}                        ▷ Addition to the attack name set
            w(α_nbAtt) ← RandomFloatInRange(0, 1)    ▷ Selection of the weight for this attack
            s(α_nbAtt) ← cycle[j]                    ▷ Update of the attack source function
            t(α_nbAtt) ← cycle[j + 1]                ▷ Update of the attack target function
        end if
    end for
end for
return H = (A, R, s, t, w)
```

Algorithm 6 Selection of the elements belonging to a cycle (function SELECTELEMFORCYCLE)

Require: *sizeCyc*, the size of the cycle
Require: $\mathbf{H} = (\mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w})$, a wHO-AF already containing arguments and attacks

```
nbAtt ← |R|                                          ▷ The number of attacks existing in H
x ← RANDOMSELECTIONINSET(A)                          ▷ Selection of the first element for this cycle
cycle ← cycle ∪ {x}                                  ▷ A cycle is a list of elements (arguments or attacks)
sizeCyc ← sizeCyc − 1
while sizeCyc > 1 do
    x ← RANDOMSELECTIONINSET(A ∪ R)                  ▷ Selection of an element for this cycle
    while sizeCyc > 1 and x ∈ R do                   ▷ x is an attack, its target will be the next element of the cycle
        cycle ← cycle ∪ {x}
        sizeCyc ← sizeCyc − 1
        x ← GETTARGET(x)
    end while
    if sizeCyc > 1 and x ∈ A then                    ▷ x is an argument, the next element will be randomly chosen
        cycle ← cycle ∪ {x}
        sizeCyc ← sizeCyc − 1
    end if
end while
if the last element x in the cycle is an attack then ▷ For closing the cycle we must create a new attack with
                                                     ▷ the same source and weight, but a different target
    nbAtt ← nbAtt + 1
    CREATEATTACKNAME(nbAtt)
    R ← R ∪ {α_nbAtt}
    w(α_nbAtt) ← w(x)
    s(α_nbAtt) ← s(x)
    t(α_nbAtt) ← cycle[0]
    cycle ← cycle \ {x} ∪ {α_nbAtt}                  ▷ In the cycle, the last occurrence of x is replaced by α_nbAtt
end if
cycle ← cycle ∪ {cycle[0]}                           ▷ The last element in the cycle is the first one
return cycle
```

Implementation. The benchmark generation procedure and the eight semantics were implemented in Python 3.10 and executed on a standard laptop (Dell Latitude 5510, 16 GiB RAM, Intel Core i5-10210U CPU at 1.60 GHz × 8) running Ubuntu 22.04.

Regarding benchmark generation, Table 11 summarizes the 160 generated wHO-AFs. For each of the eight benchmark types, we report the minimum, maximum, and average values of the considered parameters across the 20 generated wHO-AFs. Due to the stochastic nature of the generation process, the final number of attacks in a wHO-AF may significantly exceed the initial parameter values specified in Table 10. This is mainly because additional attacks may be introduced during the construction of cycles. Conversely, depending on the evolving graph structure and the imposed constraints (e.g., avoiding duplicate attacks), it is not always possible to achieve the desired number of cycles or the targeted maximum cycle size.

Regarding semantics computation, the threshold value ϵ (see Algorithm 1) was set to 10^{-12}, which proved sufficient to ensure convergence and complete the computation of all semantics without timeout or execution failure. Each of the eight semantics was applied to compute the strengths of elements in all generated wHO-AFs. For this computation, the following performance metrics were recorded:

- *Execution time* (in seconds): minimum, maximum, and average values;
- *Memory usage* (in KiB): minimum, maximum, and average values;[7]
- *Number of iterations* required for convergence: minimum, maximum, and average values.

Table 11. Description of the generated benchmarks

Type of benchmark	nb of Arg	nb of Att (min,max,avg)	min % of HO-att	min nb of cycles (min,max,avg)	max size of cycles (min,max,avg)
0	10	(15, 15, 15)	0	(0, 0, 0)	(0, 0, 0)
1	10	(17, 20, 18)	20	(1, 2, 2)	(1, 3, 2)
2	100	(169, 185, 177)	20	(10, 10, 10)	(4, 5, 4.6)
3	1000	(2003, 2081, 2042)	30	(99, 100, 100)	(10, 10, 10)
4	10000	(30131, 30827, 30454)	50	(998, 1000, 999)	(20, 20, 20)
5	100000	(200000, 200000, 200000)	0	(0, 0, 0)	(0, 0, 0)
6	100000	(210253, 210821, 210549)	60	(999, 1000, 1000)	(20, 20, 20)
7	100000	(684464, 708477, 696173)	60	(995, 1000, 997)	(997, 1000, 999)

Results on Semantics Computation. Tables 12, 13, and 14 report the metric values obtained for Benchmarks 1, 6, and 7, respectively. The results show that even for large-scale wHO-AFs—containing up to 100000 arguments, 710000 attacks, and 1000 cycles, with cycle sizes reaching up to 1000 elements—the computation of *all* gradual semantics, whether hybrid or not, consistently converged in fewer than 30 iterations. Execution times remained below 83 seconds in the worst case and below 42 seconds in the best case, while average memory consumption was approximately 110 GiB. Furthermore, no significant differences were observed among the tested semantics with respect to either execution time or memory usage. The experiments also indicate that, although the number and size of cycles in a wHO-AF affect execution time and memory consumption, they do not influence the number of iterations required for convergence. This might suggest that these structural properties impact the computational cost of each iteration rather than the convergence process itself. It is worth recalling that both the benchmark generator and the eight algorithms computing the semantics are based on straightforward, non-optimized implementations. In particular, no effort was made to optimize data structures or improve computational efficiency. Such optimizations, left for future work, could substantially improve performance.

Some empirical studies on gradual semantics for basic argumentation frameworks were previously conducted in (Amgoud, Doder, and Vesic 2022). Our results are consistent with those earlier findings:

- the number of iterations never exceeded 30 (compared with 20 in (Amgoud, Doder, and Vesic 2022)), even for basic graphs containing 100,000 arguments;
- the number of iterations remained stable regardless of graph size;
- no execution resulted in a timeout or crash;
- all algorithms converged successfully;
- the worst-case execution time was approximately 83 seconds, substantially lower than the 2,250 seconds (37 minutes) reported in (Amgoud, Doder, and Vesic 2022). This discrepancy may stem from several factors, including differences in programming language (Java in (Amgoud, Doder, and Vesic 2022) versus Python 3.10 in our work), operating system (CentOS versus Ubuntu 22.04), and hardware configuration (the

[7] 1 KiB = 1024 bytes. Note that KiB and KB are distinct units: 1 KB = 1000 bytes.

experiments in (Amgoud, Doder, and Vesic 2022) were conducted on dual Intel Xeon E5-2643 processors running at 3.3 GHz with 32 GB RAM).

Table 12. Measured metrics for benchmark type 0

Gradual Semantics	time (in seconds) (min,max,avg)	used place (in KiB) (min,max,avg)	nb of iterations before convergence (min,max,avg)
Type of Benchmark = 1			
eHbs^{h_1}	(0.00034, 0.00115, 0.00075)	(3665, 4129, 3899)	(5, 23, 15)
eHbs^{h_6}	(0.00043, 0.00092, 0.00070)	(3737, 4177, 3957)	(8, 17, 12)
eHbs^{h_5}, $w = 0.5$	(0.00047, 0.00090, 0.00074)	(3805, 4245, 4025)	(8, 17, 12)
eMbs^{h_1}	(0.00019, 0.00124, 0.00063)	(3642, 4106, 3880)	(4, 28, 13)
eMbs^{h_6}	(0.00030, 0.00096, 0.00066)	(3714, 4154, 3934)	(5, 18, 12)
eMbs^{h_5}, $w = 0.5$	(0.00032, 0.00105, 0.00071)	(3807, 4247, 4027)	(5, 18, 12)
Hyb_1	(0.00039, 0.00119, 0.00077)	(3702, 4190, 3938)	(5, 23, 15)
Hyb_2	(0.00047, 0.00082, 0.00069)	(3751, 4191, 3974)	(8, 17, 12)

Table 13. Measured metrics for benchmark type 6

Gradual Semantics	time (in seconds) (min,max,avg)	used place (in KiB) (min,max,avg)	nb of iterations before convergence (min,max,avg)
Type of Benchmark = 6			
eHbs^{h_1}	(18.106, 27.969, 23.829)	(30863109, 30921669, 30895080)	(18, 27, 23)
eHbs^{h_6}	(14.451, 22.556, 18.496)	(30863277, 30921861, 30895243)	(13, 21, 17)
eHbs^{h_5}, $w = 0.5$	(14.849, 23.402, 19.142)	(30863345, 30921929, 30895311)	(13, 21, 17)
eMbs^{h_1}	(17.759, 28.637, 24.667)	(30863086, 30921646, 30895057)	(18, 29, 25)
eMbs^{h_6}	(15.044, 22.519, 19.024)	(30863254, 30921838, 30895220)	(14, 21, 18)
eMbs^{h_5}, $w = 0.5$	(15.722, 23.212, 19.781)	(30863347, 30921931, 30895313)	(14, 21, 18)
Hyb_1	(18.859, 28.888, 24.617)	(30863266, 30921826, 30895248)	(18, 27, 23)
Hyb_2	(14.436, 22.319, 18.325)	(30863291, 30921875, 30895251)	(13, 21, 17)

The most significant results reported in Table 14 are also illustrated graphically for easier reading: execution times are shown in Figure 13, and the number of iterations in Figure 14. Since the data consist of triplets (minimum, maximum, and average values), we use error bars for representing the obtained data: the central point represents the average, while the bars extend from the minimum to the maximum value for each semantics.

In summary, although higher-order attacks require a more sophisticated theoretical treatment and introduce additional challenges in the definition of gradual semantics, the empirical results indicate that they do not substantially increase the computational cost of computing argument and attack strengths. In particular, the proposed semantics exhibit strong practical efficiency, with all computations completed in under two minutes even for very large and highly cyclic graphs. One possible explanation for this observed efficiency is that the

Table 14. Measured metrics for benchmark type 7

Gradual Semantics	time (in seconds) (min,max,avg)	used place (in KiB) (min,max,avg)	nb of iterations before convergence (min,max,avg)
Type of Benchmark = 7			
eHbs^{h_1}	(45.574, 73.739, 54.914)	(110175693, 110376109, 110272436)	(15, 26, 18)
eHbs^{h_6}	(42.709, 65.452, 47.410)	(110175693, 110376109, 110272363)	(14, 20, 14)
eHbs^{h_5}, $w = 0.5$	(44.596, 68.113, 49.483)	(110175761, 110376177, 110272431)	(14, 20, 14)
eMbs^{h_1}	(41.751, 82.831, 54.644)	(110175694, 110376110, 110272364)	(14, 28, 18)
eMbs^{h_6}	(44.894, 64.522, 47.570)	(110175694, 110376110, 110272364)	(14, 20, 15)
eMbs^{h_5}, $w = 0.5$	(46.279, 67.303, 49.750)	(110175763, 110376179, 110272433)	(14, 20, 15)
Hyb_1	(47.248, 82.983, 56.448)	(110175706, 110376122, 110272376)	(15, 26, 18)
Hyb_2	(42.617, 64.137, 46.750)	(110175707, 110376123, 110272377)	(14, 20, 14)

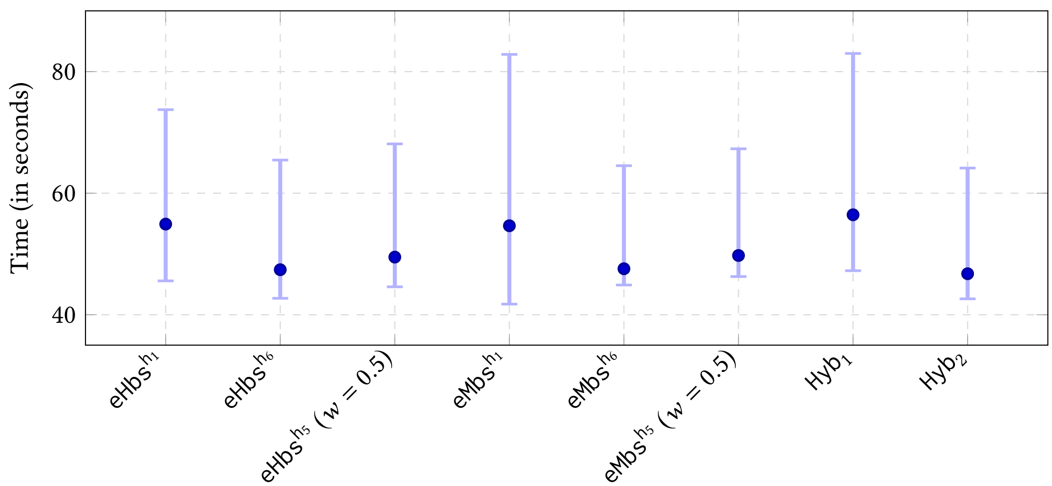


Fig. 13. Execution time by semantics (Min, Max, Average) for benchmark type 7 (extraction from Table 14)

underlying systems of equations remain structurally similar in both the basic and higher-order settings, which may preserve favorable convergence and computational properties.

These empirical observations remain preliminary. They still need to be formally validated through a rigorous theoretical analysis of the computational complexity of gradual semantics, in order to precisely characterize the impact of higher-order attacks on worst-case and average-case computational behavior.

8 Related Work

Argumentation frameworks using higher-order interactions are numerous and diverse in nature with respect to the type of interactions used (attacks only or attacks and supports), the order of these interactions (first-order, second-order, or at any level), and the form of the source or target of these interactions (one element or a set of elements). No approach defined in the literature is general enough to cover all possibilities. In this paper we

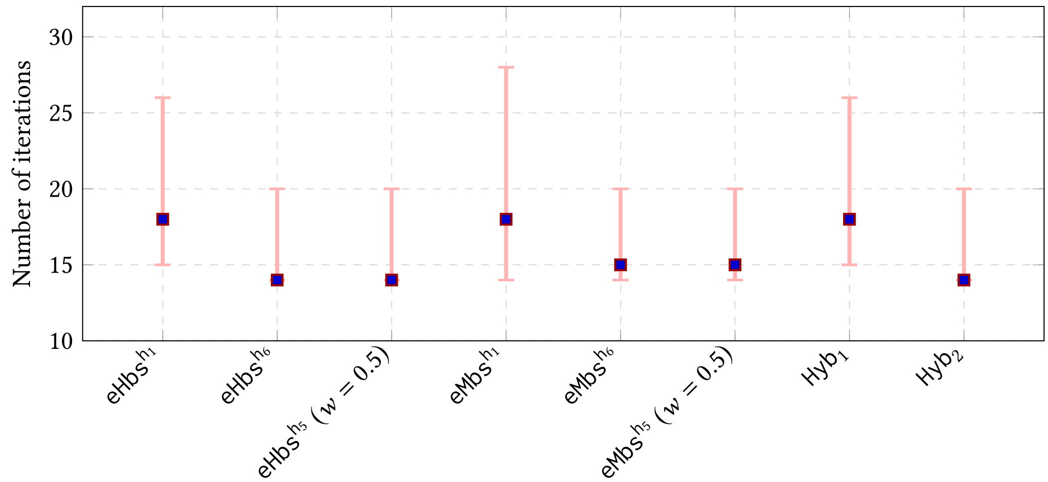


Fig. 14. Number of iterations before convergence (Min, Max, Average) by semantics for benchmark type 7 (extraction from Table 14)

focused on argumentation frameworks that use only attacks whose source is an argument and whose target is either an argument or an attack (i.e. higher-order attacks at any level). In this context, six main approaches can be distinguished:

Barringer *et al*, **in (Barringer et al. 2005):** To the best of our knowledge, this is the first and only attempt to define gradual semantics for wHO-AFs—and more generally, for frameworks in which arguments may support and/or attack either arguments or support/attack relations, and where both arguments and relations are assigned initial weights. While the authors advocated for computing the strength of elements by propagating their initial weights through the graph, they did not provide any concrete semantics. To illustrate their ideas, they used the following formula in an example: For $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathtt{AF}$, $x \in \mathcal{A} \cup \mathcal{R}$ such that $\mathtt{Att}_{\mathbf{H}}(x) = \{\pi_1, \ldots, \pi_k\}$,

$$v(x) = \mathbf{w}(x) \times \Pi_{i=1}^{n}(1 - (v(\pi_i) \times v(\mathbf{s}(\pi_i))).$$

However, this attempt has a serious drawback since the system of equations (one formula per argument or relation) may have several solutions as shown in the next example.

Example 5. *Consider in Figure 15 the wHO-AF made of two arguments a and b which attack each other (attacks denoted π and μ), with $\mathbf{w} \equiv 1$:*

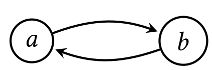


Fig. 15. Illustration of the gradual approach described in (Barringer et al. 2005)

So $v(a) = 1 - v(b)$, $v(b) = 1 - v(a)$ (as $v(\pi) = v(\mu) = 1$). The equation $v(a) + v(b) = 1$ has an infinite number of solutions.

Modgil, in (Modgil 2009b): The aim of this approach (named EAF for "Extended Argumentation Framework") is to explicitly represent the impact of the preferences between arguments in the argumentation framework by the introduction of specific attacks that target simple attacks (attacks between 2 arguments).

So in an EAF only first-order and second-order attacks can exist. Moreover no weight is taken into account and the defined semantics are either extension-based or labelling-based (see (Modgil 2009a)) with a correspondence between these two kinds of semantics (as in the case of basic AF). However, only arguments can belong to an extension whereas arguments and attacks can be labelled. This is an important difference with our work. So, even if gradual semantics for EAF could be defined using EAF extension-based semantics following the same methodology used in sections 5.1 or 6.1 (cf. Def. 11), the resulting gradual semantics for EAF will be completely different to the ones we defined (in term of satisfied *R*-properties) and so will not treat equally arguments and attacks.

Gabbay, in (D. Gabbay 2009): In his approach, Gabbay extends EAF by introducing Higher-Level Argumentation Frames (HLAF), which permit higher-order attacks at arbitrary levels but do not incorporate any weighting function **w**. He focused exclusively on labelling-based semantics. One key difference from EAF is that, in HLAF, the interactions (i.e., attacks) are also labelled. Another notable distinction is that, in certain examples, EAF and HLAF yield different outcomes. To understand the source of these discrepancies, it would be necessary to develop gradual semantics grounded in labelling-based semantics—specifically, by defining an aggregator that operates on labellings rather than on extensions.

Baroni ***et al*****, in (Baroni, Cerutti, et al. 2011):** The introduced framework in this work is called AFRA for "Argumentation Framework with Recursive Attacks" and has been motivated by the will to generalize basic AFs for a more realistic representation of reasoning situations (for instance in the area of modelling decision processes). We have already described succinctly this framework in Section 3.1.2 explaining how to obtain AFRA extensions (that contain arguments and attacks). It is interesting to note that AFRA semantics generalize AF semantics when we only consider arguments, but this does not hold for attacks since an unattacked attack does not always belong to an extension. In Section 6.1 we proposed gradual semantics from AFRA extension-based semantics and Table 5 gave the properties satisfied or not by these gradual semantics. Our study highlights the fact that these gradual semantics does not treat equally arguments and attacks.

Hanh ***et al*****, in (Hanh et al. 2011):** These authors proposed new "inductive" semantics for HLAF (based on the notion of "inductive Defence"). So it is also an extension of EAF but with higher-order attacks at any level. As in EAF, and unlike HLAF, the produced extensions contain only arguments and not attacks. So we have the following difference between our approach and Hanh *et al*'s approach: even if gradual semantics could be defined from inductive semantics (as in sections 5.1 or 6.1 with Def. 11), these resulting gradual semantics will be completely different to the ones we defined (in term of satisfied *R*-properties) and so did not treat equally arguments and attacks.

Cayrol ***et al*****, in (Cayrol, Fandinno, et al. 2020):** This work aligns with the notion of RAF (Recursive Argumentation Framework), which, like HLAF and AFRA, allows higher-order attacks at any level. The RAF framework was introduced in Section 3.1.1, and in Section 5.1, we proposed gradual semantics derived from RAF-based structural semantics. Table 5 summarizes the principles that these gradual semantics satisfy or fail to satisfy. Our analysis shows that these semantics treat arguments and attacks uniformly, thereby satisfying the Equality principle.

All these approaches have been described and compared in (Cayrol, Cohen, et al. 2021), which shows that while certain methods share commonalities from specific perspectives, none can be regarded as the most general or as fully subsuming the others. Table 15 summarizes the main differences between the six existing approaches and our own. The last column and the final four rows highlight our contributions.

Note that Definition 11, which is used to build gradual semantics from RAF structure-based semantics and AFRA extension-based semantics, could also be applied to other approaches in the literature—such as EAF and HLAF with inductive semantics. Furthermore, it would be valuable to develop an equivalent definition tailored to

Table 15. Links with wHO-AF semantics. "ND" means "Not Defined". ✓ in green (resp. × in red) means that the corresponding gradual semantics satisfy (resp. do not satisfy) the Equality property.

Approach	wHO-AF with weights	Order Max	Type of semantics	Outputs of semantics	Equality
Seminal work by Barringer *et al* (Barringer et al. 2005)	Yes	$\forall$	ND	ND	ND
EAF (Modgil 2009b)	No	2	ext-based	set of arg	ND
			lab-based	set of arg+att	ND
HLAF (D. Gabbay 2009)	No	$\forall$	lab-based	set of arg+att	ND
Inductive semantics for HLAF (Hanh et al. 2011)	No	$\forall$	ext-based	set of arg	ND
AFRA (Baroni, Cerutti, et al. 2011)	No	$\forall$	ext-based	set of arg+att	ND
RAF (Cayrol, Fandinno, et al. 2020)	No	$\forall$	ext-based	set of arg+att	ND
wHO-AF (our work)	Yes	$\forall$	struct-based gradual (RAF+Def.11)	valued arg+att	✓
	Yes	$\forall$	pure gradual (Def.14)	valued arg+att	✓
	Yes	$\forall$	ext-based gradual (AFRA+Def.11)	valued arg+att	×
	Yes	$\forall$	hybrid gradual (Def.20)	valued arg+att	×

labelling-based semantics, in order to generate gradual semantics from labelled frameworks. In that case, our approach could be extended to cover the labelled versions of both EAF and HLAF.

9 Conclusion

This paper presented the first systematic investigation of gradual semantics capable of handling (weighted) higher-order attacks. It has shown that such attacks raise two key challenges: how to treat attacks relative to arguments, and how to incorporate their strength into the evaluation of argument strength. To address these issues, we introduced four novel principles, each proposing a distinct strategy for tackling the two challenges. These principles motivated the definition of four families of semantics—two derived from the AFRA and RAF extension-based semantics, and two purely gradual ones. We then conducted a comprehensive analysis of all semantics with respect to sixteen principles, thereby establishing a unified theoretical basis for comparing them and providing the first formal account of the differences between AFRA and RAF semantics. The results highlight fundamental divergences between these (AFRA, RAF) frameworks.

The final outcome of this work is a catalogue of semantics, each offering distinct formal guarantees and behavioral characteristics. Depending on the requirements of a given application, the most appropriate semantics can be selected accordingly. Figure 16 provides a synthesis of this catalogue. Given a weighted higher-order argumentation framework **H**, it distinguishes two main branches of semantics: those that treat arguments and attacks equally, and those that treat them differently. It also shows that AFRA semantics modify the input **H**, whereas all others operate directly on the initial framework. Moreover, some semantics are defined only for unweighted graphs, while others apply to the richer wHO-AF setting. Finally, the branch of semantics violating Equality includes those that satisfy either A-Precedence or R-Precedence.

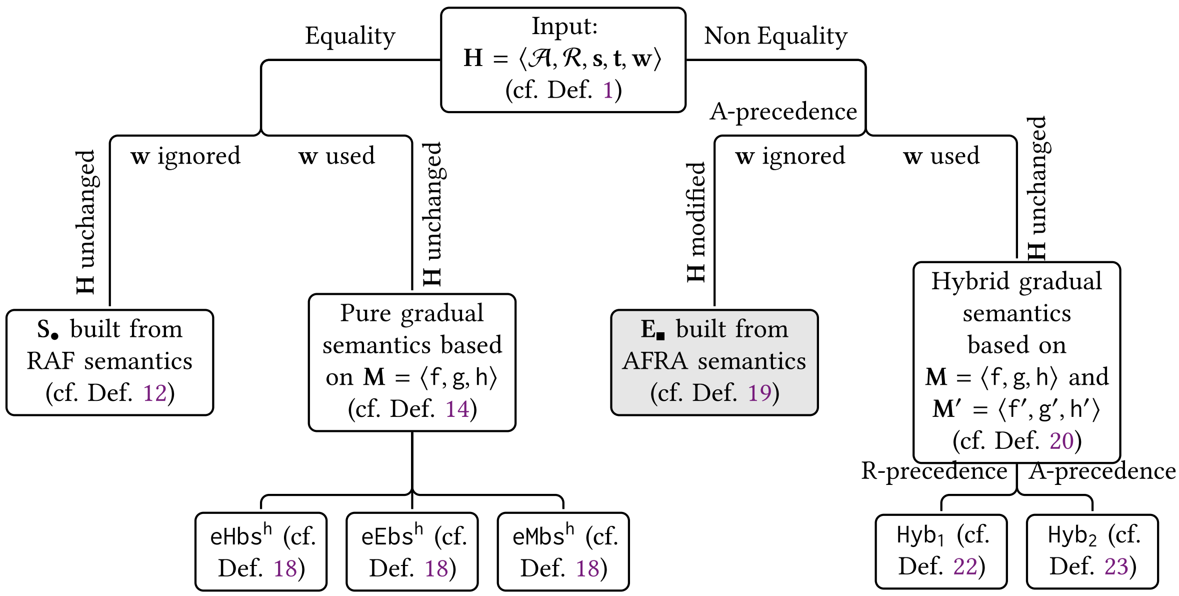


Fig. 16. Synthesis of the defined gradual semantics. The grey box indicates the only gradual semantics constructed on a modified input.

In addition to the theoretical evaluation of the new semantics, we conducted a preliminary empirical assessment of several gradual semantics. The results indicate that the approach is efficient even for large and complex frameworks, and that the inclusion of higher-order attacks does not significantly increase computation time, which is somehow unsurprising, given that some gradual semantics are defined similarly in both the basic and higher-order settings.

This work opens several promising directions for future research. A primary objective is to analyze the theoretical complexity of gradual semantics. Another line of future research concerns the identification of principles characterizing AFRA semantics, since they are the only semantics that violate R-Maximality as a consequence of modifying the input weighted argumentation framework. We also plan to identify subfamilies of semantics that satisfy specific subsets of principles, and to investigate semantics capable of handling frameworks in which attacks may originate from other attacks or from sets of elements. In addition, we intend to develop a more comprehensive and optimized implementation, and to explore applications of the proposed semantics in various domains, including the explainability of machine learning models.

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A Proofs

In this appendix, the proofs are presented following the order of the paper, with the exception of Theorems 3 and 15, which are treated jointly.

THEOREM 1. *The following properties hold.*

- *Equality $\Longleftrightarrow$ A-precedence and R-precedence.*
- *The set of all above-defined principles is compatible.*

PROOF. Let S be a gradual semantics, $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $x \in \mathcal{A}$ and $y \in \mathcal{R}$ such that:

- $\mathbf{w}(x) = \mathbf{w}(y)$, $\quad \mathtt{Sr}(x) = \mathtt{Sr}(y)$ and
- $\forall z \in \mathtt{Sr}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$.

If S satisfies Equality, then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$, hence $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) \geq \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) \leq \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$, which means that S satisfies A-Precedence and R-Precedence.

Moreover if S satisfies A-Precedence and R-Precedence, then $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) \geq \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) \leq \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$; this implies that $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$; and so S satisfies Equality.

To show that a set of principles is compatible, it is sufficient to find one gradual semantics that satisfies all the elements of the set. Theorem 15 shows that the semantics $\mathsf{Hbs}^{\mathsf{h}_6}$ (where h_6 is the modified average m-avg.) satisfies all the introduced principles (see blue line in Table 5). □

PROPERTY 1. *For fixed* $\mathbf{H}$ *and* $\mathbf{X}$*, there is a unique aggregator on* $\mathbf{X}$.

PROOF. The property follows straightforwardly from the uniqueness of the scale $\{0, \alpha, \beta, 1\}$ and Definition 11. □

THEOREM 2. *For any* $\bullet \in \{c, p, g, s\}$, $\mathbf{S}_{\bullet}$ *is a gradual semantics.*

PROOF. The result follows from Property 1. □

Theorem 3. *Let* $\mathbf{S}_\bullet$ *be such that* $\bullet \in \{c, p, g, s\}$. *Table 5 gives the principles satisfied/violated by* $\mathbf{S}_\bullet$.

Theorem 15. *Let* $\mathbf{E}_\blacksquare$ *be such that* $\blacksquare \in \{c, p, g, s\}$. *Table 5 gives the principles satisfied/violated by* $\mathbf{E}_\blacksquare$

Remark: The following proof combines the proofs of Theorems 3 and 15. The former concerns the RAF semantics while the latter concerns the AFRA semantics.

Proof. Let us analyse each principle for $\mathbf{S}_\bullet/\mathbf{E}_\blacksquare$ semantics.
Recall that:

- $\texttt{Struc}_\bullet(\mathbf{H})$ (resp. $\texttt{Ext}_\blacksquare(\mathbf{H})$) denotes the set of RAF structures (resp. AFRA extensions) of $\mathbf{H}$ for $\bullet/\blacksquare \in \{g, c, p, s\}$ (see Def. 5 and 7);
- $\mathbf{S}_\bullet$ (resp. $\mathbf{E}_\blacksquare$) denotes the gradual semantics defined from the aggregator $\mathcal{V}^{\mathbf{X}}_{\mathbf{H}}$ (see Def. 11) using $\mathbf{X} = \texttt{Struc}_\bullet(\mathbf{H})$ – see Def. 12 – (resp. $\mathbf{X} = \texttt{Ext}_\blacksquare(\mathbf{H})$ – see Def. 19 –).

Throughout the proof, the strength of the elements x of $\mathbf{H}$ using $\mathbf{S}_\bullet$ (resp. $\mathbf{E}_\blacksquare$) will be denoted by $\mathbf{V}^\bullet_\mathbf{H}$ in place of $\mathbf{V}^{\mathbf{S}_\bullet}_\mathbf{H}$ (resp. $\mathbf{V}^\blacksquare_\mathbf{H}$ in place of $\mathbf{V}^{\mathbf{E}_\blacksquare}_\mathbf{H}$) in order to simplify the notations. Moreover, in order to agglomerate the RAF and AFRA cases where possible, the notation $\bullet/\blacksquare$ will denote the use of either RAF or AFRA semantics; for instance, $\mathbf{V}^{\bullet/\blacksquare}_\mathbf{H}(x) = v$ will denote the fact that the element x in $\mathbf{H}$ has the same value v in both cases (RAF or AFRA).

Anonymity is satisfied by the four semantics and for wHO-AF built as RAF or as AFRA. This obviously follows from the definition of these semantics for both kinds of wHO-AF (see definitions of RAF and AFRA semantics).

Independence is satisfied for $\bullet/\blacksquare \in \{g, p, c\}$ but not for $\bullet/\blacksquare = s$ in both cases (AFRA or RAF). Moreover the A-Independence and the R-Independence have the same status.

Consider the first case for RAF ($\bullet \in \{g, p, c\}$). Following definitions of RAF semantics, the $\bullet$ structures of $\mathbf{H} \oplus \mathbf{H}'$ are obtained by the cartesian product of $\texttt{Struc}_\bullet(\mathbf{H})$ (not empty) with $\texttt{Struc}_\bullet(\mathbf{H}')$ (not empty) since $\mathbf{H}$ and $\mathbf{H}'$ have no common elements. Let $x \in \mathbf{H}$. If x belongs to a structure U of $\mathbf{H}$ it also belongs to any structure of $\mathbf{H} \oplus \mathbf{H}'$ created using U. Moreover if x is defeated by a structure U of $\mathbf{H}$ it is also defeated by any structure of $\mathbf{H} \oplus \mathbf{H}'$ created using U. Consequently, $\mathbf{V}^\bullet_\mathbf{H}(x) = \mathbf{V}^\bullet_{\mathbf{H}\oplus\mathbf{H}'}(x)$.

Consider now the second case for RAF ($\bullet = s$). For this semantics, $\texttt{Struc}_\bullet(\mathbf{H})$ can be empty and this prevents the respect of the independence principle. For example, consider that $\mathbf{H}$ contains two arguments a and c, and the attack $\pi = (a, c)$, and $\mathbf{H}'$ contains only the argument b and the attack (b, b). In this case, $\texttt{Struc}_\bullet(\mathbf{H})$ contains only one stable structure $\{a, \pi\}$ and $\texttt{Struc}_\bullet(\mathbf{H}')$ is empty. So $\texttt{Struc}_\bullet(\mathbf{H} \oplus \mathbf{H}')$ is also empty. Thus we have $\mathbf{V}^\bullet_\mathbf{H}(a) = \mathbf{V}^\bullet_\mathbf{H}(\pi) = 1 \neq \mathbf{V}^\bullet_{\mathbf{H}\oplus\mathbf{H}'}(a) = \mathbf{V}^\bullet_{\mathbf{H}\oplus\mathbf{H}'}(\pi) = \alpha$.

The same reasoning can be done for both cases ($\blacksquare \in \{g, p, c\}$ and $\blacksquare = s$) considering AFRA extensions and the sets $\texttt{Ext}_\blacksquare(\mathbf{H})$ and $\texttt{Ext}_\blacksquare(\mathbf{H} \oplus \mathbf{H}')$.

Directionality is satisfied for $\bullet/\blacksquare \in \{g, p, c\}$ but not for $\bullet/\blacksquare = s$ in both cases (AFRA and RAF). Moreover the A-Directionality and the R-Directionality have the same status.

Consider the first case for RAF ($\bullet \in \{g, p, c\}$). Following definitions of RAF semantics, given an element z, the fact that z belongs or not to a given structure and the fact that a given structure defeats it or not only depend on the elements that belong to a path leading to z. So if there is no such a path from x and z, $\mathbf{V}^\bullet_\mathbf{H}(z) = \mathbf{V}^\bullet_{\mathbf{H}'}(z)$.

Consider now the second case for RAF ($\bullet = s$). Once again, the fact that the set of stable structures can be empty is the reason of the violation of the Directionality. Consider for instance that $\mathbf{H}$ contains three arguments a, b and c and the attack $\pi = (a, c)$, and $\mathbf{H}'$ is obtained by the adding of the attack (b, b). In this case, $\texttt{Struc}_\bullet(\mathbf{H})$ contains only one stable structure $\{a, b, \pi\}$ and $\texttt{Struc}_\bullet(\mathbf{H}')$ is empty. Thus we have $\mathbf{V}^\bullet_\mathbf{H}(a) = \mathbf{V}^\bullet_\mathbf{H}(\pi) = 1 \neq \mathbf{V}^\bullet_{\mathbf{H}'}(a) = \mathbf{V}^\bullet_{\mathbf{H}'}(\pi) = \alpha$ whereas there is no path from b to a or from b to π.

The same reasoning can be done for both cases considering AFRA extensions, the e-defeat relation and the sets $\texttt{Ext}_\blacksquare(\mathbf{H})$ and $\texttt{Ext}_\blacksquare(\mathbf{H}')$.

Equivalence is violated by $\bullet/\blacksquare \in \{p, c, s\}$ in both cases (RAF or AFRA). For the grounded semantics, $\bullet/\blacksquare$ satisfies the principle in the RAF case but not in the AFRA case. Moreover the A-Equivalence and the R-Equivalence have not always the same status.

For the violation by $\bullet/\blacksquare \in \{p, c, s\}$, consider the following example in which x and y can be arguments or attacks[8] (see Fig. 17):

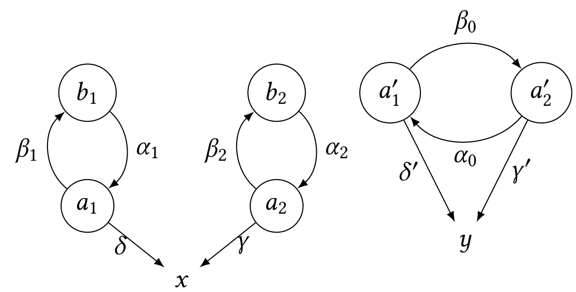

Fig. 17. Equivalence: counter-example for preferred, complete and stable RAF/AFRA semantics

In this case, there is a bijection F defined by: (a_1, a'_1), (a_2, a'_2), (δ, δ'), (γ, γ'). Moreover, for any semantics $\bullet/\blacksquare \in \{p, c, s\}$, there are several structures (for RAF case) or extensions (for AFRA case) and we have $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(a_1) = \mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(a'_1) = \beta$, $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(a_2) = \mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(a'_2) = \beta$, $\mathbf{V}^{\bullet}_{\mathrm{H}}(\delta) = \mathbf{V}^{\bullet}_{\mathrm{H}}(\delta') = 1$ (RAF case) or $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(\delta) = \mathbf{V}^{\blacksquare}_{\mathrm{H}}(\delta') = \beta$ (AFRA case), $\mathbf{V}^{\bullet}_{\mathrm{H}}(\gamma) = \mathbf{V}^{\bullet}_{\mathrm{H}}(\gamma') = 1$ (RAF case) or $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(\gamma) = \mathbf{V}^{\blacksquare}_{\mathrm{H}}(\gamma') = \beta$ (AFRA case). Among the structures (or the extensions) some of them contain x, some of them do not contain x and attack x but none contains y. So $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(x) \neq 1$ and $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(x) \neq 0$, whereas $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(y) = 0$ and thus $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(x) \neq \mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(y)$.

Consider now the case of the grounded semantics for $\bullet$ in the RAF case whatever the nature of x and y. Several cases must be studied. The first one is trivial: if the grounded structure U is empty, the elements have all the same degree α and the principle is satisfied. The second one ($U \neq \emptyset$) can be divided in the following subcases:

- either x is unattacked, so y is also unattacked and by definition x and y belong to U and so they have the same degree (1).
- or x is attacked by at least one attack denoted π; so by assumption y is also attacked by at least one attack denoted $F(\pi)$. Two cases can occur:
  - Either there exists π such that $\mathbf{V}^{\bullet}_{\mathrm{H}}(\pi) = \mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{s}(\pi)) = 1$ then $\mathbf{V}^{\bullet}_{\mathrm{H}}(F(\pi)) = \mathbf{V}^{\bullet}_{\mathrm{H}}(F(\mathbf{s}(\pi))) = 1$ and so $\mathbf{V}^{\bullet}_{\mathrm{H}}(x) = \mathbf{V}^{\bullet}_{\mathrm{H}}(y) = 0$.
  - Or, for any π, $\mathbf{V}^{\bullet}_{\mathrm{H}}(\pi) \neq 1$ or $\mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{s}(\pi)) \neq 1$ then π or its source does not belong to U. Let denote z the element π or $\mathbf{s}(\pi)$ such that $\mathbf{V}^{\bullet}_{\mathrm{H}}(z) \neq 1$. For z, two possibilities exist: either z is attacked by U, or z is not attacked by U. In the first case, x is defended by U against z and $\mathbf{V}^{\bullet}_{\mathrm{H}}(z) = 0$; moreover $\mathbf{V}^{\bullet}_{\mathrm{H}}(F(z)) = 0$ and so y is also defended by U against $F(z)$. In the second case, x is not defended by U against z and $\mathbf{V}^{\bullet}_{\mathrm{H}}(z) = \alpha$; moreover $\mathbf{V}^{\bullet}_{\mathrm{H}}(F(z)) = \alpha$ and so y is also not defended by U against $F(z)$. So if x is defended by U against all its attackers, x belongs to U and the same thing holds for y; so $\mathbf{V}^{\bullet}_{\mathrm{H}}(x) = \mathbf{V}^{\bullet}_{\mathrm{H}}(y) = 1$.

[8] Note that neither its source, nor its target can affect the strength of an attack. Thus, if x and y are attacks, then it is implicitly assumed that the example is completed in the simplest possible way, adding a source and a target without affecting the other arguments and attacks and hence the validity of the proof.

Otherwise there exists at least an attack π such that x is not defended by U against π, then x does not belong to U and the same thing holds for y; so $\mathbf{V}^{\bullet}_{\mathbf{H}}(x) = \mathbf{V}^{\bullet}_{\mathbf{H}}(y) = \alpha$ since either π ($F(\pi)$) or its source is not in U.

In conclusion, for each case, we have $\mathbf{V}^{\bullet}_{\mathbf{H}}(x) = \mathbf{V}^{\bullet}_{\mathbf{H}}(y)$ for the grounded semantics in the RAF case.
Consider now the case of the grounded semantics for ■ in the AFRA case when x and y are attacks with the following example (see Fig. 18):

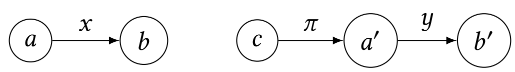

Fig. 18. Equivalence: counter-example for AFRA grounded semantics

In this case, the attacks x and y are unattacked. The attack x belongs to the grounded extension U whereas $y \notin U$. And so $\mathbf{V}^{■}_{\mathbf{H}}(x) \neq \mathbf{V}^{■}_{\mathbf{H}}(y)$.
And finally, consider the last case of the grounded semantics for ■ in the AFRA case when x and y are arguments. A direct correspondence exists between RAF structures and AFRA extensions when we only consider arguments (see (Cayrol, Fandinno, et al. 2020)). This correspondence says that an argument belongs to a RAF structure iff it belongs to an AFRA extension. So using the case of the grounded semantics for • in the RAF case, we can deduce that we also have $\mathbf{V}^{■}_{\mathbf{H}}(x) = \mathbf{V}^{■}_{\mathbf{H}}(y)$ for the grounded semantics in the AFRA case when x and y are arguments.

Maximality is satisfied for $\bullet \in \{g, p, c\}$ but not for $\bullet = s$ in the RAF case whatever the nature of x. And in the AFRA case, Maximality is violated by the four semantics when x is an attack but has the same status as in the RAF case when x is an argument.
Consider the first case ($\bullet \in \{g, p, c\}$) for RAF case. By the definitions of RAF semantics, an element x that is not attacked belongs to any structure; so $\mathbf{V}^{\bullet}_{\mathbf{H}}(x) = 1$.
Consider now the second case ($\bullet = s$) for the RAF case and the following example: $\mathbf{H}$ contains three arguments a, b and c and the attacks (b, b) and $\pi = (a, c)$. So there is no stable extension and $\mathbf{V}^{\bullet}_{\mathbf{H}}(a) = \mathbf{V}^{\bullet}_{\mathbf{H}}(\pi) = \alpha \neq 1$ whereas a and π are unattacked.
Consider now the case AFRA case when x is an argument. With the correspondence between AFRA extensions and RAF structures when arguments are considered, the results of the A-Maximality in the AFRA case are similar to those of the Maximality in the RAF case.
And finally consider the AFRA case when x is an attack and the following example: $\mathbf{H}$ contains three arguments a, b and c and the attacks π from a to b and μ from b to c. In this case, for any semantics, μ is e-defeated by π (indirect attack) and does not belong to the unique extension. So $\mathbf{V}^{■}_{\mathbf{H}}(\mu) = 0 \neq 1$ whereas μ is unattacked.

Neutrality is satisfied by g and s but not by p and c in both cases (RAF and AFRA) whatever the nature of x and y (arguments or attacks).
The initial assumptions are: $\forall x \in \mathbf{H}$, $\mathbf{w}(x) = 1$ and we consider x, y and z such that $\mathtt{Att}(y) = \mathtt{Att}(x) \cup \{z\}$, with $z \notin \mathtt{Att}(x)$ and its degree or the degree of its source is 0. So, $\forall U$ being a RAF structure or an AFRA extension, $z \notin U$ and z is defeated/e-defeated by at least one structure/extension U (an attack z' and its source $\mathbf{s}(z')$ belong to U and z' targets z or its source in the case of RAF, or an attack z' belongs to U and z' targets z or its source in the case of AFRA).
Case of g semantics: there is a unique grounded structure/extension U. Several cases must be considered:

- if $x \in U$, then U defends x and so y against all their common attackers; moreover z' defends y against z; so $y \in U$ and $\mathbf{V}^{\bullet/■}_{\mathbf{H}}(x) = \mathbf{V}^{\bullet/■}_{\mathbf{H}}(y) = 1$.

- if $x \notin U$, x is not defended by U then y is also not defended by U against their common attackers. So $y \notin U$. Moreover, either x is defeated/e-defeated by U, or it is not. In the first case, y is also defeated/e-defeated by U and so $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(x) = \mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(y) = 0$. In the second case, y is also not defeated/e-defeated by U since its additional attacker z or its source is not in U and so $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(x) = \mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(y) = \alpha$.

Case of s semantics: several cases must be considered

- There is no stable structure/extension. For $\bullet/\blacksquare$, all elements have the same degree (α); so $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(z)$ or $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(\mathbf{s}(z))$ cannot be equal to 0 and this is an impossible case.
- There is a unique stable structure/extension. If $x \in U$, then any attacker of x is defeated/e-defeated by U and so, since z is also defeated/e-defeated by U, $y \in U$ and $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(x) = \mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(y) = 1$. If $x \notin U$, then x is defeated/e-defeated by U and y is also defeated/e-defeated by U; so $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(x) = \mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(y) = 0$.
- There are at least 2 distinct stable structures/extensions. Since the degree of z (or its source) is 0, then z (or its source) does not belong to any structure and z (or its source) is defeated/e-defeated by at least one structure/extension. Note that, in the AFRA case, the fact that the source of z is e-defeated by U implies that z is also e-defeated by U. So if y is defeated/e-defeated by a given structure this cannot be because of z. So, for each structure/extension, we have either $x \in U$ or $x \notin U$ defeated/e-defeated by U; in the first case ($x \in U$), we also have $y \in U$ since all its attackers are defeated/e-defeated by U; and in the second case (x is defeated/e-defeated by U), we also have y defeated/e-defeated by the same U. So $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(x) = \mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(y)$.

Case of p and c semantics. Consider the following example in which the nature of x and y is unknown (see Fig. 19):

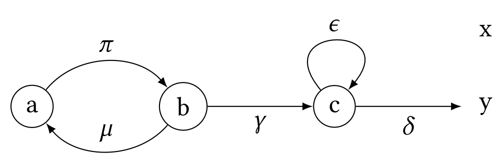

Fig. 19. Neutrality: counter-example for preferred and stable RAF/AFRA semantics

In this case, there are 2 preferred structures (resp. extensions): $U_1 = \{a, x, \pi, \mu, \gamma, \delta, \epsilon\}$ (resp. $U_1 = \{a, x, \pi\}$) and $U_2 = \{b, x, y, \pi, \mu, \gamma, \delta, \epsilon\}$ (resp. $U_2 = \{b, x, y, \mu, \gamma\}$) for RAF (resp. AFRA), U_2 being also stable. For the complete semantics there is an additional structure (corresponding to the grounded one): $U_3 = \{x, \pi, \mu, \gamma, \delta, \epsilon\}$ (resp. $U_3 = \{x\}$) for RAF (resp. AFRA). So, considering c that is the source of δ, $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(c) = 0$, since c does not belong to any structure and it is defeated/e-defeated by U_2. Moreover $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(x) = 1$ since x belongs to any structure. Consider now y: $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(y) = \beta$. since y belongs to U_2 but neither to U_1, nor to U_3. So for p and c semantics, $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(x) \neq \mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(y)$.

Weakening is satisfied by the four semantics in both cases (RAF and AFRA) whatever the nature of x (argument or attack).

Let assume by reduction ad absurdum that $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(x) = \mathbf{w}(x) = 1$. So, for any structure/extension U, $x \in U$. Moreover, following the definition of semantics (RAF and AFRA), $\forall \gamma$ being attackers of x, we have: either $\gamma \notin U$ and γ defeated/e-defeated by U (AFRA and RAF cases), or $\mathbf{s}(\gamma) \notin U$ and $\mathbf{s}(\gamma)$ defeated by U (RAF case); thus either $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(\gamma) = 0$ or $\mathbf{V}_{\mathrm{H}}^{\bullet/\blacksquare}(\mathbf{s}(\gamma)) = 0$. Contradiction.

Proportionality is trivially satisfied by the four semantics in both cases (RAF and AFRA) whatever the nature of x (argument or attack) since $\forall x \in \mathcal{A} \cup \mathcal{R}$, $\mathbf{w}(x) = 1$ and so we cannot identify two elements whose initial weights are different.

Resilience is violated by the four semantics in both cases (RAF and AFRA). Consider the following counter-example: an argument a that attacks an element x (this attack is named γ and x can be an argument or another attack). We have: $\mathbf{w}(x) = 1$ whereas $\mathbf{V}_{\mathbf{H}}^{\bullet/\blacksquare}(x) = 0$ since it is not in the unique structure (extension) U and it is defeated/e-defeated by U that contains γ (AFRA and RAF cases) and a (RAF case).

Reinforcement is violated for the four semantics in both cases (RAF and AFRA) whatever the nature of x and y (arguments or attacks).

Consider the following example in which x and y can be either arguments or attacks (see Fig. 20):

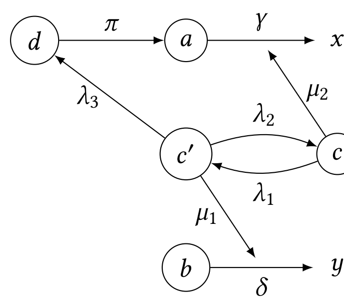

Fig. 20. Reinforcement: counter-example for any RAF/AFRA semantics

Under preferred and stable semantics and for any framework (RAF or AFRA), there are two RAF structures (resp. AFRA extensions); for instance for RAF: $U_1 = \{b, c, d, x, \pi, \mu_1, \mu_2, \lambda_1, \lambda_2, \lambda_3, \delta\}$ and $U_2 = \{b, c', a, y, \pi, \mu_1, \mu_2, \lambda_1, \lambda_2, \lambda_3, \gamma\}$. Under the complete semantics there is an additional RAF structure (resp. AFRA extension); for instance for RAF: $U_3 = \{b, \pi, \mu_1, \mu_2, \lambda_1, \lambda_2, \lambda_3\}$. Under any semantics $\bullet/\blacksquare$, $\mathbf{V}_{\mathbf{H}}^{\bullet/\blacksquare}(b) = 1 > \mathbf{V}_{\mathbf{H}}^{\bullet/\blacksquare}(a)$ (b belongs to every structure/extension whereas a does not) and $\mathbf{V}_{\mathbf{H}}^{\bullet/\blacksquare}(\delta) = \mathbf{V}_{\mathbf{H}}^{\bullet/\blacksquare}(\gamma) > 0$ since either they belong to the same number of structures (resp. extensions), or they do not belong to any structure (resp. extension) but they are not defeated (resp. e-defeated) by these structures (resp. extensions). Moreover, $\mathbf{V}_{\mathbf{H}}^{\bullet/\blacksquare}(x) = \mathbf{V}_{\mathbf{H}}^{\bullet/\blacksquare}(y) > 0$ (for the same reasons as those given for δ and γ). So in each case, $\mathbf{V}_{\mathbf{H}}^{\bullet/\blacksquare}(x) \not> \mathbf{V}_{\mathbf{H}}^{\bullet/\blacksquare}(y)$.

Counting is violated by any semantics in both cases (RAF and AFRA) whatever the nature of x and y (arguments or attacks).

First let us consider the following example in which x and y can be either arguments or attacks (see Fig. 21):

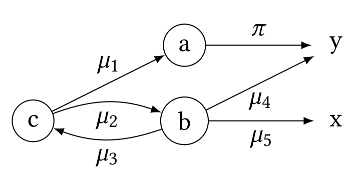

Fig. 21. Counting: counter-example for any RAF/AFRA semantics

Here, there are 2 preferred and stable RAF structures or AFRA extensions (in the case of RAF: $U_1 = \{c, x, y\} \cup \mathcal{R}$ and $U_2 = \{b, a\} \cup \mathcal{R}$). And for the complete semantics there is an additional RAF structure or AFRA extension that is also grounded (for RAF: $U_1 = \mathcal{R}$). In this example, for $\bullet/\blacksquare$ corresponding to p, s or c, $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(a) > 0$ and $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(\pi) > 0$ (note that, in the AFRA case, $\pi \in U_2$). Moreover for these semantics $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(x) > 0$ and $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(x) = \mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(y)$ even if y has one more attacker than x. Consider now the case of the grounded semantics with the same example: $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(a) = \mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(\pi) = \alpha > 0$, $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(x) = \alpha > 0$ and $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(x) = \mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(y) = \alpha$. So the principle is also violated in this case.

Attack-sensitivity is violated for the four semantics in both cases (RAF and AFRA) whatever the nature of x and y (arguments or attacks).

Consider the following example in which x and y can be either arguments or attacks (see Fig. 22):

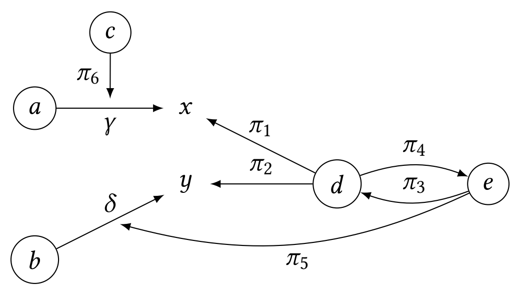

Fig. 22. Attack-sensitivity: counter-example for any RAF/AFRA semantics

Here, there are 3 complete structures (resp. extensions) for RAF (resp. AFRA); here is the result for RAF: $U_1 = \{a, b, c, e, x, y\} \cup \mathcal{R} \setminus \{\gamma, \delta\}$, $U_2 = \{a, b, c, d\} \cup \mathcal{R} \setminus \{\gamma\}$, $U_3 = \{a, b, c\} \cup \mathcal{R} \setminus \{\gamma, \delta\}$; the two first structures are also preferred and stable and the last one is the grounded structure; in the case of AFRA, the result concerning the arguments and γ, δ is the same. For any semantics $\bullet/\blacksquare \in \{p, c, s, g\}$, $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(b) = \mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(a) = 1 > 0$ and $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(\mathbf{n}(b, y)) > \mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(\mathbf{n}(a, x)) = 0$ ($\gamma = \mathbf{n}(a, x)$ is attacked by any structure whereas $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(\delta) = \alpha$ if $\bullet/\blacksquare = g$ or $= \beta$ for the other semantics). Moreover, $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(x) = \mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(y) > 0$ (this degree being α for $\bullet/\blacksquare = g$ and β for the other semantics). So in each case, $\mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(x) \not> \mathbf{V}^{\bullet/\blacksquare}_{\mathbf{H}}(y)$.

Equality is obviously satisfied for any semantics in the RAF case. Indeed these semantics do not make a difference between the defeat (or the defence) of arguments and the defeat (or the defence) of attacks.

Moreover Equality is violated in the AFRA case for any semantics. Let us consider the following example: three arguments a, b and c and two attacks $\pi = (a, b)$ and $\nu = (b, c)$. In this example there is only one extension for $\sigma \in \{c, g, p, s\}$: $\{a, \pi, c\}$. For $x = a \in \mathtt{Arg}$ and $y = \nu \in \mathtt{Att}$, we have $\mathbf{w}(x) = \mathbf{w}(y)$, $\mathtt{Sr}(x) = \mathtt{Sr}(y) = \emptyset$ and so $\forall z \in \mathtt{Sr}(x)$, $\mathbf{V}^{\blacksquare}_{\mathbf{H}}(\mathbf{n}(z, x) = \mathbf{V}^{\blacksquare}_{\mathbf{H}}(\mathbf{n}(z, y))$. Nevertheless $\mathbf{V}^{\blacksquare}_{\mathbf{H}}(x) = 1$ whereas $\mathbf{V}^{\blacksquare}_{\mathbf{H}}(y) = 0$.

A-precedence is obviously satisfied for any semantics in the RAF case (since Equality is satisfied).

It is also satisfied by any AFRA semantics. Indeed, for $\sigma \in \{c, g, p, s\}$, let us assume that there exist

$x \in \mathtt{Arg}$ and $y \in \mathtt{Att}$ such that $\mathbf{w}(x) = \mathbf{w}(y)$, $\mathtt{Sr}(x) = \mathtt{Sr}(y)$, $\forall z \in \mathtt{Sr}(x)$, $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(z,x) = \mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(z,y))$[9] and $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) < \mathbf{V}^{\blacksquare}_{\mathrm{H}}(y)$. The following cases must be studied:

- $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) = 1$: by definition of $\mathbf{V}^{\blacksquare}_{\mathrm{H}}$, we have $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) \leq 1$ and thus a contradiction with $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) < \mathbf{V}^{\blacksquare}_{\mathrm{H}}(y)$.
- $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) = \beta$: since $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) < \mathbf{V}^{\blacksquare}_{\mathrm{H}}(y)$ then $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) = 1$; but x and y have the same attackers in the AFRA; so if y can belong to any extension then x can also belong to any extension and so $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) = 1$ that is a contradiction with $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) = \beta$.
- $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) = \alpha$: since $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) < \mathbf{V}^{\blacksquare}_{\mathrm{H}}(y)$ then $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) = 1$ or $= \beta$; in the first case, using the same reasoning that the one given in the previous item we obtain a contradiction; let us consider now that $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) = \beta$; so there exists at least one extension containing y; but x and y have the same attackers in the AFRA; so if y can belong to this extension then x can also belong to this extension and so $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) \geq \beta$ that is a contradiction with $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) = \alpha$.
- $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) = 0$: since $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) < \mathbf{V}^{\blacksquare}_{\mathrm{H}}(y)$ then $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) = 1$ or $= \beta$ or $= \alpha$; for the two first cases we can use the same reasoning as the one used in the previous item and so we obtain a contradiction; for the last case ($\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) = \alpha$), we know that there is no extension that attacks y; but x and y have the same attackers in the AFRA; thus there is no extension that attacks x and thus $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) \geq \alpha$ that is a contradiction with $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) = 0$.

So in each case, the assumption $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) < \mathbf{V}^{\blacksquare}_{\mathrm{H}}(y)$ leads to a contradiction. So $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) \geq \mathbf{V}^{\blacksquare}_{\mathrm{H}}(y)$.

R-precedence is obviously satisfied for any semantics in the RAF case (since Equality is satisfied). And it is violated in the AFRA case for any semantics. The example used for proving the violation of Equality can also be used here for proving that the AFRA semantics do not give precedence to attacks over arguments.

Local-equality is satisfied for the grounded semantics in the RAF case whatever the nature of x and y (arguments or attacks) and in the AFRA if these elements are arguments. Let us consider the following cases (the grounded structure/extension is denoted with U):

- $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(x) = 1$: so $x \in U$. Since U is grounded, x is defended against all its attackers, so U defeats/e-defeats the attackers that are common with y and also defeats/e-defeats a.
  Thus, in the case of RAF, U attacks either a or $\mathbf{n}(a,x)$ and either $\mathbf{V}^{\bullet}_{\mathrm{H}}(a) = 0$, or $\mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{n}(a,x)) = 0$. Since $\mathbf{V}^{\bullet}_{\mathrm{H}}(a) = \mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{n}(b,y))$ and $\mathbf{V}^{\bullet}_{\mathrm{H}}(b) = \mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{n}(a,x))$, either $\mathbf{V}^{\bullet}_{\mathrm{H}}(b) = 0$, or $\mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{n}(b,y)) = 0$. So U defeats the attack from b to y and thus defends y against all its attackers (including b). Since U is the grounded structure, $\mathbf{V}^{\bullet}_{\mathrm{H}}(y) = 1$.
  Similarly, in the case of AFRA, U e-defeats $\mathbf{n}(a,x)$ and so $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(a,x)) = 0$. Since $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(b) = \mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(a,x))$, we have that $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(b) = 0$. So U attacks b. Moreover in the AFRA case any attack to b is also an indirect attack to $\mathbf{n}(b,y)$. Thus U e-defeats $\mathbf{n}(b,y)$ and thus defends y against all its attackers (including b). Since U is the grounded structure, $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) = 1$.
- $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(x) = \beta$: impossible since U is unique.
- $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(x) = \alpha$: so $x \notin U$ and U does not attack x. So the common attackers and/or the corresponding attacks are not in U. Let us consider the case of a.
  In the RAF case, this means that either $\mathbf{V}^{\bullet}_{\mathrm{H}}(a) \neq 1$, or $\mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{n}(a,x)) \neq 1$. Thus either $\mathbf{V}^{\bullet}_{\mathrm{H}}(b) \neq 1$, or $\mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{n}(b,y)) \neq 1$. So U does not attack y. Moreover if we assume that $y \in U$ then, following the first item of this proof (that about local-equality), we could prove that $\mathbf{V}^{\bullet}_{\mathrm{H}}(x) = 1$. So $y \notin U$ and thus, since U does not attack y, $\mathbf{V}^{\bullet}_{\mathrm{H}}(y) = \alpha$.
  The same result holds in the AFRA case considering only that $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(a,x)) \neq 1$, thus that $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(b) \neq 1$. If we assume that $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(b,y)) = 1$, then U defends $\mathbf{n}(b,y)$ and so must also defend b; so a contradiction

[9] With these assumptions, following the definition of AFRA semantics, if y is not e-defeated by an extension then x is also not e-defeated by this extension; and similarly, if y is defended by an extension then x is also defended by this extension.

and thus $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(b,y)) \neq 1$. So U does not attack y and $y \notin U$ (same reasoning as in the RAF case). So $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) = \alpha$.

- $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(x) = 0$: so $x \notin U$ and U attacks x. If these attacks come from a common attacker, then U also attacks y and so $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(y) = 0$. Otherwise, let us consider the case of a.
  In the RAF case, if U attacks x from a then $\mathbf{V}^{\bullet}_{\mathrm{H}}(a) = 1$ and $\mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{n}(a,x)) = 1$. Thus $\mathbf{V}^{\bullet}_{\mathrm{H}}(b) = 1$ and $\mathbf{V}^{\bullet}_{\mathrm{H}}(\mathbf{n}(b,y)) = 1$. So U attacks y and $\mathbf{V}^{\bullet}_{\mathrm{H}}(y) = 0$.
  The same result holds for AFRA case considering only that $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(a,x)) = 1$. Thus $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(b) = 1$. If we assume that $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(b,y)) \neq 1$, then $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(a) \neq 1$, and so a is not defended by U and so $\mathbf{n}(a,x)$ cannot be defended by U; contradiction. So $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(a) = \mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(b,y)) = 1$. Thus U attacks y and $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) = 0$.

For the grounded semantics in the AFRA case, if x and y are attacks, Local-equality is not satisfied.[10] Let us consider the following example (see Fig. 23):

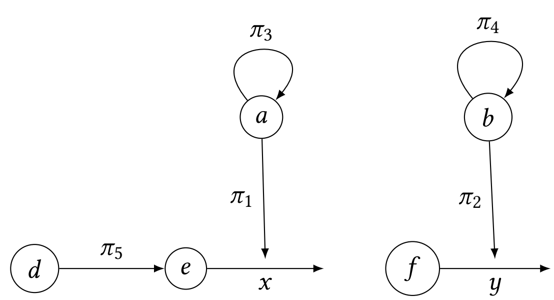

Fig. 23. Local-equality: counter-example for AFRA grounded semantics

In this example, the grounded extension U contains only d and π_5. So x is attacked by U (since there is an indirect attack from π_5 to x) and $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) = 0$. Moreover, $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(a) = \mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(a,x)) = \mathbf{V}^{\blacksquare}_{\mathrm{H}}(b) = \mathbf{V}^{\blacksquare}_{\mathrm{H}}(\mathbf{n}(b,y)) = \alpha$ and $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(y) = \alpha$. Thus $\mathbf{V}^{\blacksquare}_{\mathrm{H}}(x) \neq \mathbf{V}^{\blacksquare}_{\mathrm{H}}(y)$.
For the preferred, complete and stable semantics, Local-equality is not satisfied in both cases (RAF and AFRA) whatever the nature of x and y (see Fig. 24):

In this case there are 3 complete structures for RAF (extensions for AFRA):
- the first structure/extension that is also preferred and stable contains (among other elements) a, y and π_1, but not b, x and π_2,
- the second preferred and stable structure/extension contains (among other elements) b and π_2, but not a, x, y and π_1,
- and the last one is the grounded structure/extension and does not include a, b, π_1, π_2, x and y.

So for each $\bullet/\blacksquare \in \{p,c,s\}$, we have $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(b) = \mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(\mathbf{n}(a,x)) = \beta$ and $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(a) = \mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(\mathbf{n}(b,y)) = \beta$. Moreover we also have that $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(y) = \beta$ whereas $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(x) = 0$ since none structure/extension contains x and x is attacked by at least one (complete, preferred, stable) structure/extension. So $\mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(x) \neq \mathbf{V}^{\bullet/\blacksquare}_{\mathrm{H}}(y)$

□

[10]For the grounded semantics in the case of AFRA, note that the difference in behaviour concerning Local-equality depending on the nature of x and y comes from the fact that an attack can also be the target of an indirect attack (π e-defeats any attack ν whose source is the target of π) and an argument cannot.

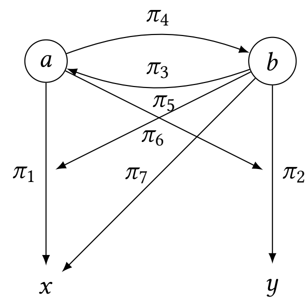

Fig. 24. Local-equality: counter-example for preferred, complete, stable RAF/AFRA semantics

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Theorem 4. *Let* S *be an extended gradual semantics and* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$ *a wHO-AF such that* $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$. *The following properties hold:*

- $\forall \alpha \in \mathcal{R}$, $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha) = \mathbf{w}(\alpha)$.
- $\forall a \in \mathcal{A}$, $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(a) = \mathsf{f}\Bigg(\mathbf{w}(a), \mathsf{g}\Bigg(\mathsf{h}\Big(\mathbf{w}(\alpha_1), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{w}(\alpha_n), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_n))\Big)\Bigg)\Bigg)$, *where* $\mathtt{Att}_{\mathbf{H}}(a) = \{\alpha_1, \ldots, \alpha_n\}$.

Proof. Let S be an extended gradual semantics that is based on the evaluation method $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$. Let also $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$ be a wHO-AF such that $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$. Thus, $\forall \alpha \in \mathcal{R}$, $\mathtt{Att}_{\mathbf{H}}(\alpha) = \varnothing$; so, $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha) = \mathsf{f}(\mathbf{w}(\alpha), \mathsf{g}())$. By Definition 13, $\mathsf{g}() = 0$, and $\mathsf{f}(\mathbf{w}(\alpha), 0) = \mathbf{w}(\alpha)$.

Let now $a \in \mathcal{A}$ such that $\mathtt{Att}_{\mathbf{H}}(a) = \{\alpha_1, \ldots, \alpha_k\}$.

$$\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(a) = \mathsf{f}\Bigg(\mathbf{w}(a), \mathsf{g}\Big(\mathsf{h}\big(\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha_1), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_1))\big), \ldots, \mathsf{h}\big(\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha_k), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_k))\big)\Big)\Bigg).$$

From above, for any $i \in [1, k]$, $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha_i) = \mathbf{w}(\alpha_i)$. Hence,

$$\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(a) = \mathsf{f}\Bigg(\mathbf{w}(a), \mathsf{g}\Big(\mathsf{h}\big(\mathbf{w}(\alpha_1), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_1))\big), \ldots, \mathsf{h}\big(\mathbf{w}(\alpha_k), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_k))\big)\Big)\Bigg).$$

□

Theorem 5. *For any well-behaved evaluation method* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, *there exists an extended gradual semantics that is based on* $\mathbf{M}$.

Proof. Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \sigma, s, t \rangle$ be a wHO-AF. Its sets of arguments and attacks are finite. Let $k = |\mathcal{A}| + |\mathcal{R}|$ (i.e., the number of elements in $\mathbf{H}$). An extended gradual semantics corresponds to a solution of the system of k equations, with variables $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(x)$ (for every $x \in \mathcal{A} \cup \mathcal{R}$):

$$\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(x) = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\Bigg(\mathsf{h}\Big(\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha_1), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\alpha_n), \mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_n))\Big)\Bigg)\Bigg), \tag{6}$$

where $\{\alpha_1, \ldots, \alpha_n\} = \mathtt{Att}_{\mathbf{H}}(x)$ (note that we have one equation per element of $x \in \mathcal{A} \cup \mathcal{R}$). Let us assume a fixed enumeration of the elements $\mathcal{A} \cup \mathcal{R} = \{x_1, \ldots, x_n\}$. Then the Equations 6 are of the form $\mathbf{V}^{\mathrm{S}}_{\mathbf{H}}(x_j) = F_j(x_1, \ldots, x_k)$, where each F_j is defined as a composition of **continuous** functions f, g and h as presented in Equation 5, therefore each F_j is a continuous function. So, the k-ary function $F : [0,1]^k \rightarrow [0,1]^k$, defined by $F = (F_1, \ldots, F_k)$, is a continuous function on the compact set $[0,1]^k$. By Brouwer's fixed point theorem, F has a fixed point, and that point is a k-tuple which is a solution of the considered system of equations. □

Theorem 6. *For any evaluation method* $\mathbf{M} \in \mathbb{M}$*, there exists a single extended gradual semantics that is based on* $\mathbf{M}$.

Proof. Let $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle \in \mathbb{M}$ be an evaluation method. We need to prove that there is one and only one semantics $\mathbf{S}$ of wHO-AF such that $\mathbf{S}$ is based on $\mathbf{M}$. The proof contains two steps:

- First, we define one semantics, denoted by $\mathbf{S}'$, that is based on $\mathbf{M}$. For that, for each wHO-AF $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$, we need to define a weighting on its *elements* (arguments or attacks). Let us fix a wHO-AF $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$ and assume a fixed enumeration of the elements $\mathcal{A} \cup \mathcal{R} = \{x_1, \ldots, x_n\}$. Note that if $\mathbf{w}(x_k) = 0$ for some $k \in \{1, \ldots, n\}$, then by Definition 13.1.d the degree of x_k can only be 0. Definition 13 and Definition 14 also ensure that x_k doesn't affect the degrees of other arguments, so, without any loss of generality, in this proof we assume that $w(x_k) > 0$ for all k. Moreover, if $I_{\mathcal{R}} = \{i \in \{1, \ldots, n\} \mid x_i \in \mathcal{R}\}$, then we define the function $z : I_{\mathcal{R}} \rightarrow \{1, \ldots, n\}$ such that

$$x_{z(i)} = \mathbf{s}(x_i). \tag{7}$$

Then we define a sequence $\{\mathbf{u}^{(k)}\}_{k=1}^{+\infty}$ of vectors from $[0,1]^n$ in the following way:
  - $\mathbf{u}^{(1)} = (\mathbf{w}(x_1), \ldots, \mathbf{w}(x_n))$.
  - In order to define $\mathbf{u}^{(k)}$ for each $k \geq 2$, we first define the mapping $Q : [0,1]^n \rightarrow [0,1]^n$: $Q(\mathbf{v}) = [Q_1(\mathbf{v}) \ldots Q_n(\mathbf{v})]$, where for every $\mathbf{v} = (v_1, \ldots, v_n)$,

$$Q_k(\mathbf{v}) = f(\mathbf{w}(x_k), g(h(v_{\ell_k(1)}, v_{z(\ell_k(1))})), \ldots, h(v_{\ell_k(n_k)}, v_{z(\ell_k(n_k))}))), \tag{8}$$

  where the indices are set in the following way: if $\{y_1^k, \ldots, y_{n_k}^k\} = \mathtt{Att}_{\mathbf{H}}(x_k)$ and $y_{i'}^k \neq y_{i''}^k$ whenever $i' \neq i''$, then for every $j \leq n_k$, $\ell_k(j) = i$ iff $y_j^k = x_i$. Then, for each $k \geq 2$, we define

$$\mathbf{u}^{(k)} = (u_1^{(k)}, \ldots, u_n^{(k)}) = Q(\mathbf{u}^{k-1}) \tag{9}$$

  We will prove that the sequence $\{\mathbf{u}^{(k)}\}_{k=1}^{+\infty}$ converges. Then we define the semantics $\mathbf{S}'$ by assigning to the graph the weighting $\mathbf{V}_{\mathbf{H}}^{\mathbf{S}'}$ such that

$$(\mathbf{V}_{\mathbf{H}}^{\mathbf{S}'}(x_1), \ldots, \mathbf{V}_{\mathbf{H}}^{\mathbf{S}'}(x_n)) = \lim_{n \rightarrow +\infty} \mathbf{u}^{(n)}.$$

  (We define a weighting on elements, in the previously described way, for every graph.)
- Then we prove that $\mathbf{S}'$ is based on $\mathbf{M}$. Finally, using the same sequence $\{\mathbf{u}^{(k)}\}_{k=1}^{+\infty}$, we show that every semantics based on $\mathbf{M}$ coincides with $\mathbf{S}'$.

Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$ be a wHO-AF and let $\mathcal{A} \cup \mathcal{R} = \{x_1, \ldots, x_n\}$. Let $g^* = \sup_{\overline{x} \in \bigcup_{n=0}^{+\infty} [0,1]^n} g(\overline{x})$. For every $j \leq n_k$, let $\mathtt{Att}_{\mathbf{H}}(x_k) = \{y_1^k, \ldots, y_{n_k}^k\} \subseteq \mathcal{R}$, and let $\ell_k(j) = i$ iff $y_j^k = x_i$. If $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ are two vectors, we write $\mathbf{u} \leq \mathbf{v}$ if and only if for every $k \in \{1, \ldots, n\}$ it holds that $u_k \leq v_k$.

Suppose that $\mathbf{u} \leq \mathbf{v}$. We will use the fact that $\mathbf{M}$ is a (well-behaved) evaluation method. For arbitrary $k \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n_k\}$, from Definition 13.3.d we conclude

$$h(u_{\ell_k(j)}, u_{z(\ell_k(j))}) \leq h(v_{\ell_k(j)}, v_{z(\ell_k(j))}).$$

Then, applying Definition 13.2.d and Definition 13.2.e we obtain:

$$g(h(u_{\ell_k(1)}, u_{z(\ell_k(1))}), \ldots, h(u_{\ell_k(n_k)}, u_{z(\ell_k(n_k))})) \leq g(h(v_{\ell_k(1)}, v_{z(\ell_k(1))}), \ldots, h(v_{\ell_k(n_k)}, v_{z(\ell_k(n_k))})).$$

Finally, from Definition 13.1.b and Definition 13.1.e we have:

$$f(\mathbf{w}(x_k), g(h(v_{\ell_k(1)}, v_{z(\ell_k(1))}), \ldots, h(v_{\ell_k(n_k)}, v_{z(\ell_k(n_k))}))) \leq$$
$$f(\mathbf{w}(x_k), g(h(u_{\ell_k(1)}, u_{z(\ell_k(1))}), \ldots, h(u_{\ell_k(n_k)}, u_{z(\ell_k(n_k))}))),$$

i.e.,

$$Q_k(\mathbf{v}) \leq Q_k(\mathbf{u}).$$

Thus, we proved

$$\mathbf{u} \leq \mathbf{v} \implies Q(\mathbf{v}) \leq Q(\mathbf{u}). \tag{10}$$

Consequently, we obtain

$$\mathbf{u} \leq \mathbf{v} \implies Q(Q(\mathbf{u})) \leq Q(Q(\mathbf{v})). \tag{11}$$

by applying Q to (10).

Now let us consider the sequence of vectors $\{\mathbf{u}^{(k)}\}_{k=1}^{+\infty}$ that we constructed in the beginning of the proof using Q, namely by employing Equation (9) and setting $\mathbf{u}^{(1)} = (\mathbf{w}(x_1), \ldots, \mathbf{w}(x_n))$. Let us show that for every $\mathbf{u}^{(k)} = (u_1^{(k)}, \ldots, u_n^{(k)})$ and for every $t \in \{1, \ldots, n\}$ we have $u_t^{(k)} > 0$. Indeed, for $k = 1$, let us recall that in the beginning of this proof we emphasized that we can assume, without loss of generality, that $w(x_t) > 0$ for every $t \in \{1, \ldots, n\}$. On the other hand, for $k > 1$ let us recall that $\mathbf{u}^{(k)} = Q(\mathbf{u}^{k-1})$, where Q is introduced by the Equation (8). Now, $u_t^{(k)} > 0$ follows from $w(x_t) > 0$ and the fact that $f(x, y) > 0$ whenever $x > 0$ (this follows directly from Definition 13.1.a and Definition 13.1.d).

Next we will show that for every $k \in \mathbb{N}$

$$\mathbf{u}^{(2)} \leq \mathbf{u}^{(4)} \leq \cdots \leq \mathbf{u}^{(2k)} \leq \mathbf{u}^{(2k+1)} \leq \cdots \leq \mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}. \tag{12}$$

From the properties of f stated in Definition 13.1 and from Equation (8) we obtain that

$$(\forall k \in \mathbb{N})\ \mathbf{u}^{(k)} \leq (\mathbf{w}(x_1), \ldots, \mathbf{w}(x_n)) = \mathbf{u}^{(1)}.$$

Specially, $\mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}$. From (11) we obtain

$$\cdots \leq \mathbf{u}^{(2k+1)} \leq \cdots \leq \mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}. \tag{13}$$

From $\mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}$ we conclude $\mathbf{u}^{(2)} \leq \mathbf{u}^{(4)}$ by (10). From (11) we obtain

$$\mathbf{u}^{(2)} \leq \mathbf{u}^{(4)} \leq \cdots \leq \mathbf{u}^{(2k)} \leq \cdots \tag{14}$$

From $\mathbf{u}^{(2)} \leq \mathbf{u}^{(1)}$, by (11) we obtain

$$(\forall k \in \mathbb{N})\ \mathbf{u}^{(2k)} \leq \mathbf{u}^{(2k+1)}. \tag{15}$$

Now (12) follows from (13), (14) and (15).

In particular, from (12) we obtain $\mathbf{u}^{(2k)} \leq \mathbf{u}^{(2k-1)}$, so for every $t \in \{1, \ldots, n\}$ the inequality

$$0 < u_t^{(2k)} \leq u_t^{(2k-1)} \leq 1 \tag{16}$$

holds. Let $\varphi_t = \frac{u_t^{(2k)}}{2u_t^{(2k-1)}}$. Then obviously $\varphi_t u_t^{(2k-1)} \leq u_t^{(2k)}$. Setting $\varphi = \min\{\varphi_1, \ldots, \varphi_n\}$, we directly obtain that for every $k \in \mathbb{N}$, there exists $0 < \varphi \leq 1$, such that

$$\varphi \mathbf{u}^{(2k-1)} \leq \mathbf{u}^{(2k)}.$$

Note that φ depends on k.

Now we define $\pi_k = \sup\{\pi \mid \pi \mathbf{u}^{(2k-1)} \leq \mathbf{u}^{(2k)}\}$. Obviously, for every $k \in \mathbb{N}$,

$$\pi_k \mathbf{u}^{(2k-1)} \leq \mathbf{u}^{(2k)}.$$

Also, if $\pi\mathbf{u}^{(2k-1)} \leq \mathbf{u}^{(2k)}$, from

$$\mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+2)} \leq \mathbf{u}^{(2i+1)} \leq \mathbf{u}^{(2i-1)},$$

we obtain $\pi\mathbf{u}^{(2k+1)} \leq \mathbf{u}^{(2k+2)}$. Consequently,

$$(\forall k \in \mathbb{N})\ \pi_k \leq \pi_{k+1}.$$

Thus, the sequence $\{\pi_k\}_{k=1}^{+\infty}$ is non-decreasing. Since it is also bounded by 1, we obtain that it converges. Let us denote $\pi = \lim_{k\to+\infty} \pi_k$.

Next we prove that $\pi = 1$. Let $\mathbf{v} = (v_1, \ldots, v_n) \in [0,1]^n$ be a vector and let $\lambda \leq 1$ be a positive number. From Definition 15.2, for every $k \in \{1, \ldots, n\}$ we obtain

$$\begin{aligned} &g(h(\lambda v_{\ell_k(1)}, \lambda v_{z(\ell_k(1))}), \ldots, h(\lambda v_{\ell_k(n_k)}), \lambda v_{z(\ell_k(n_k))}) \\ &\geq \lambda g(h(v_{\ell_k(1)}, v_{z(\ell_k(1))}), \ldots, h(v_{\ell_k(n_k)}), v_{z(\ell_k(n_k))}). \end{aligned}$$

From the definition of an EM, we derive

$$\begin{aligned} &f(\mathbf{w}(x_k),\ g(h(\lambda v_{\ell_k(1)}, \lambda v_{z(\ell_k(1))}), \ldots, h(\lambda v_{\ell_k(n_k)}), \lambda v_{z(\ell_k(n_k))})) \\ &\leq f(\mathbf{w}(x_k), \lambda g(h(v_{\ell_k(1)}, v_{z(\ell_k(1))}), \ldots, h(v_{\ell_k(n_k)}), v_{z(\ell_k(n_k))}))). \end{aligned} \tag{17}$$

Now let us, for a given number r, consider the function $f(r, x)$, denoted by φ_r, obtained by f by fixing the first variable to be r. That function with one variable is a continuous and strictly monotonic function (by Definition 13.1) defined on an interval of reals, so it has an inverse which is also continuous. Moreover, since f is decreasing on the second variable, φ_r is decreasing as well. We formally introduce the domain of φ_r to be $(f_r^*, f_r^{**}]$, where $f_r^* = \lim_{y\to g_-^*} f(r, y)$, and $f_r^{**} = \lim_{y\to 0_+} f(r, y)$. Obviously, the following equation holds:

$$f(r, \varphi_r(y)) = y. \tag{18}$$

It is easy to check that for every $k \in \{1, \ldots, n\}$ and every $\mathbf{v}' = (v_1', \ldots, v_n') \in [0,1]^n$

$$\varphi_{\mathbf{w}(x_k)}(Q_k(\mathbf{v}')) = g(h(v'_{\ell_k(1)}, v'_{z(\ell_k(1))}), \ldots, h(v'_{\ell_k(n_k)}, v'_{z(\ell_k(n_k))})). \tag{19}$$

If we denote $\lambda\mathbf{v} = (\lambda v_1, \ldots, \lambda v_n)$, note that

$$Q_k(\lambda\mathbf{v}) = f(\mathbf{w}(x_k), g(h(\lambda v_{\ell_k(1)}, \lambda v_{z(\ell_k(1))}), \ldots, h(\lambda v_{\ell_k(n_k)}, \lambda v_{z(\ell_k(n_k))}))). \tag{20}$$

Using (19) and (20), we can rewrite (17) as

$$Q_k(\lambda\mathbf{v}) \leq f(\mathbf{w}(x_k), \lambda\varphi_{\mathbf{w}(x_k)}(Q_k(\mathbf{v})). \tag{21}$$

Let us fix $i \in \mathbb{N}$. Since $\pi_i\mathbf{u}^{(2i-1)} \leq \mathbf{u}^{(2i)}$, from (10) we obtain $Q(\mathbf{u}^{(2i)}) \leq Q(\pi_i\mathbf{u}^{(2i-1)})$. Thus,

$$Q_k(\mathbf{u}^{(2i)}) \leq Q_k(\pi_i\mathbf{u}^{(2i-1)}), \tag{22}$$

for every $k \in \{1, \ldots, n\}$. On the other hand, from (21) we obtain

$$Q_k(\pi_i\mathbf{u}^{(2i-1)}) \leq f(\mathbf{w}(x_k), \pi_i\varphi_{\mathbf{w}(x_k)}(Q_k(\mathbf{u}^{(2i-1)})). \tag{23}$$

From (22) and (23) we obtain

$$Q_k(\mathbf{u}^{(2i)}) \leq f(\mathbf{w}(x_k), \pi_i\varphi_{\mathbf{w}(x_k)}(Q_k(\mathbf{u}^{(2i-1)})), \tag{24}$$

i.e.,

$$u_k^{2i+1} \leq f(\mathbf{w}(x_k), \pi_i\varphi_{\mathbf{w}(x_k)}(u_k^{2i})). \tag{25}$$

From (25) we obtain

$$\frac{u_k^{2i+2}}{f(\mathbf{w}(x_k), \pi_i \varphi_{\mathbf{w}(x_k)}(u_k^{2i}))} u_k^{2i+1} \leq u_k^{2i+2}, \tag{26}$$

for every $k \in \{1, \ldots, n\}$.

Let us denote $\frac{u_k^{2i+2}}{f(\mathbf{w}(x_k), \pi_i \varphi_{\mathbf{w}(x_k)}(u_k^{2i}))}$ by π^k. Then for every k we have $\pi^k u_k^{2i+1} \leq u_k^{2i+2}$, so for $\pi^* = \min\{\pi^1, \ldots, \pi^n\}$ we obtain $(\forall k)\pi^* u_k^{2i+1} \leq u_k^{2i+2}$, i.e., $\pi^* \mathbf{u}^{(2i+1)} \leq \mathbf{u}^{(2i+2)}$.

Since $\pi_{i+1} = \sup\{\pi \mid \pi \mathbf{u}^{(2i+1)} \leq \mathbf{u}^{(2i+2)}\}$, we obtain $\pi^* \leq \pi_{i+1}$. Let us recall that $\pi^* \in \{\pi^1, \ldots, \pi^n\}$, so we proved that $\pi^k \leq \pi_{i+1}$ for some k. In other words, we proved that for every $i \in \mathbb{N}$, there exists $k_i \in \{1, \ldots, n\}$ such that

$$\frac{u_{k_i}^{2i+2}}{f(\mathbf{w}(x_{k_i}), \pi_i \varphi_{\mathbf{w}(x_{k_i})}(u_{k_i}^{2i}))} \leq \pi_{i+1}. \tag{27}$$

Note that the sequence $\{\mathbf{u}^{(2i)}\}_{i=1}^{+\infty}$ converges, since it is non-decreasing (by (14)) and bounded by $\mathbf{u}^{(1)}$. Let us denote the limit of the sequence by $\mathbf{u}^*$, i.e.,

$$\lim_{i \to +\infty} \mathbf{u}^{(2i)} = \mathbf{u}^*.$$

Note also that the sequence $\{k_i\}_{i=1}^{+\infty}$ is an infinite sequence, and that all the elements of the sequence belong to the finite set $\{1, \ldots, n\}$. Thus, there exists $l \in \{1, \ldots, n\}$ such that l appears infinitely many times in the sequence. Let's apply limit to the inequality (27) using the subsequence obtained by taking only those i for which $k_i = l$. We obtain

$$\frac{u_l^*}{f(\mathbf{w}(x_l), \pi \varphi_{\mathbf{w}(x_l)}(u_l^*))} \leq \pi, \tag{28}$$

i.e.,

$$u_l^* \leq \pi f(\mathbf{w}(x_l), \pi \varphi_{\mathbf{w}(x_l)}(u_l^*)). \tag{29}$$

We know that $\pi \leq 1$. If $\pi < 1$, from Definition 15.1 we obtain

$$\pi f(\mathbf{w}(x_l), \pi \varphi_{\mathbf{w}(x_l)}(u_l^*)) < f(\mathbf{w}(x_l), \varphi_{\mathbf{w}(x_l)}(u_l^*)). \tag{30}$$

By (18), we have

$$f(\mathbf{w}(x_l), \varphi_{\mathbf{w}(x_l)}(u_l^*)) = u_l^*. \tag{31}$$

From (28), (30) and (31), we obtain $u_l^* < u_l^*$; a contradiction. Thus, $\pi = 1$.

Note that the sequence $\{\mathbf{u}^{(2i+1)}\}_{i=1}^{+\infty}$ converges, since it is non-increasing (by (13)) and bounded (for example, by $\mathbf{u}^{(2)}$). Let us denote the limit of the sequence by $\mathbf{u}_*$, i.e.,

$$\lim_{i \to +\infty} \mathbf{u}^{(2i+1)} = \mathbf{u}_*.$$

From (12) we obtain

$$\mathbf{u}^* \leq \mathbf{u}_*.$$

On the other hand,

$$\pi_k \mathbf{u}^{(2k-1)} \leq \mathbf{u}^{(2k)}$$

for every $k \in \mathbb{N}$. Letting $k \longrightarrow +\infty$ we obtain

$$\mathbf{u}_* \leq \mathbf{u}^*.$$

Consequently, $\mathbf{u}_* = \mathbf{u}^*$, so the sequence $\{\mathbf{u}^{(k)}\}_{k=1}^{+\infty}$ converges.

Note that the wHO-AF $\mathbf{H}$ is arbitrary chosen. We define the semantics $\mathbf{S}'$ by

$$(\mathbf{V}^{\mathbf{S}'}_{\mathbf{H}}(x_1), \ldots, \mathbf{V}^{\mathbf{S}'}_{\mathbf{H}}(x_n)) = \lim_{k \to +\infty} \mathbf{u}^{(k)},$$

for every $\mathbf{H}$.

If we let $k \longrightarrow +\infty$ in (9), using the first three conditions of the definition of $\mathbb{M}$, we obtain directly from (8) and (7) that

$$\mathbf{V}^{\mathbf{S}'}_{\mathbf{H}}(x_k) = f(\mathbf{w}(x_k), g(h(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(y^k_1), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{s}(y^k_1))), \ldots, h(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(y^k_{n_k}), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{s}(y^k_{n_k}))))),$$

for every $k \in \{1, \ldots, n\}$, where $\{y^k_1, \ldots, y^k_{n_k}\} = \mathtt{Att}_{\mathbf{H}}(x_k)$ and $y^k_{i'} \neq y^k_{i''}$ whenever $i' \neq i''$. (Recall that the indices in (8) are chosen such that for every $j \leq n_k$, $\ell_k(j) = i$ iff $y^k_j = x_i$, where $\{x_1, \ldots, x_n\}$ is the enumeration of $\mathcal{A} \cup \mathcal{R}$ chosen in the beginning of this proof.) Thus, the semantics $\mathbf{S}'(\mathbf{M})$ is based on the evaluation method $\mathbf{M}$.

Suppose now that there is another semantics $\mathbf{S}^*$ such that $\mathbf{S}^*$ is also based on the method $\mathbf{M}$. Then, for the vector

$$\mathbf{v} = (\mathbf{V}^{\mathbf{S}^*}_{\mathbf{H}}(x_1), \ldots, \mathbf{V}^{\mathbf{S}^*}_{\mathbf{H}}(x_n))$$

we have $Q(\mathbf{v}) = \mathbf{v}$. Let us define the constant sequence $\mathbf{v}^{(i)} = \mathbf{v}$, for every $i \in \mathbb{N}$. Note that, by definition of EM, $\mathbf{v} \leq \mathbf{u}^{(1)}$.

Since both $Q(\mathbf{v}^{(i)}) = \mathbf{v}^{(i+1)}$ and $Q(\mathbf{u}^{(i)}) = \mathbf{u}^{(i+1)}$, applying Q to $\mathbf{v}^{(1)} \leq \mathbf{u}^{(1)}$ and using (10) and (11) we obtain

$$(\forall i \in \mathbb{N})\ \mathbf{v}^{(2i+1)} \leq \mathbf{u}^{(2i+1)}$$

and

$$(\forall i \in \mathbb{N})\ \mathbf{v}^{(2i)} \geq \mathbf{u}^{(2i)}.$$

By letting $i \longrightarrow +\infty$ we obtain $\lim_{i \to +\infty} \mathbf{v}^{(i)} = \lim_{i \to +\infty} \mathbf{u}^{(i)}$, i.e.,

$$\mathbf{V}^{\mathbf{S}^*}_{\mathbf{H}} \equiv \mathbf{V}^{\mathbf{S}'}_{\mathbf{H}}.$$

□

Theorem 7. *Let* $\mathbf{S} \in \mathbf{ES}$ *and* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$. *For every* $x \in \mathcal{A} \cup \mathcal{R}$, $\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) \in [0, \mathbf{w}(x)]$.

Proof. Let $\mathbf{S}$ be an extended gradual semantics based on the evaluation method $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, and $x \in \mathcal{A} \cup \mathcal{R}$ with $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots, \alpha_n\}$. From Definition 14, we have:

$$\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\bigg(\mathsf{h}\Big(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\alpha_1), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\alpha_n), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_n))\Big)\bigg)\Bigg).$$

By definition of $\mathbf{M}$, $\mathsf{f}(x_1, x_2) \in [0, 1]$, hence $\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) \geq 0$. Furthermore, $\mathsf{g}(y_1, \ldots, y_k) \in [0, +\infty)$. By definition of $\mathbf{M}$, $\mathsf{f}(y, x_1) \geq \mathsf{f}(y, x_2)$ whenever $x_1 < x_2$. This means that $\forall y > 0$, it holds that $\mathsf{f}(\mathbf{w}(x), 0) \geq \mathsf{f}(\mathbf{w}(x), y)$. Hence, $\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) \leq \mathsf{f}(\mathbf{w}(x), 0) = \mathbf{w}(x)$. □

Theorem 8. *Let* $\mathbf{S} \in \mathbf{ES}$ *be based on the evaluation method* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, *and* $x \in \mathcal{A} \cup \mathcal{R}$ *such that* $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots \alpha_k\}$. *We define the sequence* $\{\sigma(x)^{(n)}\}_{n=1}^{+\infty}$ *in the following way:*

- $\sigma(x)^{(1)} = \mathbf{w}(x)$,
- $\sigma(x)^{(n+1)} = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\bigg(\mathsf{h}\Big(\sigma(\alpha_1)^{(n)}, \sigma(\mathbf{s}(\alpha_1))^{(n)}\Big), \ldots, \mathsf{h}\Big(\sigma(\alpha_k)^{(n)}, \sigma(\mathbf{s}(\alpha_k))^{(n)}\Big)\bigg)\Bigg)$.

Then, for every $x \in \mathcal{A} \cup \mathcal{R}$, *the following hold:*

(1) $\{\sigma(x)^{(n)}\}_{n=1}^{+\infty}$ *converges,*

(2) $\lim_{n\to+\infty} \sigma(x)^{(n)} = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x)$.

Proof. Trivial consequence of Theorem 6. □

Theorem 9. *Let* S *be a semantics based on a well-behaved evaluation method* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h}\rangle$.

- S *satisfies Equivalence, Maximality, Neutrality, Weakening, Proportionality, Equality, A-Precedence, and R-Precedence.*
- *If* $\mathbf{M} \in \mathbb{M}$, *then* S *satisfies Anonymity, Independence, and Directionality.*
- *If* f *is positive, then* S *satisfies Resilience.*
- *If* g *is strictly monotonic, then:*
  - S *satisfies Counting.*
  - *If* h *is strictly monotonic on the first variable (F1), then* S *satisfies Attack-sensitivity.*
  - *If* h *is strictly monotonic on the second variable (F2), then* S *satisfies Reinforcement.*
- *If* h *is symmetric,* $\mathsf{h}(x, y) = \mathsf{h}(y, x)$, *then* S *satisfies Local-Equality.*

Proof. Let S be a semantics based on a well-behaved evaluation method $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h}\rangle$ and $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w}\rangle$ be an arbitrary wHO-AF. Consider the following properties.

(C1) $\mathsf{f}(x_1, x_2) > 0$ whenever $x_1 > 0$.
(C2) $\mathsf{g}(x_1, \ldots, x_n, y) > \mathsf{g}(x_1, \ldots, x_n, z)$ whenever $y > z$.
(C3) $\mathsf{h}(x_1, y) > \mathsf{h}(x_2, y)$ whenever $x_1 > x_2$, $y \neq 0$.
(C4) $\mathsf{h}(y, x_1) > \mathsf{h}(y, x_2)$ whenever $x_1 > x_2$, $y \neq 0$.
(C5) $\mathsf{h}(x, y) = \mathsf{h}(y, x)$.

Equivalence: Let $x, y \in \mathcal{A}$ (or $x, y \in \mathcal{R}$) such that $\mathbf{w}(x) = \mathbf{w}(y)$. Suppose that $\mathtt{Att}_{\mathbf{H}}(x) = \{x_1, \ldots, x_n\}$ and $\mathtt{Att}_{\mathbf{H}}(y) = \{y_1, \ldots, y_n\}$, and that there exists a bijective function $F : \mathtt{Att}_{\mathbf{H}}(x) \to \mathtt{Att}_{\mathbf{H}}(y)$ given by $F(x_i) = y_i$ such that the following conditions hold: $\forall i \in \{1, \ldots, n\}$,

i) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x_i) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(F(x_i))$,
ii) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{s}(x_i)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{s}(F(x_i)))$,

Then, $\forall i \in [1, n]$, it holds that:
$$u_i = \mathsf{h}\Big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x_i), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{s}(x_i))\Big) = u'_i = \mathsf{h}\Big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(F(x_i)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{s}(F(x_i)))\Big).$$
So, $\mathsf{g}\big(u_1, \ldots, u_n\big) = \mathsf{g}\big(u'_1, \ldots, u'_n\big)$. Since $\mathbf{w}(x) = \mathbf{w}(y)$, then:
$$\mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\big(u_1, \ldots, u_n\big)\Bigg) = \mathsf{f}\Bigg(\mathbf{w}(y), \mathsf{g}\big(u'_1, \ldots, u'_n\big)\Bigg).$$
Thus, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Maximality: Let $x \in \mathcal{A} \cup \mathcal{R}$ such that $\mathtt{Att}_{\mathbf{H}}(x) = \emptyset$. From Definition 13, $\mathsf{g}() = 0$, and $\mathsf{f}(v, 0) = v$. Hence, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), 0) = \mathbf{w}(x)$.

Neutrality: Let $x, y \in \mathcal{A}$ (or $x, y \in \mathcal{R}$) such that:

i) $\mathbf{w}(x) = \mathbf{w}(y)$,
ii) $\mathtt{Sr}_{\mathbf{H}}(y) = \mathtt{Sr}_{\mathbf{H}}(x) \cup \{a\}$ such that $a \in \mathcal{A} \setminus \mathtt{Sr}_{\mathbf{H}}(x)$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = 0$ or $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, y)) = 0$,
iii) $\forall z \in \mathtt{Sr}_{\mathbf{H}}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$,

Let $\mathtt{Sr}_{\mathbf{H}}(x) = \{b_1, \ldots, b_k\}$. From Definition 14,
$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}\big(\mathbf{w}(x), \mathsf{g}(u_1, \ldots, u_k)\big)$$

where for any $i \in [1,k]$, $u_i = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b_i,x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b_i))$. We have also:

$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}\big(\mathbf{w}(y), \mathsf{g}(u'_1,\dots,u'_k,v)\big)$$

where for any $i \in [1,k]$, $u'_i = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b_i,y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b_i))$ and $v = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a))$. From the condition iii), it follows that for any $i \in [1,k]$, $u_i = u'_i$. From ii) and from Definition 13, namely the fact that $\mathsf{h}(x,0) = \mathsf{h}(0,x) = 0$, it follows that $v = 0$. Hence, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}\big(\mathbf{w}(y), \mathsf{g}(u_1,\dots,u_k,0)\big)$. From the same definition, namely the condition 2c), we get $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}\big(\mathbf{w}(y), \mathsf{g}(u_1,\dots,u_k)\big)$. From i) above, we have $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Proportionality: Let $x, y \in \mathcal{A}$ (resp. $x, y \in \mathcal{R}$) such that:

i) $\mathbf{w}(x) > \mathbf{w}(y)$,
ii) $\mathsf{Sr}(x) = \mathsf{Sr}(y)$,
iii) $\forall z \in \mathsf{Sr}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,y))$,
iv) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$,

Let $\mathsf{Sr}(x) = \mathsf{Sr}(y) = \{a_1,\dots,a_k\}$.
$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), v)$ where $v = \mathsf{g}(u_1,\dots,u_k)$ and for $i \in [1,k]$, $u_i = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i,x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)\big)$.
$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), v')$ where $v' = \mathsf{g}(u'_1,\dots,u'_k)$. For $i \in [1,k]$, $u'_i = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i,y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)\big)$.
It follows from iii) that $\forall i \in [1,k]$, $u_i = u'_i$, and thus $v = v'$. Hence,

$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), v) \text{ and } \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), v).$$

Since f is strictly increasing on the first variable and from i), it follows that $\mathsf{f}(\mathbf{w}(x), v) > \mathsf{f}(\mathbf{w}(y), v)$, thus $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Weakening: Let $x \in \mathcal{A}$ (resp. $x \in \mathcal{R}$) such that $\mathbf{w}(x) > 0$. Let $\mathsf{Att}_{\mathbf{H}}(x) = \{\alpha_1,\dots,\alpha_k\}$ and $\mathsf{Sr}_{\mathbf{H}}(x) = \{a_1,\dots,a_k\}$, i.e., $\mathbf{s}(\alpha_i) = a_i$ for $i \in [1,k]$. Assume also that for some $i \in [1,k]$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\alpha_i) > 0$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i) > 0$. Since $\mathbf{M}$ is well-behaved, then $\mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\alpha_i), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)) > 0$. Let $u_i = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\alpha_i), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)) > 0$.

By monotonicity of g, $\mathsf{g}(u_1,\dots,u_k) \geq \mathsf{g}(u_i)$. Since $\mathsf{g}(u_i) = u_i$ (condition 2b), then $\mathsf{g}(u_1,\dots,u_k) > 0$. Since f is decreasing in the second variable (condition 1b) and $\mathbf{w}(x) \neq 0$, then $\mathsf{f}(\mathbf{w}(x), \mathsf{g}(u_1,\dots,u_k)) < \mathsf{f}(\mathbf{w}(x), 0)$. Hence, $\mathsf{f}(\mathbf{w}(x), \mathsf{g}(u_1,\dots,u_k)) < \mathbf{w}(x)$, which shows that $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) < \mathbf{w}(x)$.

Equality, A-Precedence and R-Precedence: Let us first show that S satisfies Equality. Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $a \in \mathcal{A}$ and $\alpha \in \mathcal{R}$ such that the following hold:

i) $\mathbf{w}(a) = \mathbf{w}(\alpha)$,
ii) $\mathsf{Sr}_{\mathbf{H}}(a) = \mathsf{Sr}_{\mathbf{H}}(\alpha)$,
iii) $\forall z \in \mathsf{Sr}_{\mathbf{H}}(a)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,a)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,\alpha))$,

Assume that $\mathsf{Sr}_{\mathbf{H}}(a) = \emptyset$. Hence, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathsf{f}(\mathbf{w}(a), \mathsf{g}())$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\alpha) = \mathsf{f}(\mathbf{w}(\alpha), \mathsf{g}())$. From Definition 13, $\mathsf{g}() = 0$, $\mathsf{f}(\mathbf{w}(a), 0) = \mathbf{w}(a)$ and $\mathsf{f}(\mathbf{w}(\alpha), 0) = \mathbf{w}(\alpha)$. From i), $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\alpha)$.

Assume now that $\mathsf{Sr}_{\mathbf{H}}(a) = \mathsf{Sr}_{\mathbf{H}}(\alpha) = \{a_1,\dots,a_k\}$. Then,

$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathsf{f}(\mathbf{w}(a), \mathsf{g}(u_1,\dots,u_k)) \text{ and } \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\alpha) = \mathsf{f}(\mathbf{w}(\alpha), \mathsf{g}(u'_1,\dots,u'_k)),$$

where for $i \in [1,k]$, the following hold:

- $u_i = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i,a)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i))$
- $u'_i = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i,\alpha)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i))$

From iii), for any $i \in [1,k]$, $u_i = u'_i$. Thus, $\mathsf{g}(u_1,\dots,u_k) = \mathsf{g}(u'_1,\dots,u'_k)$. From i), $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\alpha)$.

From Theorem 1, since S satisfies Equality, then it also satisfies A-Precedence and R-Precedence.

Anonymity: Assume that $\mathbf{M} \in \mathbb{M}$.

Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle, \mathbf{H}' = \langle \mathcal{A}', \mathcal{R}', \mathbf{s}', \mathbf{t}', \mathbf{w}' \rangle \in \mathsf{AF}$ and F an isomorphism between $\mathbf{H}$ and $\mathbf{H}'$. Let us show that:

$$\forall x \in \mathcal{A} \cup \mathcal{R}, \qquad \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(F(x)) \qquad (1).$$

From Theorem 8, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(.)$ is calculated in an iterative way. Let σ_a and σ_b be the scoring functions used to calculate $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x)$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(F(x))$. Hence,

$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \lim_{n\to\infty} \sigma_a(x)^{(n)} \text{ and } \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(F(x)) = \lim_{n\to\infty} \sigma_b(F(x))^{(n)}.$$

We prove the equality (1) by induction on $i \in \{1, 2, 3, \ldots\}$.

For $i = 1$: $\sigma_a(x)^{(1)} = \mathbf{w}(x)$ and $\sigma_b(F(x))^{(1)} = \mathbf{w}'(F(x))$. Since $\mathbf{w}(x) = \mathbf{w}'(F(x))$, then $\sigma_a(x)^{(1)} = \sigma_b(F(x))^{(1)}$.

For $i > 1$: Assume that $\forall x \in \mathcal{A} \cup \mathcal{R}$, $\sigma_a(x)^{(i)} = \sigma_b(F(x))^{(i)}$, and let us show that the equality holds at step $i + 1$. Assume that $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots, \alpha_k\}$ and $\mathtt{Att}'_{\mathbf{H}}(F(x)) = \{\beta_1, \ldots, \beta_l\}$.

$$\sigma_a(x)^{(i+1)} = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\Big(\mathsf{h}\big(\sigma_a(\alpha_1)^{(i)}, \sigma_a(\mathbf{s}(\alpha_1))^{(i)}\big), \ldots, \mathsf{h}\big(\sigma_a(\alpha_k)^{(i)}, \sigma_a(\mathbf{s}(\alpha_k))^{(i)}\big)\Big)\Bigg),$$

$$\sigma_b(F(x))^{(i+1)} = \mathsf{f}\Bigg(\mathbf{w}'(F(x)), \mathsf{g}\Big(\mathsf{h}\big(\sigma_b(\beta_1)^{(i)}, \sigma_b(\mathbf{s}'(\beta_1))^{(i)}\big), \ldots, \mathsf{h}\big(\sigma_b(\beta_l)^{(i)}, \sigma_b(\mathbf{s}'(\beta_l))^{(i)}\big)\Big)\Bigg).$$

Since F is an isomorphism, then $k = l$ and for any $j \in [1, k]$, $\beta_j = F(\alpha_j)$, $\mathbf{s}'(\beta_j) = \mathbf{s}(F(\alpha_j))$ and $\mathbf{t}'(\beta_j) = \mathbf{t}(F(\alpha_j))$. So, we get:

$$\sigma_b(F(x))^{(i+1)} = \mathsf{f}\Bigg(\mathbf{w}'(F(x)), \mathsf{g}\Big(\mathsf{h}\big(\sigma_b(F(\alpha_1))^{(i)}, \sigma_b(\mathbf{s}'(F(\alpha_1)))^{(i)}\big), \ldots, \mathsf{h}\big(\sigma_b(F(\alpha_k)^{(i)}, \sigma_b(\mathbf{s}'(F(\alpha_k)))^{(i)}\big)\Big)\Bigg).$$

From the assumption: $\forall x \in \mathcal{A} \cup \mathcal{R}$, $\sigma_a(x)^{(i)} = \sigma_b(F(x))^{(i)}$ and the fact that $\mathbf{w}'(F(x) = \mathbf{w}(x)$, we get $\sigma_a(x)^{(i+1)} = \sigma_b(F(x))^{(i+1)}$. Letting $i \to +\infty$, we obtain $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(F(x))$ by Theorem 8.

Independence: Assume that $\mathbf{M} \in \mathbb{M}$.

Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle, \mathbf{H}' = \langle \mathcal{A}', \mathcal{R}', \mathbf{s}', \mathbf{t}', \mathbf{w}' \rangle \in \mathsf{AF}$ such that $\mathcal{A} \cap \mathcal{A}' = \varnothing$ and $\mathcal{R} \cap \mathcal{R}' = \varnothing$. Let $x \in \mathcal{A} \cup \mathcal{R}$. We show that $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}''}(x)$, where $\mathbf{H}'' = \mathbf{H} \oplus \mathbf{H}' = \langle \mathcal{A} \cup \mathcal{A}', \mathcal{R} \cup \mathcal{R}', \mathbf{s}'', \mathbf{t}'', \mathbf{w}'' \rangle$.

From the definition of $\oplus$, $\mathbf{w}(x) = \mathbf{w}''(x)$ and $\mathtt{Att}_{\mathbf{H}}(x) = \mathtt{Att}_{\mathbf{H}''}(x)$. Let $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots, \alpha_k\}$.

From Theorem 8, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(.)$ is calculated in an iterative way. Let σ_a and σ_b be the scoring functions used to calculate $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x)$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}''}(x)$ respectively. Hence,

$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \lim_{n\to\infty} \sigma_a(x)^{(n)} \text{ and } \mathbf{V}^{\mathsf{S}}_{\mathbf{H}''}(x) = \lim_{n\to\infty} \sigma_b(x)^{(n)}.$$

We show by induction that $\forall\, i \in \{1, 2, 3, \ldots\}$, the following property holds:

$$P(i) : \forall y \in \mathcal{A} \cup \mathcal{R}, \qquad \sigma_a(y)^{(i)} = \sigma_b(y)^{(i)}.$$

For $i = 1$, $\sigma_a(x)^{(1)} = \mathbf{w}(x)$ and $\sigma_b(x)^{(1)} = \mathbf{w}''(x) = \mathbf{w}(x)$.

For $i > 1$,

$$\sigma_a(x)^{(i+1)} = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\Big(\mathsf{h}\big(\sigma_a(\alpha_1)^{(i)}, \sigma_a(\mathbf{s}(\alpha_1))^{(i)}\big), \ldots, \mathsf{h}\big(\sigma_a(\alpha_k)^{(i)}, \sigma_a(\mathbf{s}(\alpha_k))^{(i)}\big)\Big)\Bigg),$$

$$\sigma_b(x)^{(i+1)} = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\Big(\mathsf{h}\big(\sigma_b(\alpha_1)^{(i)}, \sigma_b(\mathbf{s}(\alpha_1))^{(i)}\big), \ldots, \mathsf{h}\big(\sigma_b(\alpha_k)^{(i)}, \sigma_b(\mathbf{s}(\alpha_k))^{(i)}\big)\Big)\Bigg).$$

Since $P(i)$ holds, then for any $j = 1, \ldots, k$, the following hold: $\sigma_a(\alpha_j)^{(i)} = \sigma_b(\alpha_j)^{(i)}$ and $\sigma_a(\mathbf{s}(\alpha_j))^{(i)} = \sigma_b(\mathbf{s}(\alpha_j))^{(i)}$. So, $\sigma_a(x)^{(i+1)} = \sigma_b(x)^{(i+1)}$. Letting $i \longrightarrow +\infty$, we obtain $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}''}(x)$ by Theorem 8.

Directionality: Assume that $\mathbf{M} \in \mathbb{M}$.

Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ and $\mathbf{H}' = \langle \mathcal{A}', \mathcal{R}', \mathbf{s}', \mathbf{t}', \mathbf{w}' \rangle \in \mathsf{AF}$ such that $\mathcal{A}' = \mathcal{A}$, $\mathcal{R}' = \mathcal{R} \cup \{\pi\}$ with $\pi \notin \mathcal{R}$, $\pi = \mathbf{n}(c, x)$, $c \in \mathcal{A}$, $x \in \mathcal{A} \cup \mathcal{R}$, for any $y \in \mathcal{A} \cup \mathcal{R}$, $\mathbf{w}'(y) = \mathbf{w}(y)$. Let $z \in \mathcal{A} \cup \mathcal{R}$ such that there is no path from x to z. Let us show that $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(z) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(z)$.

For the chosen x, let $p(x) = \{y \in \mathcal{A} \cup \mathcal{R} \mid \text{there is a path from } x \text{ to } y\}$. Note that since $\mathcal{R}' = \mathcal{R} \cup \{\pi\}$, there is a path from x to y in $\mathbf{H}$ ***iff*** there is a path from x to y in $\mathbf{H}'$, so we use the same notation $p(x)$ for both graphs.

From Theorem 8, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(.)$ is calculated in an iterative way. Let σ_a and σ_b be the scoring functions used to calculate the scores in $\mathbf{H}$ and $\mathbf{H}'$ respectively. Hence,

$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(z) = \lim_{n \to \infty} \sigma_a(z)^{(n)} \text{ and } \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(z) = \lim_{n \to \infty} \sigma_b(z)^{(n)}.$$

We show by induction that $\forall\, i \in \{1, 2, 3, \ldots\}$, the following holds:

$$P(i): \quad \forall\, e \notin p(x), \qquad \sigma_a(e)^i = \sigma_b(e)^i.$$

Let $e \notin p(x)$.

- Case $i = 1$. We have that $\sigma_a(e)^1 = \mathbf{w}(e)$ and $\sigma_b(e)^1 = \mathbf{w}'(e)$. Since $\mathbf{w}'(e) = \mathbf{w}(e)$ (by assumption), then $P(1)$ holds.
- Case $i > 1$. Suppose that $\forall\, j \in \{1, \ldots, i\}$, $P(j)$ holds. We show that $P(i + 1)$ holds.
  Since $\mathcal{R}' = \mathcal{R} \cup \{\pi\}$, we have that $\mathtt{Att}_{\mathbf{H}}(e) = \mathtt{Att}_{\mathbf{H}'}(e)$. Let $\mathtt{Att}_{\mathbf{H}}(e) = \{\alpha_1, \ldots, \alpha_k\}$. We have that:

$$\sigma_a(e)^{(i+1)} = \mathsf{f}\Bigg(\mathbf{w}(e), \mathsf{g}\Big(\mathsf{h}\big(\sigma_a(\alpha_1)^{(i)}, \sigma_a(\mathbf{s}(\alpha_1))^{(i)}\big), \ldots, \mathsf{h}\big(\sigma_a(\alpha_k)^{(i)}, \sigma_a(\mathbf{s}(\alpha_k))^{(i)}\big)\Big)\Bigg),$$

$$\sigma_b(e)^{(i+1)} = \mathsf{f}\Bigg(\mathbf{w}'(e), \mathsf{g}\Big(\mathsf{h}\big(\sigma_b(\alpha_1)^{(i)}, \sigma_b(\mathbf{s}(\alpha_1))^{(i)}\big), \ldots, \mathsf{h}\big(\sigma_b(\alpha_k)^{(i)}, \sigma_b(\mathbf{s}(\alpha_k))^{(i)}\big)\Big)\Bigg).$$

  Since $\mathcal{R}' = \mathcal{R} \cup \{\pi\}$, we have that $\mathtt{Att}_{\mathbf{H}}(e) = \mathtt{Att}_{\mathbf{H}'}(e)$. From $e \notin p(x)$ we deduce that for every $y \in \mathtt{Att}_{\mathbf{H}}(e)$, we have $y \notin p(x)$, so $\sigma_a(y)^{(i)} = \sigma_b(y)^{(i)}$ by $P(i)$. Thus we proved that $\sigma_a(e)^{(i+1)} = \sigma_b(e)^{(i+1)}$. Since $z \notin p(x)$, then $\forall i \in \{1, 2, \ldots\}$, $\sigma_a(z)^{(i)} = \sigma_b(z)^{(i)}$. Letting $i \longrightarrow +\infty$, we obtain $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(z) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}'}(z)$ by Theorem 8.

Resilience: Assume that the function f of the evaluation method $\mathbf{M}$ satisfies the condition **(C1)**.

Let $x \in \mathcal{A} \cup \mathcal{R}$ such that $\mathbf{w}(x) > 0$ and $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots, \alpha_k\}$. By definition, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), v)$ where $v = \mathsf{g}(u_1, \ldots, u_k)$ with for $i \in [1, k]$, $u_i = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\alpha_i), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{s}(\alpha_i)))$. Since $\mathbf{w}(x) > 0$, then from **(C1)**, $\mathsf{f}(\mathbf{w}(x), v) > 0$, thus $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$.

Counting: Assume that the evaluation method $\mathbf{M}$ satisfies the condition **(C2)**.

Let $x, y \in \mathcal{A}$ (resp. $x, y \in \mathcal{R}$) such that:

i) $\mathbf{w}(x) = \mathbf{w}(y)$,
ii) $\mathsf{Sr}_{\mathbf{H}}(y) = \mathsf{Sr}_{\mathbf{H}}(x) \cup \{a\}$ with $a \in \mathcal{A} \setminus \mathsf{Sr}_{\mathbf{H}}(x)$,
iii) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) > 0$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, y)) > 0$,
iv) $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$
v) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$,

From Theorem 7, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) \leq \mathbf{w}(x)$ and from $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$, it follows that $\mathbf{w}(x) > 0$.

Assume $\mathsf{Sr}_{\mathbf{H}}(x) = \emptyset$. Hence, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}()) = \mathsf{f}(\mathbf{w}(x), 0) = \mathbf{w}(x)$ (since $\mathbf{M}$ is well behaved). $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), \mathsf{g}(u)) = \mathsf{f}(\mathbf{w}(y), u)$ where $u = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a)\big)$. From iii) and by Definition 13 of $\mathbf{M}$, $u > 0$. From $\mathbf{w}(x) > 0$ and the fact that f is strictly decreasing on the second variable, it follows that $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Let now $\mathsf{Sr}_{\mathbf{H}}(x) = \{a_1, \ldots, a_k\}$. $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), v)$ where $v = \mathsf{g}(u_1, \ldots, u_k)$ and for any $i \in [1, k]$, $u_i = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i, x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)\big)$. From Definition 13 of $\mathbf{M}$, $v = \mathsf{g}(u_1, \ldots, u_k) = \mathsf{g}(u_1, \ldots, u_k, 0)$.
$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), v')$ where $v' = \mathsf{g}(u'_1, \ldots, u'_k, u'_{k+1})$ and for any $i \in [1, k]$, $u'_i = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i, y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)\big)$ and $u'_{k+1} = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a)\big)$.

From ii) and iv), for any $i \in [1, k]$, $u_i = u'_i$. From iii) and by definition of $\mathbf{M}$, $u'_{k+1} > 0$. From condition **(C2)**, $\mathsf{g}(u_1, \ldots, u_k, 0) < \mathsf{g}(u_1, \ldots, u_k, u'_{k+1})$. Hence, $v < v'$. From $\mathbf{w}(x) > 0$ and the fact that f is strictly decreasing on the second variable, it follows that $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.
Attack-sensitivity: Assume that the evaluation method $\mathbf{M}$ satisfies the conditions **(C2)** and**(C3)**.
Let $x, y \in \mathcal{A}$ (resp. $x, y \in \mathcal{R}$) such that:

i) $\mathbf{w}(x) = \mathbf{w}(y)$,
ii) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$,
iii) $\mathsf{Sr}_{\mathbf{H}}(x) \setminus \mathsf{Sr}_{\mathbf{H}}(y) = \{a\}$, $\mathsf{Sr}_{\mathbf{H}}(y) \setminus \mathsf{Sr}_{\mathbf{H}}(x) = \{b\}$,
iv) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b) > 0$,
v) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b, y)) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, x))$,
vi) $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x) \cap \mathsf{Sr}_{\mathbf{H}}(y)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z, y))$.

From Theorem 7, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) \leq \mathbf{w}(x)$ and from $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$, it follows that $\mathbf{w}(x) > 0$.

Assume that $\mathsf{Sr}_{\mathbf{H}}(x) \cap \mathsf{Sr}_{\mathbf{H}}(y) = \emptyset$. Hence, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}(u)) = \mathsf{f}(\mathbf{w}(x), u)$ (since $\mathbf{M}$ is well behaved) and $u = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a)\big)$.
$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), \mathsf{g}(u')) = \mathsf{f}(\mathbf{w}(y), u')$ where $u' = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b, y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b)\big)$. From iv), v), and the condition **(C3)**, it follows that $u' > u$. From i), $\mathbf{w}(x) \neq 0$ and the fact that f is strictly decreasing on the second variable, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Assume now that $\mathsf{Sr}_{\mathbf{H}}(x) \cap \mathsf{Sr}_{\mathbf{H}}(y) = \{a_1, \ldots, a_k\}$. $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), v)$ where $v = \mathsf{g}(u_1, \ldots, u_k, u_{k+1})$ s.t. for any $i \in [1, k]$, $u_i = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i, x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)\big)$ and $u_{k+1} = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a)\big)$.

$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), v')$ where $v' = \mathsf{g}(u'_1, \ldots, u'_k, u'_{k+1})$ such that for any $i \in [1, k]$, $u'_i = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i, y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)\big)$ and $u'_{k+1} = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b, y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b)\big)$. From vi), it holds that for any $i \in [1, k]$, $u_i = u'_i$. Hence, we get:

- $v = \mathsf{g}(u_1, \ldots, u_k, u_{k+1})$
- $v' = \mathsf{g}(u_1, \ldots, u_k, u'_{k+1})$

From iv), $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b)$. Let $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = t$. So,

- $u_{k+1} = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a, x)), t\big)$
- $u'_{k+1} = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b, y)), t\big)$

From v) and since $t > 0$ (from iv)), then from condition **(C3)**, $u'_{k+1} > u_{k+1}$. From condition **(C2)**, $v' > v$. From i), we have $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), v)$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(x), v')$. From $\mathbf{w}(x) > 0$ and the fact that f is strictly decreasing on the second variable, it follows that $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.
Reinforcement: Assume that the evaluation method $\mathbf{M}$ satisfies the conditions **(C2)** and **(C4)**.
Let $x, y \in \mathcal{A}$ (resp. $x, y \in \mathcal{R}$) such that:

i) $\mathbf{w}(x) = \mathbf{w}(y)$,
ii) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$,
iii) $\mathsf{Sr}_{\mathbf{H}}(x) \setminus \mathsf{Sr}_{\mathbf{H}}(y) = \{a\}$, $\mathsf{Sr}_{\mathbf{H}}(y) \setminus \mathsf{Sr}_{\mathbf{H}}(x) = \{b\}$,
iv) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a)$,

v) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)) > 0$,
vi) $\forall z \in \mathsf{Sr}_{\mathbf{H}}(x) \cap \mathsf{Sr}_{\mathbf{H}}(y)$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(z,y))$.

From Theorem 7, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) \leq \mathbf{w}(x)$ and from $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > 0$, it follows that $\mathbf{w}(x) > 0$.

Assume that $\mathsf{Sr}_{\mathbf{H}}(x) \cap \mathsf{Sr}_{\mathbf{H}}(y) = \emptyset$. Hence, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}(u)) = \mathsf{f}(\mathbf{w}(x), u)$ and

$$u = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a)\big).$$

$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), \mathsf{g}(u')) = \mathsf{f}(\mathbf{w}(y), u')$ where

$$u' = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b)\big).$$

From iv), v), and the condition **(C4)**, it follows that $u' > u$. From i), $\mathbf{w}(x) \neq 0$ and the fact that f is strictly decreasing on the second variable, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Assume now that $\mathsf{Sr}_{\mathbf{H}}(x) \cap \mathsf{Sr}_{\mathbf{H}}(y) = \{a_1, \dots, a_k\}$. $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), v)$ where $v = \mathsf{g}(u_1, \dots, u_k, u_{k+1})$ and for any $i \in [1,k]$, $u_i = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i,x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)\big)$ and $u_{k+1} = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a)\big)$. $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), v')$ where $v' = \mathsf{g}(u'_1, \dots, u'_k, u'_{k+1})$ and for any $i \in [1,k]$, $u'_i = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i,y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i)\big)$ and $u'_{k+1} = \mathsf{h}\big(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b)\big)$. From iii) and vi), for any $i \in [1,k]$, $u'_i = u_i$. Hence, $v = \mathsf{g}(u_1, \dots, u_k, u_{k+1})$ and $v' = \mathsf{g}(u_1, \dots, u_k, u'_{k+1})$. From v), $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)) = v''$ and $v'' > 0$. So, $u_{k+1} = \mathsf{h}(v'', \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a))$ and $u'_{k+1} = \mathsf{h}(v'', \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b))$. From iv) and condition **(C4)**, it follows that $u'_{k+1} > u_{k+1}$. From condition **(C2)**, $v < v'$. From $\mathbf{w}(x) > 0$ and the fact that f is strictly decreasing on the second variable, it follows that $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) > \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Local Equality: Assume that h is symmetric, i.e., $\mathsf{h}(x,y) = \mathsf{h}(y,x)$.

Let $x, y \in \mathcal{R}$ (resp. $\forall x, y \in \mathcal{A}$) such that:
i) $\mathbf{w}(x) = \mathbf{w}(y)$,
ii) $\mathsf{Sr}_{\mathbf{H}}(x) = X \cup \{a\}$, $\mathsf{Sr}_{\mathbf{H}}(y) = X \cup \{b\}$,
iii) $\forall c \in \mathsf{Sr}_{\mathbf{H}}(x) \setminus \{a\}$, $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(c,x)) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(c,y))$,
iv) $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x))$ and $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y))$.

If $X = \emptyset$, then by definition of S and $\mathbf{M}$, we have:

$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}(v)) = \mathsf{f}(\mathbf{w}(x), v) \text{ and } \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), \mathsf{g}(v') = \mathsf{f}(\mathbf{w}(y), v'),$$

where $v = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a))$ and $v' = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b))$. From the condition iv), we get: $v = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a))$ and $v' = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b))$. From the symmetry of h, $v = v'$. From the condition i), $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$.

Assume now that $X = \{a_1, \dots, a_k\}$. Then,

$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}(v_1, \dots, v_k, v)) \text{ and } \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), \mathsf{g}(v'_1, \dots, v'_k, v'),$$

where for any $i \in [1,k]$, the following hold: $v_i = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i,x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i))$ and $v'_i = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a_i,y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a_i))$. Furthermore, $v = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(a,x)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(a))$ and $v' = \mathsf{h}(\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)), \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(b))$. From above, $v = v'$. From the condition iii), for any $i \in [1,k]$, $v_i = v'_i$. So,

$$\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}(v_1, \dots, v_k, v)) \text{ and } \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), \mathsf{g}(v_1, \dots, v_k, v).$$

From i), we get $\mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathsf{S}}_{\mathbf{H}}(y)$. □

Property 2. *Let* $\mathsf{S} \in \mathbf{ES}$ *be based on* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$. *If* S *satisfies Local Equality and* g *is strictly monotonic, then* h *is symmetric.*

Proof. Let $\mathsf{S} \in \mathbf{ES}$ be based on $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$ such that S satisfies Local Equality and g is strictly monotonic. Let $x, y \in \mathcal{R}$ (resp. $\forall x, y \in \mathcal{A}$) such that:
i) $\mathbf{w}(x) = \mathbf{w}(y)$,

ii) $\mathsf{Sr}_{\mathbf{H}}(x) = X \cup \{a\}$, $\mathsf{Sr}_{\mathbf{H}}(y) = X \cup \{b\}$,
iii) $\forall c \in \mathsf{Sr}_{\mathbf{H}}(x) \setminus \{a\}$, $\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(c,x)) = \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(c,y))$,
iv) $\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(b) = \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(a,x))$ and $\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(a) = \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(b,y))$.

If $X = \emptyset$, then by definition of **S** and **M**, we have:

$$\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}(v)) = \mathsf{f}(\mathbf{w}(x), v) \text{ and } \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), \mathsf{g}(v') = \mathsf{f}(\mathbf{w}(y), v'),$$

where $v = \mathsf{h}(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(a,x)), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(a))$ and $v' = \mathsf{h}(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(b))$. From the condition iv), we get:

$$v = \mathsf{h}(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(b), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(a)) \text{ and } v' = \mathsf{h}(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(a), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(b)).$$

Since **S** satisfies Local Equality, then $\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) = \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(y)$. Assume that $v > v'$. From the condition 1b) of Definition 13, we get $\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) < \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(y)$, which contradicts the previous equality.

Assume now that $X = \{c_1, \ldots, c_k\}$. Then,

$$\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}(v_1, \ldots, v_k, v)) \text{ and } \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(y), \mathsf{g}(v'_1, \ldots, v'_k, v'),$$

where for any $i \in [1,k]$, the following hold: $v_i = \mathsf{h}(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(c_i,x)), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(c_i))$ and $v'_i = \mathsf{h}(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(c_i,y)), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(c_i))$. Furthermore, $v = \mathsf{h}(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(a,x)), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(a))$ and $v' = \mathsf{h}(\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(\mathbf{n}(b,y)), \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(b))$. From the conditions i) and iii), for any $i \in [1,k]$, $v_i = v'_i$. So,

$$\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}(v_1, \ldots, v_k, v)) \text{ and } \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(y) = \mathsf{f}(\mathbf{w}(x), \mathsf{g}(v_1, \ldots, v_k, v')).$$

Assume that $v > v'$. From the strict monotonicity of g, we get: $\mathsf{g}(v_1, \ldots, v_k, v) > \mathsf{g}(v_1, \ldots, v_k, v')$. From the condition 1b) of Definition 13, we have $\mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(x) < \mathbf{V}^{\mathbf{S}}_{\mathbf{H}}(y)$, which contradicts the Local Equality condition. □

PROPERTY 3. *The following properties hold.*

(1) *The function* $\mathsf{f}_{\mathsf{frac}}$ *is well-behaved, positive, and rational.*
(2) *The function* $\mathsf{f}_{\mathsf{exp}}$ *is well-behaved and positive.*
(3) $\lambda \mathsf{f}_{\mathsf{exp}}(x, \lambda y) < \mathsf{f}_{\mathsf{exp}}(x,y)$, $\forall \lambda < 1$, $x \neq 0$ *and* $y \in [0,1]$.
(4) *The functions* $\mathsf{g}_{\mathsf{sum}}$ *and* $\mathsf{g}_{\mathsf{max}}$ *are well-behaved. In addition,* $\mathsf{g}_{\mathsf{sum}}$ *is strictly monotonic.*
(5) *For any* $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$*, the following holds:*
   - h *is well-behaved,*
   - *for any* $\mathsf{g} \in \{\mathsf{g}_{\mathsf{sum}}, \mathsf{g}_{\mathsf{max}}\}$*, the pair* $\langle \mathsf{g}, \mathsf{h} \rangle$ *is rational.*
(6) *The functions* $\mathsf{h}_2, \mathsf{h}_3, \mathsf{h}_4$ *are not well-behaved.*
(7) *The function* h_7 *is well-behaved; for any* $\mathsf{g} \in \{\mathsf{g}_{\mathsf{sum}}, \mathsf{g}_{\mathsf{max}}\}$*, the pair* $\langle \mathsf{g}, \mathsf{h}_7 \rangle$ *is not rational.*
(8) *The functions* $\mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3, \mathsf{h}_6, \mathsf{h}_7$ *are symmetric. The weighted functions* $\mathsf{h}_4, \mathsf{h}_5$ *are not.*

PROOF. Let us show the eight items of Property 3.

(1) Theorem 5.2 in (Amgoud and Doder 2019) shows that the function $\mathsf{f}_{\mathsf{frac}}$ satisfies the first four properties in item 1 of Definition 13, the first condition in Definition 15 and is positive. The function is also known to be continuous. So, it is well-behaved in the sense of Definition 13 and rational.
(2) Theorem 5.14 in (Amgoud and Doder 2019) shows that the function $\mathsf{f}_{\mathsf{exp}}$ satisfies the first four properties in item 1 of Definition 13 and is positive. The function is also known to be continuous. So, it is well-behaved in the sense of Definition 13.
(3) From Theorem 5.14 in (Amgoud and Doder 2019), it follows that the function $\mathsf{f}_{\mathsf{exp}}$ satisfies the first condition of Definition 15 for all $\lambda < 1$, $x \neq 0$ and $y \in [0,1]$.
(4) Let us show that $\mathsf{g}_{\mathsf{sum}}$ and $\mathsf{g}_{\mathsf{max}}$ are well-behaved. It has been shown in (Amgoud and Doder 2019) that the functions $\mathsf{g}_{\mathsf{sum}}$ and $\mathsf{g}_{\mathsf{max}}$ satisfy the first five properties in item 2 of Definition 13. Both functions are known to be continuous. So, they are well-behaved. Obviously, $\mathsf{g}_{\mathsf{sum}}$ is strictly monotonic in $[0,1]^n$.

(5) Let us show that for any $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$, for any $\mathsf{g} \in \{\mathsf{g}_{\mathsf{sum}}, \mathsf{g}_{\mathsf{max}}\}$, the pair $\langle \mathsf{g}, \mathsf{h} \rangle$ is rational. The three functions $\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6$ are aggregation operator, which are thus continuous and non-decreasing. They also satisfy the conditions 3.a,b of Definition 13 in a straightforward way. They are thus well-behaved.
Let us show that every pair $\langle \mathsf{g}, \mathsf{h} \rangle$ satisfies the last condition in Definition 15.
h_1 = **min.** Let us show that $\forall \lambda \in [0,1]$,
$$\mathsf{g}_{\mathsf{sum}}(\mathsf{h}_1(\lambda y_1, \lambda x_1), \ldots, \mathsf{h}_1(\lambda y_n, \lambda x_n)) \geq \lambda \mathsf{g}_{\mathsf{sum}}(\mathsf{h}_1(y_1, x_1), \ldots, \mathsf{h}_1(y_n, x_n)).$$
Let $i = 1, \ldots, n$. $\mathsf{h}_1(\lambda y_i, \lambda x_i) = min(\lambda y_i, \lambda x_i) = \lambda min(y_i, x_i)$. Then,
$$\mathsf{g}_{\mathsf{sum}}(\mathsf{h}_1(\lambda y_1, \lambda x_1), \ldots, \mathsf{h}_1(\lambda y_n, \lambda x_n)) = \sum_{i=1}^{n} min(\lambda y_i, \lambda x_i) = \sum_{i=1}^{n} \lambda min(y_i, x_i) =$$
$$\lambda \sum_{i=1}^{n} min(y_i, x_i) = \lambda \mathsf{g}_{\mathsf{sum}}(\mathsf{h}_1(y_1, x_1), \ldots, \mathsf{h}_1(y_n, x_n)).$$
Let us show that $\forall \lambda \in [0,1]$, we have:
$$\mathsf{g}_{\mathsf{max}}(\mathsf{h}_1(\lambda y_1, \lambda x_1), \ldots, \mathsf{h}_1(\lambda y_n, \lambda x_n)) \geq \lambda \mathsf{g}_{\mathsf{max}}(\mathsf{h}_1(y_1, x_1), \ldots, \mathsf{h}_1(y_n, x_n)).$$
Indeed:
$$\mathsf{g}_{\mathsf{max}}(\mathsf{h}_1(\lambda y_1, \lambda x_1), \ldots, \mathsf{h}_1(\lambda y_n, \lambda x_n)) = \max[\lambda min(y_1, x_1), \ldots, \lambda min(y_n, x_n)] =$$
$$\lambda \max[min(y_1, x_1), \ldots, min(y_n, x_n)] = \lambda \mathsf{g}_{\mathsf{max}}(\mathsf{h}_1(y_1, x_1), \ldots, \mathsf{h}_1(y_n, x_n)).$$
h_5 = **mw-avg.** Let $w, \lambda \in [0,1]$.
$$\mathsf{g}\Big(\mathsf{h}_5(\lambda x_1, \lambda y_1), \ldots, \mathsf{h}_5(\lambda x_n, \lambda y_n)\Big) = \mathsf{g}\Big(w\lambda x_1 + (1-w)\lambda y_1, \ldots, w\lambda x_n + (1-w)\lambda y_n\Big) =$$
$$g\Big(\lambda(wx_1 + (1-w)y_1), \ldots, \lambda(wx_n + (1-w)y_n)\Big) = X.$$
So, for any $\mathsf{g} \in \{\mathsf{g}_{\mathsf{sum}}, \mathsf{g}_{\mathsf{max}}\}$,
$$X = \lambda \mathsf{g}\Big(wx_1 + (1-w)y_1, \ldots, wx_n + (1-w)y_n\Big) = \lambda \mathsf{g}\Big(\mathsf{h}_5(x_1, y_1), \ldots, \mathsf{h}_5(x_n, y_n)\Big).$$
h_6 = **m-avg.** Note that modified weighted average (h_5) generalizes the average operator, where $w = \frac{1}{2}$. Since h_5 satisfies the property, then so is for h_6.

(6) Let us show that $\mathsf{h}_2, \mathsf{h}_3, \mathsf{h}_4$ are not well-behaved. Note that they violate the properties $\mathsf{h}(0, x) = \mathsf{h}(x, 0) = 0$. Indeed, $max(x, 0) = max(0, x) = x$ while $\frac{x+0}{2} = \frac{x}{2}$. Note that h_4 generalizes h_3 and since the latter violates the properties, then so it for h_4.

(7) Regarding h_7, it satisfies straightforwardly the properties in the item 3 of Definition 13. However, for any $\mathsf{g} \in \{\mathsf{g}_{\mathsf{sum}}, \mathsf{g}_{\mathsf{max}}\}$, the pair $\langle \mathsf{g}, \mathsf{h} \rangle$ violates the second condition of Definition 15. We show the result for $\mathsf{g}_{\mathsf{sum}}$ while the proof is the same for $\mathsf{g}_{\mathsf{max}}$. Let $\lambda \in [0,1]$, and $\mathsf{g} = \mathsf{g}_{\mathsf{sum}}$.
$$X = \mathsf{g}_{\mathsf{sum}}(\mathsf{h}_7(\lambda y_1, \lambda x_1), \ldots, \mathsf{h}_7(\lambda y_n, \lambda x_n)) = \mathsf{g}_{\mathsf{sum}}(\lambda^2 y_1 x_1, \ldots, \lambda^2 y_n x_n) = \lambda^2 \sum_{i=1}^{n} y_i x_i.$$
Besides, we have:
$$Y = \lambda \mathsf{g}_{\mathsf{sum}}(\mathsf{h}_7(y_1, x_1), \ldots, \mathsf{h}_7(y_n, x_n)) = \lambda \sum_{i=1}^{n} y_i x_i.$$
Since $\lambda \in [0,1]$, then $\lambda^2 \sum_{i=1}^{n} y_i x_i \leq \lambda \sum_{i=1}^{n} y_i x_i$. For instance, let $\sum_{i=1}^{n} y_i x_i = 1$ and $\lambda = 0.5$. Then, $X = 0.25$ while $Y = 0.5$.

(8) Note that the symmetry of $\mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3, \mathsf{h}_6, \mathsf{h}_7$ is straightforward from the definitions of the functions. Recall that h_4 is defined as follows: $\forall x, y \in [0,1]$, $\mathsf{h}_4(x, y) = wx + (1-w)y$. Assume $x = 0.2, y = 0.8$ and $w = 0.7$. Hence, $\mathsf{h}_4(0.2, 0.8) = 0.7 \times 0.2 + 0.3 \times 0.8 = 0.38$ while $\mathsf{h}_4(0.8, 0.2) = 0.7 \times 0.8 + 0.3 \times 0.2 = 0.71$. The same counter-example applies to h_5.

□

THEOREM 10. *For any* $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$*, the following properties hold:*

- $\langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h} \rangle \in \mathbb{M}$,
- $\langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle \in \mathbb{M}$.
- $\langle \mathsf{f}_{\mathtt{exp}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle \in \mathbb{M}$.

PROOF. Let $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$ be an evaluation method such that $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$. From item 5 of Property 3, h satisfies all the relevant conditions in Definition 13.

- Let $\mathsf{f} = \mathsf{f}_{\mathtt{frac}}$ and $\mathsf{g} \in \{\mathsf{g}_{\mathsf{sum}}, \mathsf{g}_{\mathsf{max}}\}$. From item 1 of Property 3, $\mathsf{f}_{\mathtt{frac}}$ satisfies all the conditions concerning f in Definitions 13 and 15. From item 4 of Property 3, g satisfies all the relevant conditions in Definition 13. Furthermore, the pairs $\langle \mathsf{g}, \mathsf{h} \rangle$ satisfy the second condition of Definition 15 ($\langle \mathsf{g}, \mathsf{h} \rangle$ being rational). Consequently, $\mathbf{M} \in \mathbb{M}$.
- Let $\mathsf{f} = \mathsf{f}_{\mathtt{exp}}$ and $\mathsf{g} = \mathsf{g}_{\mathsf{max}}$. From item 2 of Property 3, f satisfies all the conditions concerning f in Definition 13. Since $\mathsf{g}_{\mathsf{max}}(x_1, \ldots, x_n) \in [0, 1]$, then from item 3 of Property 3, the pair (f, g) is rational. From above, $\mathsf{g}_{\mathsf{max}}$ is well-behaved and rational, and the pairs $\langle \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle$ are rational. Then, $\mathbf{M} \in \mathbb{M}$.

□

THEOREM 11. *For any* $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$*, the inclusion* $\{\mathsf{eHbs}^{\mathsf{h}}, \mathsf{eMbs}^{\mathsf{h}}, \mathsf{eEbs}^{\mathsf{h}}\} \subseteq \mathbf{ES}$ *holds.*

PROOF. The inclusion follows straightforwardly from Theorem 10 and the definition of $\mathbf{ES}$. □

THEOREM 12. *Let* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathtt{AF}$ *such that* $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$ *(so a basic wHO-AF). For any semantics* $\mathsf{Hbs}^{\mathsf{h}} \in \mathbb{F}$ *(respectively* $\mathsf{Mbs}^{\mathsf{h}} \in \mathbb{F}$, $\mathsf{Ebs}^{\mathsf{h}} \in \mathbb{F}$*), if* $\langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h} \rangle \in \mathbb{M}$ *(respectively* $\langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle \in \mathbb{M}$, $\langle \mathsf{f}_{\mathtt{exp}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle \in \mathbb{M}$*), then* $\forall a \in \mathcal{A}$:

- $\mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}}}(a) = \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Hbs}^{\mathsf{h}}}(a)$,
- $\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(a) = \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Mbs}^{\mathsf{h}}}(a)$,
- $\mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(a) = \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Ebs}^{\mathsf{h}}}(a)$.

PROOF. Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathtt{AF}$ such that $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$. Thus, $\forall \alpha \in \mathcal{R}$, $\mathtt{Att}_{\mathbf{H}}(\alpha) = \varnothing$. Let $\{\mathsf{Hbs}^{\mathsf{h}}, \mathsf{Mbs}^{\mathsf{h}}, \mathsf{Ebs}^{\mathsf{h}}\} \subseteq \mathbb{F}$ such that $\{\langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h} \rangle, \langle \mathsf{f}_{\mathtt{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle, \langle \mathsf{f}_{\mathtt{exp}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle\} \subseteq \mathbb{M}$. By definition, $\forall a \in \mathcal{A}$,

- $\mathbf{Deg}_{\mathbf{H}}^{\mathsf{Hbs}^{\mathsf{h}}}(a) = \frac{\mathbf{w}(a)}{1 + \sum\limits_{\mathbf{t}(\pi)=a} \mathsf{h}\left(\mathbf{w}(\pi), \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Hbs}^{\mathsf{h}}}(\mathbf{s}(\pi))\right)}$
- $\mathbf{Deg}_{\mathbf{H}}^{\mathsf{Mbs}^{\mathsf{h}}}(a) = \frac{\mathbf{w}(a)}{1 + \max\limits_{\mathbf{t}(\pi)=a} \mathsf{h}\left(\mathbf{w}(\pi), \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Mbs}^{\mathsf{h}}}(\mathbf{s}(\pi))\right)}$
- $\mathbf{Deg}_{\mathbf{H}}^{\mathsf{Ebs}^{\mathsf{h}}}(a) = \mathbf{w}(a) \times e^{-\max\limits_{\mathbf{t}(\alpha)=a} \mathsf{h}\left(\mathbf{w}(\alpha), \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Ebs}^{\mathsf{h}}}(\mathbf{s}(\alpha))\right)}$

Note that every argument is assigned a single score.

From Theorem 4, $\forall a \in \mathcal{A}$, $\mathbf{V}_{\mathbf{H}}^{\mathsf{S}}(a) = \mathsf{f}\left(\mathbf{w}(a), \mathsf{g}\left(\mathsf{h}\left(\mathbf{w}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathsf{S}}(\mathbf{s}(\alpha_1))\right), \ldots, \mathsf{h}\left(\mathbf{w}(\alpha_n), \mathbf{V}_{\mathbf{H}}^{\mathsf{S}}(\mathbf{s}(\alpha_n))\right)\right)\right)$,
where $\mathtt{Att}_{\mathbf{H}}(a) = \{\alpha_1, \ldots, \alpha_n\}$.

From Theorem 11, $\{\mathsf{eHbs}^{\mathsf{h}}, \mathsf{eMbs}^{\mathsf{h}}, \mathsf{eEbs}^{\mathsf{h}}\} \subseteq \mathbf{ES}$, hence they assign a single score to every element. Then,

- $\mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}}}(a) = \frac{\mathbf{w}(a)}{1 + \sum\limits_{\mathbf{t}(\pi)=a} \mathsf{h}\left(\mathbf{w}(\pi), \mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}}}(\mathbf{s}(\pi))\right)} = \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Hbs}^{\mathsf{h}}}(a)$
- $\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(a) = \frac{\mathbf{w}(a)}{1 + \max\limits_{\mathbf{t}(\pi)=a} \mathsf{h}\left(\mathbf{w}(\pi), \mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(\mathbf{s}(\pi))\right)} = \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Mbs}^{\mathsf{h}}}(a)$

- $\mathbf{V}_{\mathbf{H}}^{\mathsf{Ebs}^{\mathsf{h}}}(a) = \mathbf{w}(a) \times e^{-\max\limits_{\mathbf{t}(\alpha)=a} \mathsf{h}\left(\mathbf{w}(\alpha), \mathbf{V}_{\mathbf{H}}^{\mathsf{Ebs}^{\mathsf{h}}}(\mathbf{s}(\alpha))\right)} = \mathbf{Deg}_{\mathbf{H}}^{\mathsf{Ebs}^{\mathsf{h}}}(a)$.

□

THEOREM 13. *Table 5 summarizes the list of principles that are satisfied/violated by* $\mathsf{eHbs}^{\mathsf{h}}$, $\mathsf{eMbs}^{\mathsf{h}}$ *and* $\mathsf{eEbs}^{\mathsf{h}}$ *for any* $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$.

PROOF. Recall that $\mathsf{eHbs}^{\mathsf{h}}$, $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ are based on $\langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h} \rangle$, $\langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle$ and $\langle \mathsf{f}_{\mathsf{exp}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h} \rangle$ respectively, with $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_5, \mathsf{h}_6\}$.

Theorem 11 shows that $\{\mathsf{eHbs}^{\mathsf{h}}, \mathsf{eMbs}^{\mathsf{h}}, \mathsf{eEbs}^{\mathsf{h}}\} \subseteq \mathbf{ES}$. From Theorem 9, it follows that $\mathsf{eHbs}^{\mathsf{h}}$, $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ satisfy Anonymity, Independence, Directionality, Equivalence, Maximality, Neutrality, Weakening, Proportionality, Equality, A-Precedence, and R-Precedence.

From Property 3, $\mathsf{f}_{\mathsf{frac}}$ and $\mathsf{f}_{\mathsf{exp}}$ are both positive. Then, from Theorem 9, $\mathsf{eHbs}^{\mathsf{h}}$, $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ satisfy Resilience.

From Property 3, $\mathsf{g}_{\mathsf{sum}}$ is strictly monotonic. Then, from Theorem 9, $\mathsf{eHbs}^{\mathsf{h}}$ satisfies Counting. In addition, for any $\mathsf{h} \in \{\mathsf{h}_5, \mathsf{h}_6\}$, h is strictly monotonic on both variables, then from Theorem 9, $\mathsf{eHbs}^{\mathsf{h}}$ satisfies Attack-sensitivity and Reinforcement.

$\mathsf{eHbs}^{\mathsf{h}_1}$ violates Attack-sensitivity as shown in Fig. 25.

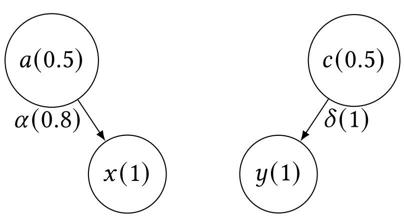


Fig. 25. Attack-sensitivity: counter example for $\mathsf{eHbs}^{\mathsf{h}_1}$

Note that $\mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}_1}}(x) = \mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}_1}}(y) = 0.66$.

$\mathsf{eHbs}^{\mathsf{h}_1}$ violates also Reinforcement as shown in Fig. 26.

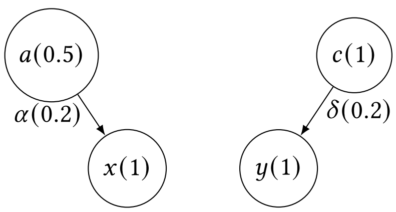


Fig. 26. Reinforcement: counter example for $\mathsf{eHbs}^{\mathsf{h}_1}$

Note that $\mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}_1}}(x) = \mathbf{V}_{\mathbf{H}}^{\mathsf{eHbs}^{\mathsf{h}_1}}(y) = 0.83$.

From Property 3, both $\mathsf{h}_1, \mathsf{h}_6$ are symmetric, then from Theorem 9, for any $\mathsf{h} \in \{\mathsf{h}_1, \mathsf{h}_6\}$, $\mathsf{eHbs}^{\mathsf{h}}$, $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ satisfy Local-Equality.

Local Equality is violated by any $\mathbf{S} \in \{\mathsf{eHbs}^{\mathsf{h}_5}, \mathsf{eMbs}^{\mathsf{h}_5}, \mathsf{eEbs}^{\mathsf{h}_5}\}$. Consider the graph in Fig. 27.

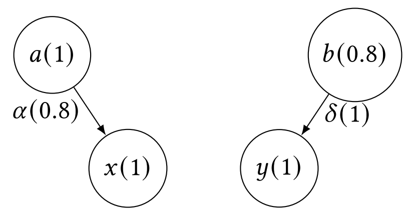


Fig. 27. Local-equality: counter example for $\mathsf{eHbs}^{\mathsf{h}_5}$, $\mathsf{eMbs}^{\mathsf{h}_5}$ and $\mathsf{eEbs}^{\mathsf{h}_5}$

Note that $\mathbf{V}_{\mathbf{H}}^{\mathbf{S}}(x) = \frac{1}{1+\mathsf{h}_5(1,0.8)}$ and $\mathbf{V}_{\mathbf{H}}^{\mathbf{S}}(y) = \frac{1}{1+\mathsf{h}_5(0.8,1)}$ for $\mathbf{S} \in \{\mathsf{eMbs}^{\mathsf{h}}, \mathsf{eEbs}^{\mathsf{h}}\}$. Note that if $w = 1$, then $\mathbf{V}_{\mathbf{H}}^{\mathbf{S}}(x) = \frac{1}{2}$ since $\mathsf{h}_5(1, 0.8) = 1$ while $\mathbf{V}_{\mathbf{H}}^{\mathbf{S}}(y) = \frac{1}{1{,}8}$ since $\mathsf{h}_5(0.8, 1) = 0.8$.

For $\mathbf{S} = \mathsf{eEbs}^{\mathsf{h}}$, we get: $\mathbf{V}_{\mathbf{H}}^{\mathbf{S}}(x) = e^{-1}$ while $\mathbf{V}_{\mathbf{H}}^{\mathbf{S}}(y) = e^{-0.8}$.

$\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ violate Counting for any h. Consider the following example (see Fig. 28).

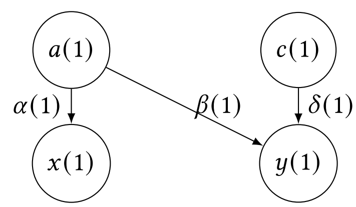


Fig. 28. Counting: counter example for $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ whatever h

Note that $\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(x) = \frac{1}{1+\mathsf{h}(1,1)}$ and $\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(y) = \frac{1}{1+\max\left(\mathsf{h}(1,1),\mathsf{h}(1,1)\right)} = \frac{1}{1+\mathsf{h}(1,1)} = \mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(x)$ for any aggregation operator h. In the same way, $\mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(x) = e^{-\mathsf{h}(1,1)}$ and $\mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(y) = e^{-\max\left(\mathsf{h}(1,1),\mathsf{h}(1,1)\right)} = e^{-\mathsf{h}(1,1)} = \mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(x)$.

$\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ violate Attack-Sensitivity for any h. Consider the graph in Fig. 29.

Note that for any h, $\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(x) = \mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(y) = \frac{1}{1+\mathsf{h}(1,1)} = 0.5$ and $\mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(x) = \mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(y) = e^{-\mathsf{h}(1,1)}$.

$\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ violate Reinforcement for any h. Consider the graph in Fig. 30.

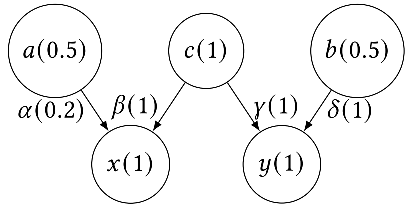


Fig. 29. Attack-sensitivity: counter example for $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ whatever h

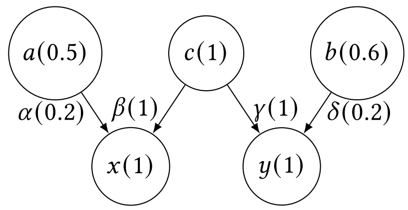


Fig. 30. Reinforcement: counter-example for $\mathsf{eMbs}^{\mathsf{h}}$ and $\mathsf{eEbs}^{\mathsf{h}}$ whatever h.

It can be checked that $\mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(x) = \mathbf{V}_{\mathbf{H}}^{\mathsf{eMbs}^{\mathsf{h}}}(y) = 0.50$ and $\mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(x) = \mathbf{V}_{\mathbf{H}}^{\mathsf{eEbs}^{\mathsf{h}}}(y) = e^{-\mathsf{h}(1,1)}$. □

Theorem 14. *For any* $\blacksquare \in \{c, p, g, s\}$, $\mathbf{E}_{\blacksquare}$ *is a gradual semantics.*

Proof. The properties follow from Property 1. □

Theorem 15. *Let* $\mathbf{E}_{\blacksquare}$ *be such that* $\blacksquare \in \{c, p, g, s\}$. *Table 5 gives the principles satisfied/violated by* $\mathbf{E}_{\blacksquare}$

This theorem is proven conjointly with Theorem 3.

Theorem 16. *For every pair of evaluation methods* $\mathbf{M}, \mathbf{M}' \in \mathbb{M}$, *there exists a unique hybrid semantics that is based on* $(\mathbf{M}, \mathbf{M}')$.

Proof. Let $\mathbf{M}_1 = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, $\mathbf{M}_2 = \langle \mathsf{f}', \mathsf{g}', \mathsf{h}' \rangle$ be two evaluation methods such that $\mathbf{M}_1, \mathbf{M}_2 \in \mathbb{M}$. We need to show that there is one and only one hybrid semantics Hyb based on $(\mathbf{M}_1, \mathbf{M}_2)$. We will prove this following closely the proof of Theorem 6.

In order to define one such hybrid semantics Hyb, first, we will fix an arbitrary wHO-AF $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle$ and we will define a weighting $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}} : \mathcal{A} \cup \mathcal{R} \longrightarrow [0, 1]$. Similarly as in the proof of Theorem 6, we fix an enumeration of the elements $\mathcal{A} \cup \mathcal{R} = \{x_1, \ldots, x_n\}$, and define the function $z : I_{\mathcal{R}} \longrightarrow \{1, \ldots, n\}$ such that $x_{z(i)} = \mathbf{s}(x_i)$, where

$$I_{\mathcal{R}} = \{i \in \{1, \ldots, n\} \mid x_i \in \mathcal{R}\}.$$

Now, for every $k \in \{1, \ldots, n\}$, let us introduce the notation:

$$\mathsf{f}^{(k)} = \begin{cases} \mathsf{f} & \text{if } k \notin I_{\mathcal{R}} \\ \mathsf{f}' & \text{if } k \in I_{\mathcal{R}} \end{cases} \qquad \mathsf{g}^{(k)} = \begin{cases} \mathsf{g} & \text{if } k \notin I_{\mathcal{R}} \\ \mathsf{g}' & \text{if } k \in I_{\mathcal{R}} \end{cases} \qquad \mathsf{h}^{(k)} = \begin{cases} \mathsf{h} & \text{if } k \notin I_{\mathcal{R}} \\ \mathsf{h}' & \text{if } k \in I_{\mathcal{R}}. \end{cases} \tag{32}$$

We also introduce the relation $\leq$ on n-tuples as follows: $(u_1, \ldots, u_n) \leq (v_1, \ldots, v_n)$ if and only if for every $k \in \{1, \ldots, n\}$ it holds that $u_k \leq v_k$.

We define the function $Q : [0,1]^n \to [0,1]^n$ with $Q(\mathbf{v}) = [Q_1(\mathbf{v}) \dots Q_n(\mathbf{v})]$, where, for every $k \in \{1, \dots, n\}$, Q maps each vector $\mathbf{v} = (v_1, \dots, v_n)$ to

$$Q_k(\mathbf{v}) = \mathsf{f}^{(k)}(\mathbf{w}(x_k), \mathsf{g}^{(k)}(\mathsf{h}^{(k)}(v_{\ell_k(1)}, v_{z(\ell_k(1))}), \dots, \mathsf{h}^{(k)}(v_{\ell_k(n_k)}, v_{z(\ell_k(n_k))}))), \tag{33}$$

where the indices are set as follows: if $\{y_1^k, \dots, y_{n_k}^k\} = \mathtt{Att}_{\mathbf{H}}(x_k)$ and $y_{i'}^k \neq y_{i''}^k$ whenever $i' \neq i''$, then for every $j \leq n_k$, $\ell_k(j) = i$ iff $y_j^k = x_i$.

Next, as in the proof of Theorem 6, we define the sequence of vectors $\{\mathbf{u}^{(m)}\}_{m=1}^{+\infty}$ as follows:

- $\mathbf{u}^{(1)} = (\mathbf{w}(x_1), \dots, \mathbf{w}(x_n))$.
- $\mathbf{u}^{(m+1)} = (u_1^{(m+1)}, \dots, u_n^{(m+1)}) = Q(\mathbf{u}^m)$, for each $m \geq 1$

Then we define the hybrid semantics Hyb which transforms $\mathbf{H}$ (we do this for each $\mathbf{H}$ - let us recall that above we fixed an arbitrary wHO-AF $\mathbf{H}$) into the weighting $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}} : \mathcal{A} \cup \mathcal{R} \to [0,1]$, such that for every $k \in \{1, \dots, n\}$,

$$\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x_k) = \lim_{m \to +\infty} u_k^{(m)}. \tag{34}$$

For that, we will first need to prove that the sequence $\{\mathbf{u}^{(m)}\}_{m=1}^{+\infty}$ necessarily converges. Then we will prove that

$$\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x_k) = \mathsf{f}^{(k)}\left(\mathbf{w}(x), \mathsf{g}^{(k)}\left(\mathsf{h}^{(k)}\left(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_1))\right), \dots, \mathsf{h}^{(k)}\left(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_n), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_n))\right)\right)\right). \tag{35}$$

Note that this means that Hyb based on $(\mathbf{M}_1, \mathbf{M}_2)$, since (35) can be equivalently rewritten as follows:

$$\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x) = \begin{cases} \mathsf{f}\left(\mathbf{w}(x), \mathsf{g}\left(\mathsf{h}\left(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_1))\right), \dots, \mathsf{h}\left(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_n), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_n))\right)\right)\right) & \text{if } x \in \mathcal{A} \\ \\ \mathsf{f}'\left(\mathbf{w}(x), \mathsf{g}'\left(\mathsf{h}'\left(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_1))\right), \dots, \mathsf{h}'\left(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\alpha_n), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_n))\right)\right)\right) & \text{if } x \in \mathcal{R}, \end{cases}$$

for every $x \in \mathcal{A} \cup \mathcal{R}$, i.e., Hyb is a hybrid semantics based on the evaluation methods $(\mathbf{M}_1, \mathbf{M}_2)$.

Similarly as in the proof of Theorem 6, using the fact that $\mathbf{M}_1$ and $\mathbf{M}_2$ are two well-behaved evaluation methods, we can show that if $(u_1, \dots, u_n) \leq (v_1, \dots, v_n)$, then for every $k \in \{1, \dots, n\}$ and $j \in \{1, \dots, n_k\}$,

$$\begin{gathered} f^{(k)}(\mathbf{w}(x_k), g^{(k)}(h^{(k)}(v_{\ell_k(1)}, v_{z(\ell_k(1))}), \dots, h^{(k)}(v_{\ell_k(n_k)}, v_{z(\ell_k(n_k))}))) \leq \\ f^{(k)}(\mathbf{w}(x_k), g^{(k)}(h^{(k)}(u_{\ell_k(1)}, u_{z(\ell_k(1))}), \dots, h^{(k)}(u_{\ell_k(n_k)}, u_{z(\ell_k(n_k))}))), \end{gathered} \tag{36}$$

i.e.,

$$(u_1, \dots, u_n) \leq (v_1, \dots, v_n) \quad \Longrightarrow \quad Q((u_1, \dots, u_n)) \geq Q((v_1, \dots, v_n)). \tag{37}$$

This can be used to show that for the sequence $\{\mathbf{u}^{(m)}\}_{m=1}^{+\infty}$ constructed above for the purpose of defining $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}$ (Equation (34)),

$$\mathbf{u}^{(2)} \leq \mathbf{u}^{(4)} \leq \dots \leq \mathbf{u}^{(2m)} \leq \mathbf{u}^{(2m+1)} \leq \dots \leq \mathbf{u}^{(3)} \leq \mathbf{u}^{(1)} \tag{38}$$

holds. From monotonicity and boundedness of the sequences $\{\mathbf{u}^{(2m)}\}_{m=1}^{+\infty}$ and $\{\mathbf{u}^{(2m+1)}\}_{m=1}^{+\infty}$, we obtain that they both converge, possibly to different values. Moreover, if we denote those values by $\mathbf{u}^*$ and $\mathbf{u}_*$, i.e.,

$$\lim_{m \to +\infty} \mathbf{u}^{(2m)} = \mathbf{u}^*, \quad \lim_{m \to +\infty} \mathbf{u}^{(2m+1)} = \mathbf{u}_*,$$

then we necessarily have

$$\mathbf{u}^* \leq \mathbf{u}_*. \tag{39}$$

Using the above equation, similarly as in the proof of Theorem 6, we can show that the set $\{\pi \mid \pi\mathbf{u}^{(2m-1)} \leq \mathbf{u}^{(2m)}\}$ is non-empty for every m, and that the sequence $\{\pi_k\}_{k=1}^{+\infty}$ defined by $\pi_m = \sup\{\pi \mid \pi\mathbf{u}^{(2m-1)} \leq \mathbf{u}^{(2m)}\}$, is a non-decreasing sequence of numbers from the unit interval of reals, so it converges. Next we prove that $\pi = \lim_{k\to+\infty} \pi_k = 1$.

If $g^* = \sup_{\overline{x}\in\bigcup_{n=0}^{+\infty}[0,1]^n} g(\overline{x})$, $g'^* = \sup_{\overline{x}\in\bigcup_{n=0}^{+\infty}[0,1]^n} g'(\overline{x})$, and $f_r^* = \lim_{y\to g_-^*} f(r,y)$, $f_r^{**} = \lim_{y\to 0_+} f(r,y)$, $f_r'^* = \lim_{y\to g_-'^*} f'(r,y)$, $f_r'^{**} = \lim_{y\to 0_+} f'(r,y)$, then for every positive r from the unit interval of reals there exist the functions $\varphi_r : (f_r^*, f_r^{**}] \longrightarrow [0,+\infty)$ and $\varphi_r' : (f_r'^*, f_r'^{**}] \longrightarrow [0,+\infty)$ such that

$$f(r,\varphi_r(y)) = y, \quad f'(r,\varphi_r'(y)) = y, \tag{40}$$

i.e., φ_r and φ_r' are the inverse functions of the functions obtained by f and f' (respectively) by fixing the first variable to be r. From Definition 13.1.b we obtain that φ_r is decreasing, and from Definition 13.1.e we obtain that φ_r is continuous. Now we introduce the notation

$$\varphi^{(k)} = \begin{cases} \varphi & \text{if } k \notin I_{\mathcal{R}} \\ \varphi' & \text{if } k \in I_{\mathcal{R}} \end{cases} \tag{41}$$

for every $k \in \{1,\ldots,n\}$. Then we can prove the inequality

$$\frac{u_k^{2i+2}}{f^{(k)}(\mathbf{w}(x_k), \pi_i\varphi^{(k)}_{\mathbf{w}(x_k)}(u_k^{2i}))} u_k^{2i+1} \leq u_k^{2i+2}, \tag{42}$$

for every $k \in \{1,\ldots,n\}$, in a similar way as we have proven the inequality (26) in the proof of Theorem 6, using the equations (17) and (19)-(25). Indeed, even if now we have two different methods for arguments and attacks, note that in the derivation, consisting of the equations (17) and (19)-(25), the number $k \in \{1,\ldots,n\}$ is fixed, so we will use only one method in the corresponding derivation for (42) (we will use only f, g and h if $k \notin I_{\mathcal{R}}$, and only f', g' and h' if $k \in I_{\mathcal{R}}$). By the inequality (42) and the definition of π_i, for every $i \in \mathbb{N}$, there exists $k_i \in \{1,\ldots,n\}$ such that

$$\frac{u_{k_i}^{2i+2}}{f^{(k_i)}(\mathbf{w}(x_{k_i}), \pi_i\varphi^{(k_i)}_{\mathbf{w}(x_{k_i})}(u_{k_i}^{2i}))} \leq \pi_{i+1}. \tag{43}$$

Let's observe, similarly as in the proof of Theorem 6, that $\{k_i\}_{i=1}^{+\infty}$ is an infinite sequence, whose elements belong to the finite set $\{1,\ldots,n\}$, so there exists $l \in \{1,\ldots,n\}$ such that l appears infinitely many times in the sequence. By applying limit to the inequality (43), using the subsequence obtained by taking only those i for which $k_i = l$, we obtain, after using some basic rewriting,

$$u_l^* \leq \pi f^{(l)}(\mathbf{w}(x_l), \pi\varphi^{(l)}_{\mathbf{w}(x_l)}(u_l^*)). \tag{44}$$

If we assume $\pi < 1$, from Definition 15. we obtain

$$\pi f^{(l)}(\mathbf{w}(x_l), \pi\varphi^{(l)}_{\mathbf{w}(x_l)}(u_l^*)) < f^{(l)}(\mathbf{w}(x_l), \varphi^{(l)}_{\mathbf{w}(x_l)}(u_l^*)). \tag{45}$$

Furthermore, from (40) and (41) we obtain

$$f^{(l)}(\mathbf{w}(x_l), \varphi^{(l)}_{\mathbf{w}(x_l)}(u_l^*)) = u_l^*. \tag{46}$$

It would follow from (44), (45) and (46) that $u_l^* < u_l^*$which is impossible, so $\pi = 1$. Finally, from the definition of π_m, for every $m \in \mathbb{N}$ we have $\pi_m\mathbf{u}^{(2m-1)} \leq \mathbf{u}^{(2m)}$, so applying $m \longrightarrow +\infty$ we have $\mathbf{u}_* \leq \mathbf{u}^*$. Together with (39), this gives us $\mathbf{u}_* = \mathbf{u}^*$, so the sequence $\{\mathbf{u}^{(m)}\}_{m=1}^{+\infty}$ converges. Then the proof that Hyb, defined by $\{\mathbf{u}^{(m)}\}_{m=1}^{+\infty}$, is really based on $(\mathbf{M}_1, \mathbf{M}_2)$, as well as that it is the only hybrid semantics based on $(\mathbf{M}_1, \mathbf{M}_2)$, is straightforward and almost identical to the corresponding part of the proof of Theorem 6. ◻

THEOREM 17. *Let* Hyb *be a hybrid semantics based on the evaluation methods* $(\mathbf{M}, \mathbf{M}')$, *where* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, $\mathbf{M}' = \langle \mathsf{f}', \mathsf{g}', \mathsf{h}' \rangle$. *Let* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, *and* $x \in \mathcal{A} \cup \mathcal{R}$ *such that* $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots \alpha_k\}$. *We define the sequence* $\{\sigma(x)^{(n)}\}_{n=1}^{+\infty}$ *in the following way:*

- $\sigma(x)^{(1)} = \mathbf{w}(x)$.
- $\sigma(x)^{(n+1)} = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\bigg(\mathsf{h}\Big(\sigma(\alpha_1)^{(n)}, \sigma(\mathbf{s}(\alpha_1))^{(n)}\Big), \ldots, \mathsf{h}\Big(\sigma(\alpha_k)^{(n)}, \sigma(\mathbf{s}(\alpha_k))^{(n)}\Big)\bigg)\Bigg)$, *if* $x \in \mathcal{A}$.
- $\sigma(x)^{(n+1)} = \mathsf{f}'\Bigg(\mathbf{w}(x), \mathsf{g}'\bigg(\mathsf{h}'\Big(\sigma(\alpha_1)^{(n)}, \sigma(\mathbf{s}(\alpha_1))^{(n)}\Big), \ldots, \mathsf{h}'\Big(\sigma(\alpha_k)^{(n)}, \sigma(\mathbf{s}(\alpha_k))^{(n)}\Big)\bigg)\Bigg)$, *if* $x \in \mathcal{R}$.

For every $x \in \mathcal{A} \cup \mathcal{R}$, *the following hold:*

(1) $\{\sigma(x)^{(n)}\}_{n=1}^{+\infty}$ *converges,*
(2) $\lim\limits_{n \to +\infty} \sigma(x)^{(n)} = \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x)$.

PROOF. The results follow straightforwardly from Theorem 16. □

THEOREM 18. *Let* Hyb *be a hybrid semantics based on the evaluation methods* $(\mathbf{M}, \mathbf{M}')$, *where* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, $\mathbf{M}' = \langle \mathsf{f}', \mathsf{g}', \mathsf{h}' \rangle$, *and* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ *such that* $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$. *For any* $x \in \mathcal{A} \cup \mathcal{R}$, *the following hold:*

- *If* $x \in \mathcal{R}$, *then* $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x) = \mathbf{w}(x)$.
- *If* $x \in \mathcal{A}$, *then* $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(x) = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\bigg(\mathsf{h}\Big(\mathbf{w}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{w}(\alpha_k), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}}(\mathbf{s}(\alpha_n))\Big)\bigg)\Bigg)$,
  where $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots \alpha_k\}$.

PROOF. Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ such that $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$. Hence, $\forall \alpha \in \mathcal{R}$, $\mathtt{Att}_{\mathbf{H}}(\alpha) = \emptyset$. Since $\mathbf{M}' \in \mathbf{M}$, then it is well-behaved. Consequently, $\mathsf{g}'() = 0$ and $\mathsf{f}'(\mathbf{w}(\alpha), 0) = \mathbf{w}(\alpha)$.

The second property is straightforward. □

COROLLARY 1. *Let* $\mathrm{S}_1 \in \mathbf{ES}$ *be an extended gradual semantics based on an evaluation method* $\mathbf{M}$, *and let* S_2 *be a hybrid semantics based on* $(\mathbf{M}', \mathbf{M}'')$. *Let* $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ *be such that* $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$.

$$\textit{If } \mathbf{M} = \mathbf{M}', \ \textit{then} \ \forall x \in \mathcal{A} \cup \mathcal{R}, \ \mathbf{V}_{\mathbf{H}}^{\mathrm{S}_1}(x) = \mathbf{V}_{\mathbf{H}}^{\mathrm{S}_2}(x).$$

PROOF. Let $\mathrm{S}_1 \in \mathbf{ES}$ be an extended gradual semantics based on $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, and let S_2 be a hybrid semantics based on $(\mathbf{M}', \mathbf{M}'')$. Consider $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$ such that $\forall \alpha \in \mathcal{R}$, $\mathbf{t}(\alpha) \in \mathcal{A}$. Assume that $\mathbf{M} = \mathbf{M}'$. From Theorems 4 and 18, $\forall x \in \mathcal{R}$, $\mathbf{V}_{\mathbf{H}}^{\mathrm{S}_1}(x) = \mathbf{V}_{\mathbf{H}}^{\mathrm{S}_2}(x) = \mathbf{w}(x)$. Furthermore, $\forall x \in \mathcal{A}$,

$$\mathbf{V}_{\mathbf{H}}^{\mathrm{S}_1}(x) = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\bigg(\mathsf{h}\Big(\mathbf{w}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathrm{S}_1}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{w}(\alpha_k), \mathbf{V}_{\mathbf{H}}^{\mathrm{S}_1}(\mathbf{s}(\alpha_n))\Big)\bigg)\Bigg),$$

and

$$\mathbf{V}_{\mathbf{H}}^{\mathrm{S}_2}(x) = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\bigg(\mathsf{h}\Big(\mathbf{w}(\alpha_1), \mathbf{V}_{\mathbf{H}}^{\mathrm{S}_2}(\mathbf{s}(\alpha_1))\Big), \ldots, \mathsf{h}\Big(\mathbf{w}(\alpha_k), \mathbf{V}_{\mathbf{H}}^{\mathrm{S}_2}(\mathbf{s}(\alpha_n))\Big)\bigg)\Bigg),$$

where $\mathtt{Att}_{\mathbf{H}}(x) = \{\alpha_1, \ldots \alpha_k\}$. From Theorem 6, since $\mathbf{M} \in \mathbb{M}$, then there exists a single extended gradual semantics that is based on $\mathbf{M}$. Then, the two equations above have a single solution and so $\mathbf{V}_{\mathbf{H}}^{\mathrm{S}_1}(x) = \mathbf{V}_{\mathbf{H}}^{\mathrm{S}_2}(x)$. □

THEOREM 19. *Let* Hyb *be a hybrid semantics based on the evaluation methods* $(\mathbf{M}, \mathbf{M}')$, *where* $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$ *and* $\mathbf{M}' = \langle \mathsf{f}', \mathsf{g}', \mathsf{h}' \rangle$.

(1) Hyb *satisfies Anonymity, Independence, Directionality, Equivalence, Maximality, Neutrality, Weakening and Proportionality.*
(2) *If* f *(resp.* f'*) is positive, then* Hyb *satisfies A- (resp. R-) Resilience.*
(3) *If* g *(resp.* g'*) is strictly monotonic, then:*
- Hyb *satisfies A- (resp. R-) Counting.*
- *If* h *(resp.* h'*) is strictly monotonic on the first variable (F1), then* Hyb *satisfies A- (resp. R-) Attack-sensitivity.*
- *If* h *(resp.* h'*) is strictly monotonic on the second variable (F2), then* Hyb *satisfies A- (resp. R-) Reinforcement.*

(4) *If* h *(resp.* h'*) is symmetric,* $\mathsf{h}(x, y) = \mathsf{h}(y, x)$*, then* Hyb *satisfies A- (resp. R-) Local-Equality.*
(5) *If* $\mathsf{f} = \mathsf{f}'$*,* $\mathsf{g} = \mathsf{g}'$*,* $\mathsf{h}' \succeq \mathsf{h}$ *(resp.* $\mathsf{h} \succeq \mathsf{h}'$*), then* Hyb *satisfies A-Precedence (resp. R-Precedence).*

Proof. Let Hyb be a hybrid semantics based on the evaluation methods $(\mathbf{M}, \mathbf{M}')$, where $\mathbf{M} = \langle \mathsf{f}, \mathsf{g}, \mathsf{h} \rangle$, $\mathbf{M}' = \langle \mathsf{f}', \mathsf{g}', \mathsf{h}' \rangle$ and $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$.

Items (1)-(4).

The proofs for Anonymity, Independence, Directionality, Equivalence, Maximality, Neutrality, Weakening, Proportionality, Resilience, Counting, Attack-Sensitivity, Reinforcement and Local Equality are similar to those detailed for Theorem 9.

Item (5). Assume that $\mathsf{f} = \mathsf{f}'$, $\mathsf{g} = \mathsf{g}'$ and $\mathsf{h}' \succeq \mathsf{h}$. Let $x \in \mathcal{A}$ and $y \in \mathcal{R}$ such that:

i) $\mathbf{w}(x) = \mathbf{w}(y)$,
ii) $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y)$,
iii) $\forall c \in \mathsf{Sr}_{\mathbf{H}}(x), \mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(\mathbf{n}(c, x)) = \mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(\mathbf{n}(c, y))$,

Due to the condition ii), let $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y) = \{a_1, \ldots, a_k\}$. By definition of Hyb and since $\mathsf{f} = \mathsf{f}'$, and $\mathsf{g} = \mathsf{g}'$, we have:

$$\mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(x) = \mathsf{f}\Bigg(\mathbf{w}(x), \mathsf{g}\Bigg(u_1, \ldots, u_k\Bigg)\Bigg) \text{ and } \mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(y) = \mathsf{f}\Bigg(\mathbf{w}(y), \mathsf{g}\Bigg(v_1, \ldots, v_k\Bigg)\Bigg)$$

where for any $i = 1, \ldots, k$,

- $u_i = \mathsf{h}(\mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(\mathbf{n}(a_i, x)), \mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(a_i))$
- $v_i = \mathsf{h}'(\mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(\mathbf{n}(a_i, y)), \mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(a_i))$

From iii), it follows that for any $i = 1, \ldots, k$,

- $v_i = \mathsf{h}'(\mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(\mathbf{n}(a_i, x)), \mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(a_i))$.

Since $\mathsf{h}' \succeq \mathsf{h}$, then for any $i = 1, \ldots, k$, $\mathsf{h}'(\mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(\mathbf{n}(a_i, x)), \mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(a_i)) \geq \mathsf{h}(\mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(\mathbf{n}(a_i, x)), \mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(a_i))$, i.e., $v_i \geq u_i$. Since $\mathbf{M}$ is well-behaved, then g is non-decreasing, therfore:

$$\mathsf{g}(v_1, \ldots, v_k) \geq \mathsf{g}(u_1, \ldots, u_k).$$

From i), $\mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(x) \geq \mathbf{V}^{\mathsf{Hyb}}_{\mathbf{H}}(y)$. Thus, Hyb satisfies **A-Precedence**.

The same reasoning holds for showing **R-Precedence** in the case $\mathsf{h} \succeq \mathsf{h}'$. □

Theorem 20. *The hybrid semantics* Hyb_1 *satisfies R-Precedence and violates Equality and A-Precedence.*

Proof. Recall that the hybrid semantics Hyb_1 is based on the evaluation methods $(\mathbf{M}_1, \mathbf{M}_2)$, where $\mathbf{M}_1 = \langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h}_6 \rangle$ and $\mathbf{M}_2 = \langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{max}}, \mathsf{h}_6 \rangle$. Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $x \in \mathcal{A}$ and $y \in \mathcal{R}$ such that:

i) $\mathbf{w}(x) = \mathbf{w}(y)$,
ii) $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y)$,
iii) $\forall c \in \mathsf{Sr}_{\mathbf{H}}(x), \mathbf{V}^{\mathsf{Hyb}_1}_{\mathbf{H}}(\mathbf{n}(c, x)) = \mathbf{V}^{\mathsf{Hyb}_1}_{\mathbf{H}}(\mathbf{n}(c, y))$,

Due to the condition ii), let $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y) = \{a_1, \dots, a_k\}$. By definition of Hyb_1,

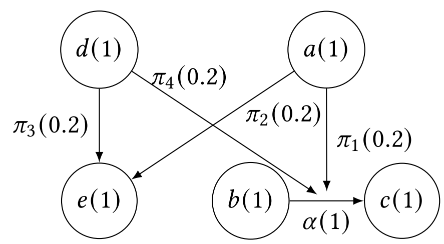
$$\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_1}(x) = \frac{\mathbf{w}(x)}{1 + \sum\limits_{i=1,k} u_i} \text{ and } \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_1}(y) = \frac{\mathbf{w}(y)}{1 + \max\limits_{i=1,k} v_i},$$

where for any $i = 1, \dots, k$,

- $u_i = \mathsf{h}_6\big(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_1}(\mathbf{n}(a_i, x)), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_1}(a_i)\big)$
- $v_i = \mathsf{h}_6\big(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_1}(\mathbf{n}(a_i, y)), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_1}(a_i)\big)$

From iii), it follows that for any $i = 1, \dots, k$, $u_i = v_i$. Since $u_1, \dots, u_k \in [0, 1]$, then $\sum\limits_{i=1,k} u_i \geq \max\limits_{i=1,k} u_i$. From $\mathbf{w}(x) = \mathbf{w}(y)$, we get $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_1}(x) \leq \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_1}(y)$. Thus, Hyb_1 satisfies R-Precedence.

Let us now show that Hyb_1 violates Equality and *A-Precedence*. It is sufficient to consider the wHO-AF $\mathbf{H}_3$ depicted in Figure 12 and recalled below:



It is easy to check that $\mathbf{V}_{\mathbf{H}_3}^{\mathsf{Hyb}_1}(e) = \frac{1}{1+2\mathsf{h}_6(0.2,1)} = 0.454$ and $\mathbf{V}_{\mathbf{H}_3}^{\mathsf{Hyb}_1}(\alpha) = \frac{1}{1+\mathsf{h}_6(0.2,1)} = 0.625$. Then, $\mathbf{V}_{\mathbf{H}_3}^{\mathsf{Hyb}_1}(e) < \mathbf{V}_{\mathbf{H}_3}^{\mathsf{Hyb}_1}(\alpha)$ which shows that Hyb violates Equality and A-Precedence. □

Theorem 21. *The hybrid semantics* Hyb_2 *satisfies A-Precedence and violates Equality and R-Precedence.*

Proof. Recall that the hybrid semantics Hyb_2 is based on the evaluation methods $(\mathbf{M}_1, \mathbf{M}_2)$, where $\mathbf{M}_1 = \langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h}_1 \rangle$ and $\mathbf{M}_2 = \langle \mathsf{f}_{\mathsf{frac}}, \mathsf{g}_{\mathsf{sum}}, \mathsf{h}_6 \rangle$. Let $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathbf{s}, \mathbf{t}, \mathbf{w} \rangle \in \mathsf{AF}$, $x \in \mathcal{A}$ and $y \in \mathcal{R}$ such that:

i) $\mathbf{w}(x) = \mathbf{w}(y)$,
ii) $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y)$,
iii) $\forall c \in \mathsf{Sr}_{\mathbf{H}}(x), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(\mathbf{n}(c, x)) = \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(\mathbf{n}(c, y))$,

Due to the condition ii), let $\mathsf{Sr}_{\mathbf{H}}(x) = \mathsf{Sr}_{\mathbf{H}}(y) = \{a_1, \dots, a_k\}$. From Definition 12,

$$\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(x) = \frac{\mathbf{w}(x)}{1 + \sum(u_1, \dots, u_k)} \text{ and } \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(y) = \frac{\mathbf{w}(y)}{1 + \sum(v_1, \dots, v_k)},$$

where for any $i = 1, \dots, k$,

- $u_i = \mathsf{h}_1(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(\mathbf{n}(a_i, x)), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(a_i))$
- $v_i = \mathsf{h}_6(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(\mathbf{n}(a_i, y)), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(a_i))$

From iii), it follows that for any $i = 1, \dots, k$,

- $v_i = \mathsf{h}_6(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(\mathbf{n}(a_i, x)), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(a_i))$.

Since $\mathsf{h}_6 \succeq \mathsf{h}_1$, then for any $i = 1, \dots, k$, $\mathsf{h}_6(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(\mathbf{n}(a_i, x)), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(a_i)) \geq \mathsf{h}_1(\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(\mathbf{n}(a_i, x)), \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(a_i))$. Hence, $\sum(v_1, \dots, v_k) \geq \sum(u_1, \dots, u_k)$. From i), $\mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(x) \geq \mathbf{V}_{\mathbf{H}}^{\mathsf{Hyb}_2}(y)$. Thus, Hyb_2 satisfies A-Precedence.

To show that the hybrid semantics Hyb_2 violates Equality and R-Precedence, it is sufficient to consider the wHO-AF $\mathbf{H}_3$ depicted in Figure 12. Indeed,

$$\mathbf{V}_{\mathbf{H}_3}^{\mathsf{Hyb}_2}(e) = \frac{1}{1 + 2\mathsf{h}_1(0.2, 1)} = \frac{1}{1 + 2\mathtt{min}(0.2, 1)} = 0.714,$$

$$\mathbf{V}_{\mathbf{H}_3}^{\mathsf{Hyb}_2}(\alpha) = \frac{1}{1 + 2\mathsf{h}_6(0.2, 1)} = \frac{1}{1 + 2((0.2 + 1)/2)} = 0.454.$$

Then, $\mathbf{V}_{\mathbf{H}_3}^{\mathsf{Hyb}_2}(e) > \mathbf{V}_{\mathbf{H}_3}^{\mathsf{Hyb}_2}(\alpha)$ which shows that Hyb_2 violates Equality and R-Precedence.

□

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