Bases for Weighted Gradual Semantics and Inverse Problems in Argumentation Theory

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Weighted gradual semantics provide an acceptability degree to each argument representing its final strength, computed based on factors including the argument's background evidence, and taking into account interactions between the argument and others.

We introduce five important problems linking gradual semantics and acceptability degrees. First, we re-examine the inverse problem, seeking to identify each argument's initial weights within the argumentation framework which lead to a specific final acceptability degree. Second, we ask whether the function mapping between argument weights and acceptability degrees is one-to-one. Third, we ask if this mapping is a homeomorphism so that small perturbations in weights lead to small perturbation in acceptability degrees and vice versa. Fourth, we ask whether argument weights can be found when preferences, rather than acceptability degrees for arguments are considered. Last, we consider the geometry of the space of valid acceptability degrees, asking whether "gaps" exist in this space.

While different gradual semantics have been proposed in the literature, in this paper, and building on the geometry of the acceptability degree space, we identify a large family of weighted gradual semantics which contains many of the existing well-known semantics while maintaining desirable properties such as convergence to a unique fixed point and solving all five aforementioned problems.

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1 Introduction

Dung's widely used approach to abstract argumentation (Dung 1995) considers a set of abstract arguments and a binary attack relation between them, encoding these as a directed graph. Given such an argumentation system, argumentation semantics identify justified sets of arguments by considering inter-argument attacks (Baroni et al. 2011; Caminada 2006; Dung et al. 2007; Verheij 1996). Given the popularity of Dung's approach, myriad semantics have been proposed. These seek to address perceived shortcomings with Dung's original semantics, or encode additional elements (e.g., supports (Cayrol and Lagasquie-Schiex 2005)) to enrich or capture some facet of reasoning.

The standard approach to argumentation semantics associates sets of arguments with justified, unjustified or an unknown justification status. However, such a classification is too strict; in particular, humans may view

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some arguments as more justified than others. Thus, there has been increasing interest within the argumentation community in so-called *ranking-based* semantics (Amgoud and Ben-Naim 2013; Bonzon et al. 2016, 2023; Yun, Vesic, and Croitoru 2020). Such semantics aim to identify a ranking over the arguments, with higher ranked arguments considered more justified than arguments ranked lower. One approach to creating such a ranking, called *gradual semantics* (Amgoud and Doder 2018; Besnard and Hunter 2001; Yun and Vesic 2021), involves associating a numerical *acceptability degrees* to all arguments within the system, with the final ranking computed according to the numeric ordering of these degrees. Furthermore, some ranking-based semantics compute the acceptability degree of an argument based not only on the topology of the argumentation framework, but also based on some *initial weight* assigned to each argument. These *weighted gradual semantics*, exemplified by the weighted *max-based*, *card-based* and *h-categorizer* semantics, are widely studied and shown to have various desirable properties (Amgoud and Doder 2018; Amgoud, Doder, and Vesic 2022; Oren, Yun, et al. 2022).

Standard argumentation semantics take an argumentation framework as input and output a set of justified arguments (or the justification status associated with arguments). Similarly, a weighted gradual semantics takes an argumentation system and initial weights for all arguments as input, and outputs the arguments' acceptability degrees. In contrast, in this work we are interested in the *inverse problem* where we seek to output the possible initial weights associated with arguments given a semantics, argumentation system and arguments' acceptability degrees.

Such inverse problems have recently garnered increasing attention. For example, Skiba et al. 2022 have studied whether, given a ranking and a ranking-based semantics, we can find an argumentation framework (without weights) such that the selected ranking-based semantics induces the ranking when applied to the framework. Several researchers (Kido and Liao 2022; Kido and Okamoto 2017; Mumford et al. 2022; Riveret 2016; Riveret and Governatori 2016) have studied how attack or defeat relations can be determined given sets of acceptable arguments. Their focus was on Dung-style classical semantics. This line of work is similar to that of Oren and Yun 2023, where the authors identify the complexity class for the problem of identifying a set of attacks that will yield the desired acceptability degrees for a set of weighted arguments with respect to three weighted gradual semantics. Mailly 2023 also studied realization problems for extension semantics but made use of auxiliary arguments, i.e., given a set of extensions S and using k auxiliary arguments, can we find m argumentation frameworks such that the union of their extensions is exactly equal to S?

Apart from their interest to the community, inverse problems may be useful in the construction of a user or domain model in an online manner. For example, if a user specifies that some arguments are justified during a dialogue, one may be able to use the techniques specified above to infer the user's defeat relation between arguments without explicit probing, with this information used in later dialogues (e.g., with the system not advancing arguments that would be defeated according to the user's knowledge model). In the context of weighted gradual semantics, initial weights could be seen to capture a user's belief in the strength of arguments before inter-argument interactions (c.f., prior probabilities in Bayesian reasoning), with argument acceptability degrees (obtained from the semantics) and their concomitant preferences capturing posterior strengths. By identifying initial argument weights, a system can again reason about the potential acceptability degrees of arguments in future interactions without needing to explicitly obtain this information from the user. Furthermore, work examining how humans reason about arguments (Cerutti et al. 2014; Rahwan et al. 2010) suggests that humans ascribe strength to arguments while considering inter-argument interactions. Therefore, eliciting arguments' initial weights rather than arguments' final acceptability degrees from humans is challenging. Thus, the ability to infer the former from the latter (under the assumption that weighted gradual semantics capture a form of human reasoning as suggested by Vesic et al. 2022) is potentially important. In this context, inverse problems could potentially contribute to both user modelling and preference elicitation (Mahesar et al. 2018, 2020).

Perhaps most closely related to this paper is the work described by Oren et al. 2022, which addressed the inverse problem in the context of weighted gradual semantics. This latter paper utilised an iterative, approximate numeric technique to identify initial weights for the weighted max-based, card-based and h-categorizer semantics. However, it did not provide any guarantees about the presence (or uniqueness) of a solution to the inverse problem. In this work, we move away from such an approximation-based approach and investigate the theory behind the inverse problem, providing important formal insights. Namely, we ask ourselves the questions: (1) Can we always find a solution to the inverse problem for the three weighted gradual semantics? (2) If yes, can we generalise this result to a more general family of gradual semantics and (if so) characterise this family? (3) For any given preference ordering on arguments (rather than simply numerical acceptability degrees), can we always find some weights on arguments so that the resultant acceptability degrees will follow the preference ordering?

Our work makes the following contributions.

- C1 We introduce and address five problems relevant to the inverse problem approach described above. The first, Problem 3.1, is a detection problem which asks to find an analytic way to recover the initial weight from a given acceptability degree, and decide whether doing so is or is not possible. The second, Problem 3.3, asks whether any acceptability degree is attained by exactly one weight. The third, Problem 3.5, asks whether the matching between weights and acceptability degrees is continuous so that a small perturbation in one causes only a small perturbation in the other. The fourth, Problem 3.7, is an existence problem which asks whether any ranking of arguments can be realised by some acceptability degree. The fifth, Problem 3.8, is a "no gap" problem: we ask how close the set of acceptability degrees is to being convex. Having introduced the problems, we discuss their importance and potential applications in Section 3.2.
- C2 We establish a new family of "weighted *based* gradual semantics". In doing so, we generalise a powerful fixed-point theorem originally described by Pu et al. (Pu et al. 2014b).
  In this context, the main technical result of this paper lies in Theorem 7.6. With this in hand, we show that all five inverse problems above are solved by all the semantics belonging to this new family of gradual semantics.
- C3 Importantly, the family contains the max-based, card-based and h-categoriser gradual semantics. In Section 8 we identify within this family a large collection of new gradual semantics which generalise the ones above. We also identify a new gradual semantics which we call the geometric mean-based weighted gradual semantics and show how it relates to the three above semantics, as detailed in Corollary 8.5 and Proposition 8.7.

Problems 3.1 and 3.3, together with contribution C2 address our first question. We build on the results, together with the remaining problems and C3 to describe and characterise a more general family of semantics (Question 2). Problem 3.7 is used to answer our third question.

(Amgoud and Doder 2019) are the only other work we are aware of which formalizes a family of gradual semantics in a manner similar to this paper. This leads to our final contribution:

- C4 We extend the family of gradual semantics from C3 for in context of weighted gradual semantics with weights ascribed to both arguments and attacks.

The remainder of this paper is structured as follows. In Section 2, we briefly recap the background notions and formalism needed in this paper. In Section 3, we formalise the five inverse problems this paper focuses on and discuss their theoretical and practical importance. In Section 4 we introduce the key notion of grading schemes and use it to give a useful alternative definition for weighted gradual semantics which is particularly suitable for this paper. We expect that this new formalism will be useful to other argumentation researchers studying the same concepts. In Section 5, we introduce the notion of grading bases which are a convenient tool in the context of argumentation theory to compute acceptability degrees. In Section 6, we introduce basing schemes which are another abstract layer formed from the concept of grading bases from Section 5. We then show how basing schemes give rise to grading schemes, the basic building blocks of gradual semantics as defined in Section 4. In Section 7, building on the formalism developed Section 6 to introduce a new family of "weighted *based* gradual semantics". We prove the main result of this paper, i.e., that under fairly general conditions all the semantics in this family are defined by means of limits and that all five inverse problems in Section 3 are answered positively for them. In Section 8, we show that the max-based, card-based and h-categoriser semantics are members of the family above and hence all five inverse problems hold true for them. We introduce two new families of weighted-based semantics, which we call the L^p-based and the card L^p-based semantics, which generalise the three semantics above. We also introduce a totally different new weighted gradual semantics, the geometric mean-based semantics. All of them are defined by convergent limits and answer positively the five inverse problems. In Section 9 we show how our methods extend to give a framework to construct weighted gradual semantics which take into account varied strengths of attacks and compare the overlap between our results and those of (Amgoud and Doder 2019). In Section 10 we summarize and conclude our work.

The paper contains a significant amount of notation to provide general results. To aid the reader, we provide a table of notation in Appendix A.

2 Background - Weighted Gradual Semantics

Definition 2.1 (AF)**.** An **argumentation framework** or an **argumentation graph** is a pair $G = (A, D)$ where A is a set referred to as the set of arguments and $D \subset A \times A$ is called the set of attacks. The set of all arguments attacking $a \in A$ is

$$\mathtt{Att}_G(a) = \{a' \in A : (a', a) \in D\}.$$

Thus, an argumentation framework can be seen as a directed graph with vertex set A and set of directed edges D.

Definition 2.2. A **grading** on a set A is a function $x \colon A \to [0, 1]$. The space of all possible gradings on A is denoted by

$$X(A) \ = \ [0,1]^A \ \stackrel{\text{def}}{=} \ \{\text{all functions } x \colon A \to [0,1]\}.$$

When A is understood from the context we will write X instead of $X(A)$.

The idea behind many gradual semantics is to compute a ranking of the arguments in $G = (A, D)$ using values between 0 and 1. In (Rago et al. 2016), the term "scores" is used for what we call grading. In (Amgoud, Doder, and Vesic 2022), the terminology used for grading is "weights". In this paper we will reserve the term "weights" for a different object we associate to argumentation graphs.

Definition 2.3 (WAF)**.** A **weighted argumentation framework** (WAF) is a triple $F = (A, D, w)$ where $G = (A, D)$ is an argumentation framework and $w \colon A \to [0, 1]$ is a function called a **weighting** on A. The space of all weightings on A is

$$W(A) = [0,1]^A.$$

When A is understood from the context we will write W instead of $W(A)$.

We emphasise that abstractly, a weighting and a grading on A is the same thing: both are functions from A to $[0, 1]$. However, they play very different roles as we explain below when describing weighted gradual semantics. Such weighted gradual semantics assign a particular grading called "the acceptability degrees" to each weighted argumentation graph. Thus, the weighting[1] of the argumentation graph is part of the data that a semantics uses to compute the acceptability degrees. Changing the weighting of an argumentation graph changes the acceptability degrees that the semantics will assign to it. To highlight this distinction we use different terminology and letters, X and W, to denote the same space $[0, 1]^A$ playing the role of the space of all gradings and weightings, respectively.

[1]Some authors refer to what we call a "weighting" as an "initial weight".

Definition 2.4 (Weighted Gradual Semantics)**.** A **weighted gradual semantics** Σ is a function that associates each weighted argumentation framework $F = (A, D, w)$ with a grading function $\Sigma(F) \in X = [0, 1]^A$ called the **acceptability degrees** of F with respect to Σ.

A weighted argumentation framework is a Dung abstract framework (with arguments and binary attacks) augmented with a weighting function. Amgound, Doder and Vesic 2022 investigated different desirable properties of gradual semantics (e.g., anonymity, directionality, independence, etc) for gradual semantics in weighted argumentation frameworks. They showed that the three weighted gradual semantics described in Examples 2.8–2.10 below satisfy many of these properties. The argumentation literature thus focuses on these three semantics, which are specific instantiations of the weighted gradual semantics of Definition 2.4 above.

From the notational point of view, we emphasise that for any weighted argumentation graph $F = (A, D, w)$, the acceptability degrees $\Sigma(F)$ is itself a function $A \to [0, 1]$. Thus, for any $a \in A$ we will write

$$\Sigma(F)(a)$$

for the value of the acceptability degree of the argument a (with respect to the weighted gradual semantics Σ).

Definition 2.4 is very abstract. It strips off any details of how acceptability degrees are calculated from $F = (A, D, w)$ and any relationship between the acceptability degrees and the weighting w which most weighted gradual semantics require. For example, a desirable requirement for a gradual semantics Σ is that it satisfies the Maximality, Weakening and Resilience principles, (Amgoud, Doder, and Vesic 2022, Principles 4, 5 and 8). Only gradings $x: A \to [0, 1]$ where $x(i) \le w(i)$ for all $i \in A$ can meet these requirements and therefore the acceptability degree of $F = (A, D, w)$ satisfies $\Sigma(F)(i) \le w(i)$ for all $i \in A$. We formalise this as follows.

Definition 2.5. Let $\mathbb{R}^A$ denote the set of functions $u: A \to \mathbb{R}$. Define a partial order on $\mathbb{R}^A$ as follows. For any $u, v \in \mathbb{R}^A$

$$u \preceq v \iff u(i) \le v(i) \text{ for all } i \in A.$$

Definition 2.6. A weighted gradual semantics Σ is **dominated by weights** if

$$\Sigma(F) \preceq w$$

for any weighted argumentation graph $F = (A, D, w)$. (Note that the most widely used weighted gradual semantics mentioned by Amgoud et al. 2022 are dominated by weights).

Given the abstract definition of weighted gradual semantics, We now turn to the question of how semantics are constructed in practice. That is, we consider how acceptability degrees are calculated from weighted argumentation graphs.

In a weighted argumentation framework $F = (A, D, w)$, arguments are allowed to attack each other. Weighted gradual semantics can adhere to diverse principles. The underlying intuition is that arguments that are attacked by a large number of arguments (Amgoud et al. 2022, Principle 19 - Cardinality Precedence) or by arguments with a high acceptability degree (Amgoud et al. 2022, Principle 20 - Quality Precedence), or a mixture of both, will experience a reduction in their grading. Given an argumentation system with a grading $x \in X = [0, 1]^A$, we can model the impact of attacks on argument gradings by computing a new grading using a function $T : X \to X$ whose input is x and whose output is the new grading $T(x)$, capturing the effect of attacking arguments based on the current grading[2]. Thus, the system transitions from one grading of arguments to another. It is reasonable to expect that the outcome of increasing the gradings of the attackers of an argument $i \in A$ is a further reduction in its grade. Doing this for all the arguments $i \in A$, we see that if $x \preceq x'$ then $T(x') \preceq T(x)$.

[2]Effectively, this function T is used to capture the iterative process used in computing gradual semantics, where an argument's acceptability degree is iteratively adjusted based on the acceptability degrees of its attackers.

It is conceivable that a weighted argumentation framework has an "optimal" grading x', i.e one which is stable under these interactions by attacks. This stable grading would mean that $T(x') = x'$, i.e. x' is a fixed point of T. This leads to the next definition.

Definition 2.7. Let $X = [0,1]^A$ be the space of gradings of A. A **grading transition function** on A is an **order reversing** function $T: X \to X$, that is, for any $x, y \in X$

$$x \preceq y \implies T(y) \preceq T(x).$$

It is **stable** if

(a) T has a *unique* fixed point, i.e there exists a unique $y \in X$ such that $T(y) = y$.
(b) For any $x \in X$

$$y = \lim_{k\to\infty} T^k(x)$$

where T^k denotes the k-fold composition $T \circ \cdots \circ T$.

Clearly, the actual choice of T depends on the model under investigation. One goal of this paper is to identify a useful procedure to construct stable grading transition functions, and we now describe how the three popular weighted gradual semantics described in (Amgoud, Doder, and Vesic 2022) can be captured in our formalism.

Example 2.8. The **weighted max-based** gradual semantics, denoted $\Sigma_{\mathbb{MB}}$. Given $F = (A, D, w)$, the acceptability degree of an argument $a \in A$ is defined by

$$\Sigma_{\mathbb{MB}}(F)(a) = \lim_{k\to\infty} \mathrm{MB}_k(a)$$

for the sequence of functions $\mathrm{MB}_k: A \to [0,1]$ defined recursively by setting $\mathrm{MB}_0(a) = w(a)$ for all $a \in A$ and

$$\mathrm{MB}_{k+1}(a) = \frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}(a)} \mathrm{MB}_k(b)}$$

Here, the grading transition function $T_{G,w}: [0,1]^A \to [0,1]^A$ used to define $\Sigma_{\mathbb{MB}}$ is defined for any grading function $x: A \to [0,1]$ by

$$T_{G,w}(x)(a) = \frac{w(a)}{1 + \max\limits_{b \in \mathtt{Att}(a)} x(b)}.$$

Example 2.9. The **weighted card-based** gradual semantics $\Sigma_{\mathbb{CB}}$. Given $F = (A, D, w)$, the acceptability degree of an argument $a \in A$ is

$$\Sigma_{\mathbb{CB}}(F)(a) = \lim_{k\to\infty} \mathrm{CB}_k(a)$$

for the sequence of functions $\mathrm{CB}_k: A \to [0,1]$ defined recursively by $\mathrm{CB}_0(a) = w(a)$ for all $a \in \mathcal{A}$ and

$$\mathrm{CB}_{k+1}(a) = \frac{w(a)}{1 + |\mathtt{Att}^*(a)| + \frac{1}{|\mathtt{Att}^*(a)|} \sum\limits_{b \in \mathtt{Att}^*(a)} \mathrm{CB}_k(b)}$$

Here, $\mathtt{Att}^*(a) = \{b \in \mathtt{Att}(a) \mid w(b) > 0\}$ and if $\mathtt{Att}^*(a) = \emptyset$ the term involving division by $|\mathtt{Att}^*(a)|$ vanishes. The grading transition function $T_{G,w}: [0,1]^A \to [0,1]^A$ used to define $\Sigma_{\mathbb{CB}}$ is defined for any grading function $x: A \to [0,1]$ by

$$T_{G,w}(x)(a) = \frac{w(a)}{1 + |\mathtt{Att}^*(a)| + \frac{1}{|\mathtt{Att}^*(a)|} \sum\limits_{b \in \mathtt{Att}^*(a)} x(b)}.$$

Example 2.10. The **weighted h-categorizer** gradual semantics $\Sigma_{\mathbb{HC}}$. Given $F = (A, D, w)$, the acceptability degree of an argument $a \in A$ is

$$\Sigma_{\mathbb{HC}}(F)(a) = \lim_{k\to\infty} \mathrm{HC}_k(a)$$

for the sequence of functions $\mathrm{HC}_k \colon A \longrightarrow [0,1]$ defined recursively by $\mathrm{HC}_0(a) = w(a)$ and

$$\mathrm{HC}_{k+1}(a) = \frac{w(a)}{1 + \sum\limits_{b\in \mathtt{Att}(a)} \mathrm{HC}_k(b)}$$

The grading transition function $T_{G,w} \colon [0,1]^A \longrightarrow [0,1]^A$ used to define $\Sigma_{\mathbb{HC}}$ is

$$T_{G,w}(x)(a) = \frac{w(a)}{1 + \sum\limits_{b\in \mathtt{Att}(a)} x(b)}$$

for any grading function $x \colon A \longrightarrow [0,1]$.

Note that $\mathrm{MB}_0, \mathrm{CB}_0$ and HC_0 are set to $w(a)$ as per the standard definition of these semantics. However, as we will see later in this paper, $\mathrm{MB}_0, \mathrm{CB}_0$ and HC_0 may be chosen to be arbitrary elements in $[0,1]^A$

One crucial issue with the above semantics is the need to prove that the relevant limits exist and yield grading functions, i.e., that that the grading transition functions $T_{G,w}$ are stable. Amgoud et al. 2022, in Theorems 7, 12, and 17, proved convergence for the three semantics on a case-by-case basis. One of the achievements of the current work is to provide a general framework to construct stable grading transitions of which the three weighted gradual semantics above are special cases, and to provide a uniform treatment to the question of convergence.

In some applications, the exact acceptability degrees assigned to arguments by a weighted gradual semantics is not required. Instead, these acceptability degrees are used to rank arguments, resulting in a preference ordering over arguments (Bonzon et al. 2016). Formally, we say that a is at least as preferred as b iff a's acceptability degree is at least as high as b's, and denote this by $a \trianglerighteq b$. We write $a \simeq b$ iff $a \trianglerighteq b$ and $b \trianglerighteq a$. We write $a \triangleright b$ iff $a \trianglerighteq b$ and $a \ntrianglelefteq b$. And finally, $a \trianglelefteq b$ iff $a \ntriangleright b$.

Example 2.11. (taken from (Oren, Yun, et al. 2022)). Let $F = \langle \mathcal{A}, \mathcal{D}, w\rangle$ be the WAF depicted in Figure 1 with weights $w(a_0) = 0.43$, $w(a_1) = 0.39$, $w(a_2) = 0.92$, and $w(a_3) = 0.3$. Table 1 lists the acceptability degrees and the associated rankings on arguments for the semantics $\Sigma_{\mathbb{MB}}, \Sigma_{\mathbb{CB}}$ and $\Sigma_{\mathbb{HC}}$ described in Examples 2.8, 2.9 and 2.10.

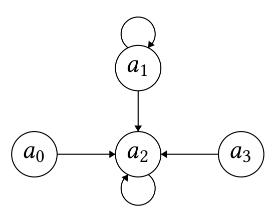

Fig. 1. Graphical representation of a WAF.

Table 1. Acceptability degrees of the arguments from Figure 1, rounded to 2 decimal numbers.

	a_0	a_1	a_2	a_3	
$w(a_i)$	0.43	0.39	0.92	0.30	Argument ranking
$\Sigma_{\mathbb{MB}}(F)(a_i)$	0.43	0.30	0.58	0.30	$a_1 \simeq a_3 \lhd a_0 \lhd a_2$
$\Sigma_{\mathbb{HC}}(F)(a_i)$	0.43	0.30	0.38	0.30	$a_1 \simeq a_3 \lhd a_2 \lhd a_0$
$\Sigma_{\mathbb{CB}}(F)(a_i)$	0.43	0.18	0.17	0.30	$a_2 \lhd a_1 \lhd a_3 \lhd a_0$

A single preference ordering captures multiple possible acceptability degrees. In addition, we may want to consider properties relating to all possible acceptability degrees. Therefore, it is useful to study the collection of all possible acceptability degrees, leading to our next definition.

Definition 2.12. Let Σ be a weighted gradual semantics. Let $G = (A, D)$ be an argumentation framework with weighting and grading spaces W and X on A. The set

$$D_{\Sigma,G} = \left\{x \in [0,1]^A : x = \Sigma((A, D, w)) \text{ for some } w \in [0,1]^A\right\}$$

is called the **space of acceptability degrees** of G.

Example 2.13. Let $\Sigma = \Sigma_{\mathbb{HC}}$ be the h-categoriser gradual semantics described in Example 2.10. Let G be a complete graph with arguments $\mathcal{A} = \{1, 2, 3\}$. A representation of the acceptability degree space $D_{\Sigma_{\mathbb{HC}},G}$ is shown in Figure 2. The boundaries of the acceptability degree space are not 2D planes in $\mathbb{R}^3$ which highlights the challenge of categorizing this space.

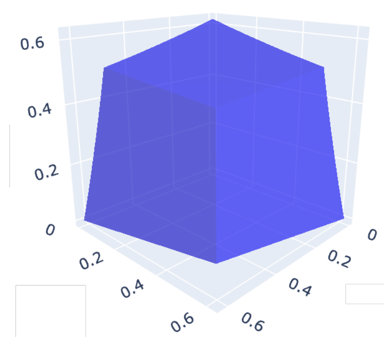

Fig. 2. Representation (in blue) of the acceptability degree space of $\Sigma_{\mathbb{HC}}$ for a complete argumentation graph with 3 arguments.

In this paper, we will study the topology and geometry of the space $D_{\Sigma,G}$ with respect to a certain class of gradual semantics Σ. We frame this in the next section by formulating multiple problems which we call "inverse problems".

3 The Inverse Problems

This paper revolves around five problems on weighted gradual semantics which we describe next. Having introduced the problems we then justify their importance in the context of argumentation and dialogue by considering their potential applications. Throughout this section Σ denotes a fixed weighted gradual semantics

3.1 Statement of the Problems

The first problem is a *detection problem*. Given a grading $x: A \to [0,1]$ on $G = (A, D)$, we want to be able to detect whether x is an acceptability degree for G, i.e whether $x \in D_{\Sigma,G}$.

Problem 3.1. Denote $X = [0,1]^A$ as the space of all gradings of $G = (A, D)$. Denote by $\mathbb{R}^A$ the set of all functions $u: A \to \mathbb{R}$ and $W = [0,1]^A$ the space of all weights on G. The **inverse problem** seeks computable functions

$$\mathrm{inv}_{\Sigma,G}: X \to \mathbb{R}^A$$

such that

(i) $x \in D_{\Sigma,G} \iff \mathrm{inv}_{\Sigma,G}(x) \in W$, and in this case
(ii) $\Sigma(A, D, w) = x$ where $w = \mathrm{inv}_{\Sigma,G}(x)$.

Example 3.2. Consider the argumentation graph G depicted in Figure 1 as per Example 2.11. The Inverse Problem seeks computable functions

$$\mathrm{inv}_{\Sigma_{\mathbb{MB}},G},\ \mathrm{inv}_{\Sigma_{\mathbb{CB}},G},\ \mathrm{inv}_{\Sigma_{\mathbb{HC}},G}\ :\ [0,1]^4 \to \mathbb{R}^4$$

which will output the weights $[0.43, 0.39, 0.92, 0.30]$ if given the inputs in the rows of Table 1 representing the relevant semantics.

Note that these functions will output a quadruple outside $[0,1]^4$ if given an input $x \in [0,1]^4$ which is *not* an acceptability degree of the respective gradual semantics studied in this example.

The second question we ask is whether an acceptability degree of a given argumentation graph is attained by only one possible weight. We call such problems *reflection* problems since one can recover the weight from the acceptability degree.

Problem 3.3. Given an argumentation graph $G = (A, D)$ with X and W the space of gradings and weights on A, we define a function $\sigma^G : W \to D_{\Sigma,G}$ that associates each $w \in W$ with the grading $\Sigma(A, D, w) \in D_{\Sigma,G}$. The **reflection problem** asks whether this σ^G is a bijection.

Another important question is whether small perturbations of weights causes small perturbations of acceptability degrees and vice versa, whether small perturbations of acceptability degree can only be the result of small perturbations of the weights. To make sense of such statements we first need to topologise $\mathbb{R}^A$, the space of all functions $u: A \to \mathbb{R}$.

Definition 3.4. Let A be a finite set. Define the Euclidean norm of $u \in \mathbb{R}^A$ by

$$\|u\| = \sqrt{\sum_{i \in A} u(i)^2}$$

The metric on $\mathbb{R}^A$ is $d(u, v) = \|u - v\|$.

Clearly, there is a natural identification (isometry) of metric spaces $\mathbb{R}^A \cong \mathbb{R}^n$ where $|A| = n$. Then $X, W \subseteq \mathbb{R}^A$ inherit the metric of subspaces. The question above about small perturbations is then rephrased as asking whether the function $\sigma^G : W \to D_{\Sigma,G}$ from the Reflection Problem 3.3 above, and the inverse function $\mathrm{inv}_{\Sigma,G}: X \to \mathbb{R}^A$ from the Inverse Problem 3.1 are continuous.

Problem 3.5. The **topological reflection problem** asks whether the function defined in Problem 3.3 is a homeomorphism[3]. Note that in this case $D_{\Sigma,G}$ is homeomorphic to the Euclidean cube $[0,1]^n$ in $\mathbb{R}^n$ for $n = |A|$.

[3] Recall that a homeomorphism is a bijective continuous function whose inverse is also continuous.

Example 3.6. In Section 8, we will show that the semantics defined by Amgoud et al. 2022 (see Examples 2.8, 2.9, 2.10) all have a positive solution to the topological reflection problem, and hence to the reflection problem. Thus, given the final acceptability degrees shown in Table 1 for the graph shown in Figure 1, it follows that the weighting [0.43, 0.39, 0.92, 0.30] is the only one which results in the acceptability degrees listed in the remaining rows in this table for each one of these semantics.

Moreover, since the topological reflection problem is also satisfied, a small change to the initial weights will result in a small change in the acceptability degrees and vice versa.

The topological reflection problem as well as the next problem we consider study the behaviour of acceptability degrees under linear rescaling of grades. Consider an arbitrary grading $x\colon A \longrightarrow [0,1]$. This grading may or may not be in the space of acceptability degrees of $G = (A, D)$. We thus ask whether it is possible to find an element x' in the space of acceptability degrees which retains the ratios between the gradings of the arguments in x. In other words, we ask whether there exists $x' \in D_{\Sigma,G}$ such that for any $a, b \in A$

$$\frac{x'(a)}{x'(b)} = \frac{x(a)}{x(b)}.$$

Observe that this is the case if and only if $x' = t \cdot x$ for some $t > 0$.

Problem 3.7. Let $G = (A, D)$ be an argumentation graph. The **projective preference ordering problem** asks whether for any grading $x \in X = [0,1]^A$ on A there exists some $t > 0$ such that $t \cdot x \in D_{\Sigma,G}$.

Practically speaking, an affirmative answer to this question for a semantics means that we can always find initial weights which result in some desired preference ordering over final acceptability degrees. In fact, a positive answer to the question yields an even stronger result, namely that we can find initial weights which result in some desired ratio of strengths between arguments (e.g., one could specify that argument a_1 should have 0.5 the acceptability degree of a_2, which should be 3 times a_3, etc). This can be achieved by creating an arbitrary grading $x \in [0,1]^A$ which obeys the prescribed ratios, and then seeking out an appropriate t so as to scale x down to an element of $D_{\Sigma,G}$. This result can be seen as a generalisation of (Oren, Yun, et al. 2022), which identified a specific ratio of acceptability degrees for each argument to ensure achievability.

Inspired by the projective preference problem, we ask whether $D_{\Sigma,G}$ is a cone[4] with apex $\mathbf{0} \in X \subseteq \mathbb{R}^n$.

Problem 3.8. Let $G = (A, D)$ be an argumentation graph. The **radiality problem** asks whether for any $x \in D_{\Sigma,G}$ the line segment $[0, x] = \{t \cdot x : 0 \le t \le 1\}$ in $X(A)$ is contained in $D_{\Sigma,G}$.

A positive answer to this question provides significant information about the geometry of the set $D_{\Sigma,G}$, guaranteeing — for example — that given a valid set of acceptability degrees, all points between 0 and that point in the acceptability degree space are themselves valid. In other words, the set $D_{\Sigma,G}$ is "solid".

Example 3.9. In Section 8, we show that the semantics in Examples 2.8, 2.9, 2.10 have a positive solution to the projective preference ordering problem and the radiality problem. Going back to Example 2.11, let us consider the gradual semantics $\Sigma_{\mathbb{CB}}$ and the case where one may want an acceptability degree in which a_2 has acceptability degree that is two times higher than all the other arguments. The grading $x = [0.5, 0.5, 1, 0.5]$ is not a valid acceptability degree (as there are no weightings w on A such that $\Sigma(A, D, w) = x$). Since argument a_2 attacks itself it cannot achieve the desired acceptability degree of 1 (rather, it's maximum value is ~ 0.618), even if all its other attackers have an acceptability degree of 0[5].

[4]We remind the reader that a cone in $\mathbb{R}^n$ with apex $\mathbf{0} \in X \subseteq \mathbb{R}^n$ is a subset C of $\mathbb{R}^n$ such that for any $x \in C$, then $t \cdot x \in C$ for any $0 \le t \le 1$. A cone does not necessary have a "flat" base. An example of a cone in $\mathbb{R}^2$ is a quarter of a circle centred at the origin.

[5]To obtain the maximum acceptability degree of a_2, we assume that all other attackers have an acceptability degree of 0 and a_2 has an initial weight of 1. With those parameters, the acceptability degree of a_2 is ~ 0.618.

The projective preference problem guarantees there exists $t > 0$ such that $t \cdot x$ is an acceptability degree. Indeed, the scaled down vector tx, with $t = 0.40$ is an acceptability degree obtained using the initial weights $[0.20, 0.24, 0.80, 0.20]$. The positive answer to the radiality problem for $\Sigma_{\mathbb{CB}}$ means that $t \cdot x$ is an acceptability degree for any $0 \leq t \leq 0.4$. For example, $0.35 \cdot [0.5, 0.5, 1, 0.5]$ is also an acceptability degree and is obtained using the initial weights [0.175, 0.205625, 0.65625, 0.175].

3.2 Inverse Problems in Argument and Dialogue

Having introduced the five problems above, we now consider their utility in the context of argumentation and dialogue based systems. We believe that the clearest application of these problems is in opponent modelling, preference elicitation (Mahesar et al. 2020; Oren and Yun 2023; Oren, Yun, et al. 2022), and dialogue strategy (Hadjinikolis et al. 2013; Rienstra et al. 2013).

The inverse problem. This problem is perhaps the most fundamental, and allows us to determine whether it is possible to move from some acceptability degrees to valid initial weights. Thus, if a weighted gradual semantics satisfies the inverse problem, then given a target set of acceptability degrees for some arguments, one can determine whether these are achievable, i.e., whether there exist valid initial weights which lead to them.

In a dialogue where participants disagree about final acceptability degrees, achievable initial weights suggest that a dialogue shift should occur to discuss and modify these initial weights. In contrast, in cases where the desired initial weights are invalid, the participant understands that the underlying argumentation graph must be modified, for example through the addition or deletion of attacks or arguments (c.f., (Coste-Marquis, Konieczny, J. Mailly, et al. 2015)).

The (topological) reflection problem. The inverse problem does not guarantee a one-to-one mapping between acceptability degrees and initial weights. Thus, when trying to learn the initial weights of a reasoner, one may identify the incorrect weights given some acceptability degree. A positive solution to the reflection problem provides this guarantee. By having a positive answer to both of the reflection and inverse problem, we can ensure that our semantics can be used to learn the initial weights used by a reasoner, allowing us to infer this element of their knowledge. A positive answer to the topological reflection problem (which also results in a positive answer to the reflection problem) ensures that a small variation in the initial weights induces a small variation of the acceptability degree. This is important if one were to consider the impact of arguments (Delobelle and Villata 2019; Kampik et al. 2024). Without this property, a small change in initial weights could lead to large changes in acceptability degrees (and vice versa), making the analysis of argument impact challenging. A semantics which satisfies topological reflection is thus — in a sense — well behaved and more amenable to impact analysis.

Thus, while these two problems have limited practical application, they specify an important property for semantics, specifying desirable behaviour when applied.

The projective preference ordering problem. The ability to achieve ratios of acceptability degrees underpins the approach described in (Oren, Yun, et al. 2022) to find initial weights which could be used for any semantics which satisfies this problem. In addition, this problem suggests an approach to argument weight revision; while argument graphs are normally modified to achieve some target argument values (e.g., justification status (J.-G. Mailly 2013) for classical frameworks, and preferences or acceptability degrees in the case of argumentation semantics (Dupuis de Tarlé et al. 2022)), any semantics using this property can instead be scaled to achieve the same effect.

The radiality problem. While the projective preference ordering ensures that one can scale down any vector of target acceptability degree to achieve a valid one, there is no guarantee that further scaling down will also yield valid answers. A positive answer to this problem provides such a guarantee, as well as providing a necessary condition for the volume of acceptability degrees in to contain no gaps. In cases where initial weights represent some form of certainty or confidence, and where evidence must be gathered (by expending resources) to strengthen these initial weights, the ability to weaken arguments arbitrarily would allow a reasoner to trade off resource consumption with argument strength (similar to the ideas described in (Skitalinskaya et al. 2023)).

One of the achievements of this paper is the identification of a large family of abstract weighted gradual semantics which includes Amgoud et al's 2022 $\Sigma_{\mathbb{HC}}$, $\Sigma_{\mathbb{MB}}$ and $\Sigma_{\mathbb{CB}}$ semantics. We demonstrate that all five problems above have a positive solution for this family of semantics, and provide an easily computable inverse function for the three semantics (unlike the numerical approach utilised by (Oren, Yun, et al. 2022)).

4 Grading Schemes and a Reformulation of Weighted Gradual Semantics

In this section we introduce the concept of grading schemes and use it in order to give an alternative, simple yet useful, definition for weighted gradual semantics which we will use throughout the paper.

Grading schemes

Recall from Definitions 2.2 and 2.3 that the spaces of all weightings and all gradings of an argumentation graph $G = (A, D)$ are denoted by W and X respectively.

Throughout this paper we will use the word **scheme** to describe a function from the set W of weightings (of G) to some other set.

Definition 4.1. Let $G = (A, D)$ be an argumentation graph with spaces of weighting and gradings $W = [0, 1]^A$ and $X = [0, 1]^A$. A **grading scheme** on A is a function

$$\sigma \colon W \to X.$$

The space of all grading schemes on A is denoted

$$\mathrm{GrScheme}(A) = \mathrm{Func}(W, X).$$

The image of a grading scheme $\sigma \colon W \to X$

$$D_\sigma = \mathrm{Im}(W \xrightarrow{\sigma} X)$$

is called the **space of acceptability degrees** of σ.

For example, keeping in mind that W and X are both equal to $[0, 1]^A$, the identity function $id \colon W \to X$ is a grading scheme. The constant function $\mathbf{0} \colon W \to X$ which sends every $w \in W$ to the $\mathbf{0} \in X = [0, 1]^A$, i.e., the constant function $\mathbf{0} \colon A \to [0, 1]$ is another example of a grading scheme.

Recall the partial order $\preceq$ on $\mathbb{R}^A$ in Definition 2.5.

Definition 4.2. A grading scheme $\sigma \colon W \to X$ is **dominated by weights** if for every $w \in W$

$$\sigma(w) \preceq w.$$

For example, both $\mathrm{id} \colon W \to X$ and $\mathbf{0} \colon W \to X$ are dominated by weights.

Reformulating the concept of weighted gradual semantics

We now turn to give a reformulation of the concept of weighted gradual semantics in terms of grading schemes.

Definition 4.3. Denote

$$\mathcal{AF} = \{\text{all argumentation graphs } G = (A, D)\}.$$
$$\mathcal{WAF} = \{\text{all weighted argumentation graphs } F = (A, D, w)\}.$$

Fix a weighted gradual semantics Σ. For any argumentation graph $G = (A, D)$ with weighting and grading spaces $W = [0, 1]^A$ and $X = [0, 1]^A$ we obtain a function

$$\sigma^G : W \to X$$

defined for any $w \in W$ by

$$\sigma^G(w) = \Sigma(A, D, w). \tag{1}$$

In our language, σ^G is a grading scheme (Definition 4.1).

By construction, for any weighted argumentation graph $F = (A, D, w)$ with underlying graph $G = (A, D)$ we have $\Sigma(F) = \sigma^G(w)$. Thus, we see that the information required to calculate the values of a weighted gradual semantics Σ is the same as the collection of the grading schemes $\sigma^G \in \mathrm{GrScheme}(A)$ defined in (1), one for each argumentation graph $G = (A, D)$.

The converse is also true. Suppose that we are given a collection of grading schemes $s^G \in \mathrm{GrScheme}(A)$, one for each $G = (A, D) \in \mathcal{AF}$. We can define a weighted gradual semantics Σ which for any $F = (A, D, w) \in \mathcal{WAF}$ with underlying graph $G = (A, D)$ is defined by $\Sigma(F) = s^G(w)$.

We see that the concept of a weighted gradual semantics is the same as a function with domain $\mathcal{AF}$ which to any argumentation graph $G = (A, D)$ assigns a grading scheme on A. This leads to the following reinterpretation of the concept of weighted gradual semantics. A reformulation which we will adopt for the rest of this paper.

Definition 4.4 (Weighted Gradual Semantics)**.** A weighted gradual semantics is a function

$$\Sigma : \mathcal{AF} \longrightarrow \bigcup_{(A,D) \in \mathcal{AF}} \mathrm{GrScheme}(A)$$

such that for any $G = (A, D)$ in $\mathcal{AF}$, the value of Σ at G, namely $\Sigma(G)$, is a grading scheme in the component G of the codomain. By definition $\Sigma(G)$ is a function $W \to X$. Its value at $w \in W$, i.e

$$\Sigma(G)(w),$$

is called the acceptability degree of the weighted argumentation graph $F = (G, w)$.

To summarise, a weighed gradual semantics is a function from the collection of all argumentation graphs to the collection of grading schemes.

In order to construct weighted gradual semantics we need a tool to construct useful grading schemes. The discussion in Section 2.1 suggests that a strategy suitable in the context of argumentation theory is to find a framework to construct stable grading transition functions as per Definition 2.7. This is the purpose of Sections 5 and 6 below.

5 Grading Bases

In this section we define *grading bases*. Intuitively, grading bases encode the specific way by which an argument's attackers grades are aggregated to adjust the argument's grade. In turn, these grading bases are then used to construct grading transition functions, and these underpin the recursive mechanism utilised by many weighted gradual semantics, for example the semantics $\Sigma_{\mathbb{HC}}$, $\Sigma_{\mathbb{MB}}$ and $\Sigma_{\mathbb{CB}}$ in Examples 2.8, 2.9 and 2.10. In Theorem 5.9 we demonstrate that the grading transition functions resulting from grading bases are stable, namely they always converge; this result is desirable for any useful weighted gradual semantics. Proofs of the results in this section are provided in Appendix B.

Definition 5.1. Let A be a set and set $X = [0, 1]^A$. Let $\varphi : X \to [0, \infty)$ be a function.

(a) Call φ **order preserving** if for any $x, y \in X$,

$$x \preceq y \implies \varphi(x) \le \varphi(y).$$

(b) Call φ **bounded** if there exist $M \geq 0$ such that $\varphi(x) \leq M$ for all $x \in X$.
(c) Call φ **super homogeneous** if for any $0 \leq t \leq 1$ and any $x \in X$,

$$\varphi(t \cdot x) \geq t \cdot \varphi(x).$$

It is called **homogeneous** if equality holds.
(d) Call φ **tame** if for any $x \in X$ and any $0 \leq t \leq 1$

$$t \cdot \varphi(t \cdot x) \leq \varphi(x).$$

The collection of all bounded, order preserving and super homogeneous functions on $X = [0,1]^A$ is denoted

$$\mathcal{BISH}(A).$$

The subspace of the tame super homogeneous bounded order preserving functions is denoted

$$\mathcal{BITSH}(A).$$

The tameness of $\mathcal{BITSH}$ over $\mathcal{BISH}$ is important in addressing Problem 3.8, as done in Theorem 7.6(c).

Remark 5.2. By construction $\mathcal{BITSH}(A) \subseteq \mathcal{BISH}(A)$ are subspaces of $\mathrm{Func}(X, [0,\infty))$, the space of all functions $f : X \to [0,\infty)$.

The collection $\mathcal{BISH}(A)$ is very rich as propositions 5.3 and 5.4 below demonstrate. For any $i \in A$, evaluation at i gives rise to a function $\mathrm{ev}_i : [0,1]^A \to [0,1]$,

$$\mathrm{ev}_i(x) = x(i).$$

We remark that X can be identified with $[0,1]^n \subseteq \mathbb{R}^n$ where $n = |A|$, and the evaluation ev_i becomes the projection function $\pi_i : [0,1]^n \xrightarrow{(x_1,\ldots,x_n)\mapsto x_i} [0,1] \subseteq [0,\infty)$.

Proposition 5.3. *The following functions belong to $\mathcal{BITSH}(A)$.*
(a) All constant functions $\psi : [0,1]^A \to [0,\infty)$.
(b) All evaluation maps $\mathrm{ev}_i : [0,1]^A \xrightarrow{x \mapsto x(i)} [0,\infty)$ for any $i \in A$.

We can combine elements of $\mathcal{BISH}(A)$ to obtain more complex functions which are also elements of $\mathcal{BISH}(A)$ as follows.

Proposition 5.4. *Let $\psi_1, \ldots, \psi_k \in \mathcal{BISH}(A)$. Then the following functions $\psi : X \to [0,\infty)$ obtained from $\psi_1, \ldots, \psi_k$ are also in $\mathcal{BISH}(A)$.*
(i) $\psi = \sum_{j=1}^{k} a_j \psi_j$ for any $a_1, \ldots, a_k \geq 0$.
(ii) $\psi = \max\{\psi_1, \ldots, \psi_k\}$, provided that $\psi_1, \ldots, \psi_k$ are homogeneous.
(iii) $\psi = \sqrt[p]{\psi_1^p + \cdots + \psi_k^p}$ for any $p > 0$ (Here ψ_i^p is the p-th power of the function ψ_i).
(iv) $\psi = \sqrt[k]{\psi_1 \cdots \psi_k}$ (geometric mean).
If $\psi_1, \ldots, \psi_k$ are tame then so are the functions ψ in (i)–(iv).

Definition 5.5. Let $G = (A, D)$ be an argumentation graph. A **grading base** on A is a function

$$\varphi : A \to \mathcal{BISH}(A).$$

It is called a **tame** if $\varphi(i) \in \mathcal{BISH}(A)$ are tame for all $i \in A$. It is called **continuous** if $\varphi(i) : X \to [0,\infty)$ are continuous.

The space of all grading bases is denoted by

$$\mathrm{GrBase}(A) := \mathcal{BISH}(A)^A = \mathrm{Func}(A, \mathcal{BISH}(A)),$$

Remark 5.6. We recall that the set Z^A of all functions from A to Z can be identified with the set of A-indexed tuples in Z:

$$Z^A \cong \prod_{i \in A} Z$$

Therefore, throughout this paper we will freely, and without mention, view grading bases, i.e functions $\varphi \in \mathrm{GrBase}(A) = \mathcal{BISH}(A)^A$, as A-indexed tuples

$$\varphi = (\varphi_i)_{i \in A}$$

where $\varphi_i \colon X \longrightarrow [0, \infty)$ are functions in $\mathcal{BISH}(A)$.

Intuitively, grading bases are the functions used to calculate the aggregate attacks on a particular argument in an argumentation graph. In Examples 2.8–2.10, the grading bases are the functions that appear in the denominators when defining the sequences $\mathbb{MB}_k$, $\mathbb{CB}_k$ and $\mathbb{HC}_k$ and are used to aggregate the scores of an argument's attackers.

The next definition and proposition formalise the way in which grading bases give rise to *stable* grading transition functions (Definition 2.7). As mentioned previously, such stability is crucial in the context of a semantics.

Definition 5.7. Let $G = (A, D)$ be an argumentation graph with X, W its spaces of gradings and weightings. Let $w \in W$ be a weighting and $\varphi \colon A \longrightarrow \mathcal{BISH}(A)$ a grading base. Define a function

$$T_{\varphi,w} \colon X \longrightarrow X$$

as follows. For any $x \in X$, let $T_{\varphi,w}(x) \in X$ be the function $T_{\varphi,w}(x) \colon A \longrightarrow [0, 1]$ defined for any $i \in A$ by

$$T_{\varphi,w}(x)(i) = \frac{w(i)}{1 + \varphi_i(x)}.$$

Proposition 5.8. *With the notation of Definition 5.7, $T_{\varphi,w}$ is a* stable *grading transition function. In addition, $T_{\varphi,w}$ is dominated by w in the sense that for any $x \in X$*

$$T_{\varphi,w}(x) \preceq w.$$

The proof of the stability of $T_{\varphi,w}$ in Proposition 5.8 relies on the next theorem which is a key result of this paper. It is an important generalisation of the work of Pu, Zhang and Luo 2014a, Theorems 1 and 2.

Let X denote $[0, 1]^A$ for some finite set A, equipped with the partial order $\preceq$ in Definition 2.7.

Theorem 5.9. *Let $f : X \longrightarrow X$ be a function such that*

(a) f is order reversing, and
(b) there is some $0 < \alpha \leq 1$ such that for any $x \in X$ and any $0 \leq t \leq 1$

$$f(tx) \preceq \frac{1}{t + \alpha(1 - t)} f(x).$$

Then f has a unique *fixed point $y \in X$. Furthermore,*

$$y = \lim_{k \to \infty} x^{(k)}$$

where $x^{(k)}$ is any sequence defined recursively by choosing $x^{(0)} \in X$ arbitrarily and

$$x^{(k+1)} = f(x^{(k)}).$$

6 Basing Schemes and their Associated Grading Schemes

In this section, we introduce basing schemes, namely functions from the weights space W of an argumentation graph $G = (A, D)$ to the set $\mathrm{GrBase}(A)$. Their usefulness comes from the fact that they give rise to grading schemes, the building blocks of gradual semantics, and from the fact that they are naturally suited to the context of argumentation theory. We show that under some weak assumptions, grading schemes obtained from basing schemes solve all five problems described in Section 3 adapted to the setting of grading schemes. Proofs of the results in this section are presented in Appendix C.

Definition 6.1. Let $G = (A, D)$ be an argumentation framework and let X and W denote its gradings and weightings spaces. A **basing scheme** on A is a function

$$b\colon W \longrightarrow \mathrm{GrBase}(A).$$

The space of all basing schemes on A is denoted

$$\mathrm{BaseScheme}(A) := \mathrm{Func}(W, \mathrm{GrBase}(A)).$$

Now consider a basing scheme b. Then for any $w \in W$ we obtain a grading base $b(w)$ and, hence by Proposition 5.8, a stable grading transition function $T_{b(w),w}$, and in turn, an element $x = \mathrm{fix}(T_{b(w),w})$ in X. This justifies the following definition.

Definition 6.2. Let $G = (A, D)$ be an argumentation graph with gradings and weightings spaces X, W. The grading scheme associated with a basing scheme $b\colon W \longrightarrow \mathrm{GrBase}(A)$ is the function

$$\sigma_b\colon W \longrightarrow X$$

defined for any $w \in W$ by

$$\sigma_b(w) = \mathrm{fix}(T_{b(w),w}).$$

This yields a procedure to obtain a grading scheme σ_b from a basing scheme b, namely a function $\tilde{\sigma}$ that maps b to σ_b, i.e

$$\tilde{\sigma}\ :\ \ \mathrm{BaseScheme}(A) \xrightarrow{b \mapsto \sigma_b} \mathrm{GrScheme}(A).$$

We depict the procedure described in Definition 6.2 to obtain σ_b in the following display.

$$\sigma_b\colon W \xrightarrow{w \mapsto (b(w),w)} \mathrm{GrBase}(A) \times W \xrightarrow{(\varphi,w)\mapsto T_{\varphi,w}} \left\{ \begin{matrix} \text{stable grading} \\ \text{transition functions} \\ T\colon X \longrightarrow X \text{ on } A \end{matrix} \right\} \xrightarrow{T \mapsto \mathrm{fix}(T)} X.$$

Proposition 6.3. *The grading scheme σ_b is dominated by weights (Definition 4.2).*

As mentioned above, our aim here is to show that under some weak conditions, the grading schemes σ_b behave in a manner that enable a solution to the five problems described in Section 3. More specifically, these weak conditions revolve around what is called the support of a function, and are formalised below.

Definition 6.4. Let A be a set. The **support** of a function $f\colon A \longrightarrow \mathbb{R}$, is the set

$$\mathrm{supp}(f) = \{a \in A : f(a) \neq 0\}.$$

Thus, $\mathrm{supp}(f)$ is a subset of A, the domain of f, and we obtain a function

$$\mathrm{supp}\ :\ \ \mathbb{R}^A \xrightarrow{f \mapsto \mathrm{supp}(f)} \wp(A)$$

where $\wp(A)$ is the powerset of A.

Definition 6.5. A function $f\colon \mathbb{R}^A \to Z$ is **constant on supports** if for any $x, y \in \mathbb{R}^A$

$$\operatorname{supp}(x) = \operatorname{supp}(y) \implies f(x) = f(y).$$

We say that a function $f\colon \mathbb{R}^A \to \mathbb{R}^A$ **preserves supports** if for every $x \in \mathbb{R}^A$

$$\operatorname{supp}(x) = \operatorname{supp}(f(x)).$$

The next lemma shows that σ_b (Definition 6.2) preserves supports.

Lemma 6.6. *Let $\sigma_b\colon W \to X$ be the grading scheme associated to a basing scheme $b \in \operatorname{BaseScheme}(A)$, see Definition 6.2. Then σ_b preserves supports. That is, for any $w \in W$*

$$\operatorname{supp}(\sigma_b(w)) = \operatorname{supp}(w).$$

Remark 6.7. The following notational comment will be used throughout and the reader is encouraged to keep it in mind. Suppose that $b \in \operatorname{BaseScheme}(A)$, i.e b is a function $b\colon W \to \operatorname{GrBase}(A)$. Then for any $w \in W$ we get $b(w) \in \operatorname{GrBase}(A) = \mathcal{BISH}(A)^A$. Thus, we write

$$\begin{aligned} b(w) &= (b(w)_i)_{i\in A} && \text{where} \\ b(w)_i &\colon X \to [0,\infty) && \text{are in } \mathcal{BISH}(A). \end{aligned}$$

Definition 6.8. A basing scheme $b\colon W \to \operatorname{GrBase}(A)$ is called **constant on supports** if it is constant on supports as a function from W, i.e., for any $w, w' \in W$

$$\operatorname{supp}(w) = \operatorname{supp}(w') \implies b(w) = b(w').$$

It is called **constant** if it is a constant function $b\colon W \to \operatorname{GrBase}(A)$.

It is called **continuous**, resp. **tame**, if for every $w \in W$ the grading base $b(w)$ is continuous, resp. tame, in the sense of Definition 5.5.

In the following theorem we identify grading schemes σ_b which will be used to build suitable semantics that solve the five problems of Section 3 (see Theorem 7.6 and Proposition 8.2). Recall that $\mathbb{R}^A$ is equipped with the Euclidean metric.

Theorem 6.9. *Let $G = (A, D)$ be an argumentation graph with weighting and grading spaces W, X. Let $b\colon W \to \operatorname{GrBase}(A)$ be a basing scheme. Assume that*

- *b is constant on supports (Definition 6.8).*

Let σ_b be the grading scheme associated to b (Definition 6.2). Define a function

$$\operatorname{inv}_b\colon X \to \mathbb{R}^A$$

as follows. For any $x \in X = [0,1]^A$ we may view x as an element of $W = [0,1]^A$ and we set $\operatorname{inv}_b(x) \in \mathbb{R}^A$ to be the function which for any $i \in A$ is defined by

$$\operatorname{inv}_b(x)(i) = x(i) \cdot \big(1 + b(x)_i(x)\big).$$

Then

(a) For any $x \in X$

$$x \in D_{\sigma_b} \iff \operatorname{inv}_b(x) \in W.$$

Moreover, $\sigma_b\colon W \to D_{\sigma_b}$ is bijective with inverse $\operatorname{inv}_b|_{D_{\sigma_b}}$.

(b) If b is constant with value $\varphi\colon A \to \mathcal{BISH}(A)$ such that the components $\varphi_i\colon X \to [0,\infty)$ are continuous then $\sigma_b\colon W \to D_{\sigma_b}$ is a homeomorphism.

(c) For any $x \in X$ there exists $t > 0$ such that $t \cdot x \in D_{\sigma_b}$.

(d) Assume in addition that b is tame (Definition 6.8). Let $x \in D_{\sigma_b}$. Then $t \cdot x \in D_{\sigma_b}$ for all $0 \le t \le 1$.

We end this section with the following proposition which gives a criterion to decide when the space of acceptability degrees of one grading scheme associated with a basing scheme is contained in another. This is a fundamental result which, in conjunction with the family of semantics introduced in the next two sections, enable us to compare the acceptability degree space of different semantics.

PROPOSITION 6.10. *Let $G = (A, D)$ be an argumentation graph. Let X, W be the grading and weighting spaces on A. Let $b, b' \in \text{BaseScheme}(A)$ be constant on supports. Suppose that for any $w \in W$ and any $i \in A$ (see Remark 6.7)*

$$b(w)_i(x) \leq b'(w)_i(x)$$

for every $x \in X$ such that $\text{supp}(x) = \text{supp}(w)$. *Then*

$$D_{\sigma_{b'}} \subseteq D_{\sigma_b}.$$

7 Weighted Based Gradual Semantics

In this section we use the reformulation of weighted gradual semantics in terms of grading schemes which we presented in Section 4 and the construction in Section 6 in order to construct a large family of weighted gradual semantics which we call weighted *based* gradual semantics. Proofs for the results in this section are deferred to Appendix D.

Definition 7.1. An **argumentation base** is a function

$$\beta \colon \mathcal{AF} \to \bigcup_{(A,D)\in\mathcal{AF}} \text{BaseScheme}(A)$$

which assigns to any argumentation graph $G = (A, D) \in \mathcal{AF}$ a basing scheme $\beta(G)$ on A in the component (A, D) of the codomain.

The construction in Definition 6.2 gives for any argumentation graph $G = (A, D)$ a function

$$\tilde{\sigma} \colon \text{BaseScheme}(A) \xrightarrow{b \mapsto \sigma_b} \text{GrScheme}(A).$$

The interpretation of weighted gradual semantics in Definition 4.4 leads to

Definition 7.2. Let β be an argumentation base. The composition

$$\Sigma_\beta \ : \ \mathcal{AF} \xrightarrow{\beta} \bigcup_{(A,D)\in\mathcal{AF}} \text{BaseScheme}(A) \xrightarrow{\bigcup_{(A,D)} \tilde{\sigma}} \bigcup_{(A,D)\in\mathcal{AF}} \text{GrScheme}(A)$$

is called the **weighted *based* gradual semantics** with base β and is denoted Σ_β.

Remark 7.3. By Definitions 7.2 and 6.2, for any argumentation graph $G \in \mathcal{AF}$

$$\Sigma_\beta(G) = \sigma_{\beta(G)}.$$

Definition 7.4. An argumentation base β is called **locally constant on supports**, **locally constant**, **locally continuous** or **locally tame** if for any argumentation graph $G = (A, D)$ the basing scheme $b = \beta(G)$ is constant on supports, constant, continuous or tame, respectively, in the sense of Definition 6.8.

Remark 7.5. Recall that if β is an argumentation base then by definition $\beta(G)$ is a basing scheme for any argumentation graph $G = (A, D)$. For any $w \in W$, by Remark 6.7 the components of $b = \beta(G)(w)$ are

$$\beta(G)(w)_i \colon X \to [0, \infty).$$

THEOREM 7.6. *Let Σ_β be a weighted based gradual semantics with base β (Definition 7.2). Then*

(a) *Σ_β is dominated by weights (Definition 2.6). That is, $\Sigma_\beta(G)(w) \preceq w$ for any argumentation graph $G = (A, D)$ and any weighting $w \in W$.*

(b) *Σ_β is* calculated by limits *in the following sense. Let $G = (A, D)$ be an argumentation graph with weighting $w \in W$. Define a sequence $y_k \in X = [0, 1]^A$ recursively by choosing an arbitrary $y_0 \in X$, and for any $k \geq 0$ define $y_{k+1} \in \mathbb{R}^A$ for any $i \in A$ by*

$$y_{k+1}(i) = \frac{w(i)}{1 + \beta(G)(w)_i(y_k)}$$

(see Remark 7.5). Then $y_k \in X$ for all $k \geq 0$ and

$$\Sigma_\beta(G)(w) = \lim_{k\to\infty} y_k.$$

Assume further that β is locally constant on supports (Definition 7.4). Then

(c) *Σ_β has positive solution to the inverse, reflection and the projective preference ordering problems, i.e Problems 3.1, 3.3, 3.7.*
If in addition β is locally tame then Σ_β also solves the radiality problem, Problem 3.8.

(d) *If β is locally constant and locally continuous then Σ_β solves the topological reflection problem, Problem 3.5.*

(e) *The inverse function $\mathrm{inv}_{\Sigma_\beta,G} : X \longrightarrow \mathbb{R}^A$ in Problem 3.1 is computed explicitly in terms of β and the argumentation graph $G = (A, D)$ by the formula*

$$\mathrm{inv}_{\beta,G}(x)(i) = x(i) \cdot (1 + \beta(G)(x)_i(x))$$

where we view any $x \in X = [0, 1]^A$ as an element in $W = [0, 1]^A$.

This theorem not only identifies the conditions under which the five problems have a positive answer, but also — and importantly — provides an analytic expression by which initial weights can be computed from final acceptability degrees. As we will see in the next section, this analytic solution is applicable to the $\mathbb{HC}$, $\mathbb{MB}$ and $\mathbb{CB}$ semantics of Amgoud et al. 2022, and stands in contrast to the numerical approach proposed in (Oren, Yun, et al. 2022). The analytic solution to the three semantics arises as they are special cases of the broad family of semantics we have introduced, which we demonstrate in the next section.

8 Applications and Examples: L^p-based and card-L^p-based Semantics

In this section we show that the max-based semantics, card-based semantics and the h-categoriser semantics introduced by (Amgoud, Doder, and Vesic 2022, Sec. 7) are special cases of two families of weighted *based* gradual semantics (see Definition 7.2) which we call the L^p-based and the card-L^p-based semantics. We investigate how the acceptability degree spaces of different semantics relate to each other, and then use the machinery of argumentation bases of gradual semantics to introduce a new semantic not yet studied in the literature.

Consider some $1 \leq p \leq \infty$. Recall that for any finite set S the L^p-norm of $u \in \mathbb{R}^S$ is

$$\|u\|_p = \left(\sum_{i\in S} |x(i)|^p\right)^{\frac{1}{p}} \qquad \text{if } 1 \leq p < \infty$$

$$\|u\|_\infty = \max_{i\in S}\{|x(i)|\} \qquad \text{if } p = \infty$$

For example, if $p = 2$ we get the Euclidean norm on $\mathbb{R}^S$. For $p = 1$ and $p = \infty$ we get

$$\|u\|_1 = \sum_{i\in S} |x(i)| \qquad \text{and} \qquad \|u\|_\infty = \max_{i\in S} |x(i)|.$$

It is well known that if $1 \leq p \leq q \leq \infty$ then

$$\|u\|_q \leq \|u\|_p.$$

Also, the L^p-norm is a continuous function, i.e., $\mathbb{R}^S \xrightarrow{u \mapsto \|u\|_p} [0, \infty)$ is a continuous function.

Recall that the restriction of a function is obtained by reducing the domain of that function. Namely, suppose that A is a finite set and $S \subseteq A$. Restriction of functions can be captured by a function

$$\mathrm{Res}^A_S : \mathbb{R}^A \xrightarrow{u \mapsto u|_S} \mathbb{R}^S.$$

Notice that if $x \in \mathbb{R}^A$ and $S \subseteq A$ is such that $\mathrm{supp}(x) \subseteq S$ then

$$\|x\|_p = \| \mathrm{Res}^A_S(x) \|_p.$$

Given an argumentation graph $G = (A, D)$ and $i \in A$ and some $u \in [0, 1]^A$ set

$$\mathtt{Att}^u_G(i) = \mathtt{Att}_G(i) \cap \mathrm{supp}(u).$$

See Definition 6.4 for the support.

We will now define two families of weighted based gradual semantics which, as we will see later, generalise the max-based, h-categoriser and card-based semantics.

Construction 8.1. Let $1 \leq p \leq \infty$. We define the argumentation bases $\beta_{L^p\mathbb{B}}$ and $\beta_{\mathbb{C}L^p\mathbb{B}}$ as follows. Given an argumentation graph $G = (A, D)$ with spaces of gradings and weightings $X, W = [0, 1]^A$, define functions

$$\beta_{L^p\mathbb{B}}(G) \colon W \to \mathrm{Func}(X, [0, \infty))^A$$
$$\beta_{\mathbb{C}L^p\mathbb{B}}(G) \colon W \to \mathrm{Func}(X, [0, \infty))^A$$

by the following formulas. For any $w \in W$, any $i \in A$, and any $x \in X$

$$\beta_{L^p\mathbb{B}}(G)(w)_i(x) = \| \mathrm{Res}^A_{\mathtt{Att}_G(i)}(x) \|_p$$
$$\beta_{\mathbb{C}L^p\mathbb{B}}(G)(w)_i(x) = \begin{cases} |\mathtt{Att}^w_G(i)| + \frac{1}{|\mathtt{Att}^w_G(i)|} \cdot \| \mathrm{Res}^A_{\mathtt{Att}^w_G(i)}(x) \|_p & \text{if } \mathtt{Att}^w_G(i) \neq \emptyset \\ 0 & \text{if } \mathtt{Att}^w_G(i) = \emptyset. \end{cases}$$

We note that $\beta_{L^p\mathbb{B}}(G)(w)_i$ and $\beta_{\mathbb{C}L^p\mathbb{B}}(G)(w)_i$ are continuous in x and therefore define bounded functions $X \to [0, \infty)$. They are also order preserving since the L^p-norm is clearly order preserving. They are in $\mathcal{BITSH}(A)$ by Propositions 5.3 and 5.4(i),(iii) and (ii). It follows that the range of $\beta_{L^p\mathbb{B}}(G)$ and $\beta_{\mathbb{C}L^p\mathbb{B}}(G)$ is $\mathrm{GrBase}(A)$, hence they are continuous and tame basing schemes (Definitions 6.1 and 6.8). Moreover, they are constant on supports because $\beta_{L^p\mathbb{B}}(G) \colon W \to \mathrm{GrBase}(A)$ is constant, and because $\mathtt{Att}^w_G(i) = \mathtt{Att}^{w'}_G(i)$ if $\mathrm{supp}(w) = \mathrm{supp}(w')$. Thus, we have shown that the argumentation bases $\beta_{L^p\mathbb{B}}$ and $\beta_{\mathbb{C}L^p\mathbb{B}}$ are locally constant on supports, locally continuous and locally tame.

The associated weighted based gradual semantics (Definition 7.2) are denoted

$$\Sigma_{L^p\mathbb{B}},$$
$$\Sigma_{\mathbb{C}L^p\mathbb{B}}$$

respectively, and are called the L^p**-based** and the **card-**L^p**-based** weighted gradual semantics.

Proposition 8.2. *For any $1 \leq p \leq \infty$, the semantics $\Sigma_{L^p\mathbb{B}}$ and $\Sigma_{\mathbb{C}L^p\mathbb{B}}$ have positive solution to the inverse problem (Problem 3.1), the reflection problem (Problem 3.3), the projective preference ordering problem (Problem 3.7), and the radiality problem (Problem 3.8). The semantics $\Sigma_{L^p\mathbb{B}}$ have a positive solution to the topological reflection problem (Problem 3.5).*

Furthermore, $\Sigma_{L^p\mathbb{B}}$ and $\Sigma_{\mathbb{C}L^p\mathbb{B}}$ are defined by means of limits:

$$\Sigma_{L^p\mathbb{B}}(G)(w), \Sigma_{\mathbb{C}L^p\mathbb{B}}(G)(w) = \lim_{k\to\infty} y_k$$

where the sequence $y_k \in X$ is defined recursively by choosing $y_0 \in X$ arbitrarily and

$$y_{k+1}(i) = \frac{w(i)}{1 + \| \operatorname{Res}^A_{\mathtt{Att}_G(i)}(y_k)\|_p} \qquad \text{for } \Sigma_{L^p\mathbb{B}} \tag{2}$$
$$y_{k+1}(i) = \frac{w(i)}{1 + |\mathtt{Att}^w_G(i)| + \frac{1}{|\mathtt{Att}^w_G(i)|} \cdot \| \operatorname{Res}^A_{\mathtt{Att}^w_G(i)}(y_k)\|_p} \qquad \text{for } \Sigma_{\mathbb{C}L^p\mathbb{B}}$$

where the last term in the denominator in the second formula is understood to be zero if $\mathtt{Att}^w_G(i) = \emptyset$. The inverse functions in Problem 3.1 for these semantics are defined by

$$\operatorname{inv}_{L^p\mathbb{B},G}(x)(i) = x(i) \cdot (1 + \| \operatorname{Res}^A_{\mathtt{Att}_G(i)}(x)\|_p) \qquad \text{for } \Sigma_{L^p\mathbb{B}} \tag{3}$$
$$\operatorname{inv}_{\mathbb{C}L^p\mathbb{B},G}(x)(i) = x(i) \cdot \left(|\mathtt{Att}^x_G(i)| + \frac{1}{|\mathtt{Att}^x_G(i)|} \| \operatorname{Res}^A_{\mathtt{Att}^x_G(i)}(x)\|_p\right) \qquad \text{for } \Sigma_{\mathbb{C}L^p\mathbb{B}}$$

with the convention that the term involving $\frac{1}{|\mathtt{Att}^x_G(i)|}$ vanishes if $\mathtt{Att}^x_G(i) = \emptyset$.

Proof. In Construction 8.1 we demonstrated that the argumentation bases $\beta_{L^p\mathbb{B}}$ and $\beta_{\mathbb{C}L^p\mathbb{B}}$ are locally constant on supports and locally tame and that $\beta_{L^p\mathbb{B}}$ are locally constant and locally continuous. The result is immediate from Theorem 7.6. □

The max-based, card-based and h-categoriser semantics are special cases of the L^p-based and the card-L^p-based semantics in Construction 8.1.

Corollary 8.3. *It is the case that*

$$\Sigma_{\mathbb{HC}} = \Sigma_{L^1\mathbb{B}}$$
$$\Sigma_{\mathbb{MB}} = \Sigma_{L^\infty\mathbb{B}}$$
$$\Sigma_{\mathbb{CB}} = \Sigma_{\mathbb{C}L^1\mathbb{B}}.$$

In particular, all five inverse problems hold true for these semantics, with the exception of $\Sigma_{\mathbb{CB}}$ which does not have a solution for the topological reflection problem. Furthermore they have inverse functions $\operatorname{inv}_{\mathbb{MB}}$, $\operatorname{inv}_{\mathbb{HC}}$, $\operatorname{inv}_{\mathbb{CB}}$ described analytically in display (4) below.

Proof. Consider the L^p-based semantics for $p = 1$ and $p = \infty$. By inspection of the definition of $\beta_{L^1\mathbb{B}}$ and $\beta_{L^\infty\mathbb{B}}$ the sequences y_k defined in Equation (2) in Construction 8.1 coincide with the sequences $\mathbb{HC}_k$ and $\mathbb{MB}_k$ that were used to define the h-categoriser and the max-based semantics. This proves, in particular, that the sequences $\mathbb{HC}_k$ and $\mathbb{MB}_k$ converge and that $\Sigma_{L^1\mathbb{B}} = \Sigma_{\mathbb{HC}}$ and that $\Sigma_{L^\infty\mathbb{B}} = \Sigma_{\mathbb{MB}}$ as required.

Similarly, by inspection of the definition of $\beta_{\mathbb{C}L^p\mathbb{B}}$ with $p = 1$, the sequence y_k for $\Sigma_{\mathbb{C}L^1\mathbb{B}}$ in Equation (2) of Construction 8.1 coincides with the sequence $\mathbb{CB}_k$ used to define $\Sigma_{\mathbb{CB}}$. Notice that $\mathtt{Att}^*_G(i)$ in that example is denoted here $\mathtt{Att}^w_G(i)$. It follows that the sequence $\mathbb{CB}_k$ is convergent and that $\Sigma_{\mathbb{C}L^1\mathbb{B}} = \Sigma_{\mathbb{CB}}$.

Explicit formulas for the inverse functions for these semantics are a straightforward application of (3). Given an argumentation graph $G = (A, D)$, for any x in the grading space, i.e $\in X = [0,1]^A$

$$\mathrm{inv}_{\mathbb{MB},G}(x)(i) = x(i) \cdot \left(1 + \max_{j \in \mathtt{Att}_G(i)} x(j)\right) \tag{4}$$

$$\mathrm{inv}_{\mathbb{HC},G}(x)(i) = x(i) \cdot \left(1 + \sum_{j \in \mathtt{Att}_G(i)} x(j)\right)$$

$$\mathrm{inv}_{\mathbb{CB},G}(x)(i) = x(i) \cdot \left(1 + |\mathtt{Att}^*_G(i)| + \tfrac{1}{|\mathtt{Att}^*_G(i)|} \sum_{j \in \mathtt{Att}^*_G(i)} x(j)\right)$$

□

Notice that our methods in the corollary above give a uniform proof for the convergence of the sequences for the max-based, h-categoriser and card-based semantics and hence gives a conceptual treatment, rather than ad-hoc one, for Theorems 7, 12 and 17 in Section 7 of (Amgoud, Doder, and Vesic 2022). Also, since we have shown that these semantics, $\Sigma_{\mathbb{MB}}$, $\Sigma_{\mathbb{CB}}$ and $\Sigma_{\mathbb{CB}}$, solve the inverse and reflection problems, this gives a conceptual and uniform proof for Theorems 9, 14 and 19 in that paper.

We now turn our attention to how the acceptability degrees space of the different semantics we consider relate to each other.

Proposition 8.4. *For any $1 \le p \le q \le \infty$ and any argumentation graph $G = (A, D)$ the following inclusions hold between the spaces of acceptability degrees.*

$$D_{\Sigma_{\mathbb{C}L^p\mathbb{B}},G} \subseteq D_{\Sigma_{\mathbb{C}L^q\mathbb{B}},G} \subseteq D_{\Sigma_{L^p\mathbb{B}},G} \subseteq D_{\Sigma_{L^q\mathbb{B}},G}.$$

Proof. By definition of $\beta_{L^p\mathbb{B}}(G)$ and $\beta_{L^q\mathbb{B}}(G)$, for any $w \in W$, and $i \in A$ and any $x \in X$

$$\beta_{L^q\mathbb{B}}(G)(w)_i(x) = \|\,\mathrm{Res}^A_{\mathtt{Att}_G(i)}(x)\|_q \le \|\,\mathrm{Res}^A_{\mathtt{Att}_G(i)}(x)\|_p = \beta_{L^p\mathbb{B}}(G)(w)_i(x).$$

The third inclusion follows from Proposition 6.10. Similarly, the first inclusion follows from Proposition 6.10 since

$$\begin{aligned}\beta_{\mathbb{C}L^q\mathbb{B}}(G)(w)_i(x) &= |\mathtt{Att}^w_G(i)| + \tfrac{1}{|\mathtt{Att}^w_G(i)|} \cdot \|\,\mathrm{Res}^A_{\mathtt{Att}^w_G(i)}(x)\|_q \\ &\le |\mathtt{Att}^w_G(i)| + \tfrac{1}{|\mathtt{Att}^w_G(i)|} \cdot \|\,\mathrm{Res}^A_{\mathtt{Att}^w_G(i)}(x)\|_p = \beta_{\mathbb{C}L^p\mathbb{B}}(G)(w)_i(x).\end{aligned}$$

Suppose that $w \in W$ and that $x \in X$ are such that $\mathrm{supp}(x) = \mathrm{supp}(w)$. Then

$$\mathrm{supp}(\mathrm{Res}^A_{\mathtt{Att}_G(i)}(x)) = \mathrm{supp}(x) \cap \mathtt{Att}_G(i) = \mathrm{supp}(w) \cap \mathtt{Att}_G(i) = \mathtt{Att}^w_G(i).$$

It follows that for any $1 \le r \le \infty$

$$\|\,\mathrm{Res}^A_{\mathtt{Att}^w_G(i)}(x)\|_r = \|\,\mathrm{Res}^A_{\mathtt{Att}_G(i)}(x)\|_r. \tag{5}$$

Since $x(i) \in [0,1]$ for all $i \in A$ it is clear that $\|\,\mathrm{Res}^A_S(x)\|_r \le |S|$ for all $S \subseteq A$. Then

$$\begin{aligned}\beta_{\mathbb{C}L^p\mathbb{B}}(G)(w)_i(x) &= |\mathtt{Att}^w_G(i)| + \tfrac{1}{|\mathtt{Att}^w_G(i)|}\|\,\mathrm{Res}^A_{\mathtt{Att}^w_G(i)}(x)\|_p \ge |\mathtt{Att}^w_G(i)| \\ &\ge \|\,\mathrm{Res}^A_{\mathtt{Att}^w_G(i)}(x)\|_q = \|\,\mathrm{Res}^A_{\mathtt{Att}_G(i)}(x)\|_q = \beta_{L^q\mathbb{B}}(G)(w)_i(x).\end{aligned}$$

The second inclusion in the statement of this proposition now follows from Proposition 6.10. □

Corollary 8.5. *For the max-based, h-categoriser and the card-based semantics, the following inclusions of acceptability degrees hold*

$$D_{\mathbb{CB}} \subseteq D_{\mathbb{HC}} \subseteq D_{\mathbb{MB}}.$$

Proof. Combine Corollary 8.3 and Proposition 8.4. □

We end this section by introducing a new weighted gradual semantics of a very different flavor (using a product to aggregate argument strength rather than summation) which we call the **geometric mean based** weighted gradual semantics. This new semantics illustrates the power of our formalism to create radically new semantics which differ significantly from the three semantics introduced by Amgoud et al. 2022 and which nevertheless is able to positively answer our five problems. This semantics also satisfies a unique set of properties in comparison to those semantics investigated in (Amgoud, Doder, and Vesic 2022).

Example 8.6. We define an argumentation base $\beta_{\mathbb{GM}}$ as follows. For any argumentation graph $G = (A, D)$ with grading and weighting spaces X, W on A, let $\beta_{\mathbb{GM}}(G) \in \text{BaseScheme}(A)$ be defined by the following formula, see Remark 7.5. For any $w \in W$, any $i \in A$ and $x \in X$

$$\beta_{\mathbb{GM}}(G)(w)_i(x) = \begin{cases} \left(\prod_{j \in \mathtt{Att}_G^w(i)} x(j) \right)^{1/|\mathtt{Att}_G^w(i)|} & \text{if } \mathtt{Att}_G^w(i) \neq \emptyset \\ 0 & \text{if } \mathtt{Att}_G^w(i) = \emptyset. \end{cases}$$

These functions $X \to [0, \infty)$ (one for each $i \in A$) are clearly continuous with compact domain X, hence bounded, and by inspection they are order preserving (increasing in the $x(j)$'s). By Proposition 5.4(iv) they are in $\mathcal{BITSH}(A)$.

Therefore $\beta_{\mathbb{GM}}(G)$ is indeed a function $W \to \text{GrBase}(A)$, i.e a basing scheme. Moreover, it is clearly constant on supports since $\mathtt{Att}_G^w(i) = \mathtt{Att}_G^{w'}(i)$ if $\text{supp}(w) = \text{supp}(w')$, it is continuous and tame (Definition 6.8). It follows that $\beta_{\mathbb{GM}}$ is an argumentation base which is locally constant on supports, locally continuous and locally tame (Definition 7.4).

Let $\Sigma_{\mathbb{GM}}$ be the weighted gradual semantics $\Sigma_{\beta_{\mathbb{GM}}}$ associated with $\beta_{\mathbb{GM}}$. By Theorem 7.6 all the five inverse problems hold true for $\Sigma_{\mathbb{GM}}$, with the exception of the topological reflection problem (because $\beta_{\mathbb{GM}}(G)$ are not constant functions). Moreover it is defined by limits and has an inverse function which can be written explicitly in terms of the functions $\beta_{\mathbb{GM}}(G)$ as detailed in Theorem 7.6(b)(e).

Proposition 8.7. *For any argumentation graph* $G = (A, D)$

$$D_{\Sigma_{\mathbb{MB}}, G} \subseteq D_{\Sigma_{\mathbb{GM}}, G}$$

Proof. We note that for any $a_1, \ldots, a_n \geq 0$

$$\sqrt[n]{a_1 \ldots a_n} \leq \max_{1 \leq i \leq n} a_i.$$

Consider an argumentation graph $G = (A, D)$ with spaces X, W of gradings and weightings. Consider some $w \in W$ and $i \in A$ and $x \in X$ such that $\text{supp}(x) = \text{supp}(w)$. We have seen in (5) that this implies that

$\| \mathrm{Res}^{A}_{\mathtt{Att}^{w}_{G}(i)}(x)\|_r = \| \mathrm{Res}^{A}_{\mathtt{Att}_{G}(i)}(x)\|_r$. Therefore

$$\begin{aligned}
\beta_{\mathbb{GM}}(G)(w)_i(x) &= \sqrt[|\mathtt{Att}^{w}_{G}(i)|]{\prod_{j\in \mathtt{Att}^{w}_{G}(i)} x(j)} \\
&\leq \max\{x(j) : j \in \mathtt{Att}^{w}_{G}(i)\} \\
&= \| \mathrm{Res}^{A}_{\mathtt{Att}^{w}_{G}(i)}(x)\|_\infty \\
&= \| \mathrm{Res}^{A}_{\mathtt{Att}_{G}(i)}(x)\|_\infty \\
&= \beta_{L^\infty\mathbb{B}}(G)(w)_i(x).
\end{aligned}$$

By Proposition 6.10 and Corollary 8.3 we obtain the desired inclusion $D_{\mathbb{MB},G} = D_{\Sigma_{L^\infty\mathbb{B}},G} \subseteq D_{\Sigma_{\mathbb{GM}}}$. □

In light of Proposition 8.4 and Corollary 8.3 this shows that the space of acceptability degrees of $\Sigma_{\mathbb{GM}}$ contains the spaces of acceptability degrees of the semantics studied in Construction 8.1. That is, for any $1 \leq p \leq \infty$

$$D_{\Sigma_{L^p\mathbb{B}}}, D_{\Sigma_{\mathbb{C}L^p\mathbb{B}}} \subseteq D_{\Sigma_{\mathbb{GM}}}.$$

Proposition 8.8. *The geometric mean based weighted gradual semantics satisfies anonymity, independence, directionality, maximality, (strict) weakening, weakening soundness, resilience, (strict) proportionality, neutrality, (strict) reinforcement, symmetry, equivalence, and compensation. It does not satisfy mononotony, (strict) invariance, cardinality precedence, quality precedence, and counting.*

Proof. Proofs for the positive properties can be seen trivially, and so we consider those properties that are not satisfied.

For *monotony*, consider the graph $G = (A, D)$ with $A = \{a, b, c_1, c_2, c_3\}$ and $D = \{(c_2, a), c_3, a), (c_1, b), c_2, b), c_3, b)\}$. Let $w(a) = w(b) = 1$ and $w(c_1) = w(c_2) = 0.1$ while $w(c_3) = 0.2$. We have that the acceptability degree of b is (strictly) greater than that of a. The same counterexample as for monotony serves as a counterexample to *cardinality precedence* and *counting*.

For (strict) *invariance*, we consider the graph $G = (A, D)$ with $A = \{a, b, a', b', x, y, z\}$ with $D = \{(z, b), (x, a'), (y, b'), (z, b')\}$. If we set $w(a) = w(b) = w(a') = w(b') = 1$, $w(x) = w(y) = 0.5$ and $w(z) = 0.1$, then we get that the degree of a is strictly greater than b but the degree of a' is strictly less than b'.

For quality precedence, we consider the graph $G = (A, D)$ with $A = \{a, b, a_1, a_2, b_1, b_2\}$ with $D = \{(a_1, a), (a_2, a), (b_1, b), (b_2, b)\}$. If we set $w(a) = w(b) = 1$, $w(a_1) = 0.9$, $w(a_2) = 0.001$, $w(b_1) = w(b_2) = 0.85$, then we get that the degree of a is strictly greater than that of b.

□

We note that the profile of this semantics (the properties satisfied or not by this semantics) is distinct to all other semantics summarized in Table 1 in (Amgoud, Doder, and Vesic 2022).

9 Gradual Semantics with Scaled Attack Strengths

The main focus of this paper revolves around the inverse problems introduced in Section 3 and identifying a framework to construct weighted gradual semantics which solve these problems. However, the ideas in this paper easily extend to a more general notion of weighted gradual semantics, namely ones which also take into account varying strengths of attacks (c.f., work such as that by Dunne et al. 2011 and Coste-Marquis et al. 2012; 2012 which builds upon it) in addition to weights of arguments. The purpose of this section is to show how our methods can be used to obtain a construction of this type of more general type of gradual semantics, following closely the line of reasoning in Sections 4–7.

This type of weighted gradual semantics were introduced by Amgoud and Doder 2019. At the end of this section we will compare our approach with their work and show how our framework gives greater flexibility than theirs.

Definition 9.1 (cf. Definitions 2.1, 2.3)**.** Let $G = (A, D)$ be an argumentation graph. A **weighting** on A is a function $w\colon A \to [0,1]$. A **scaling** for D is a function $s\colon D \to [0,1]$. The space of all weightings (resp. scalings) on A (resp. D) is denoted $W(A) = [0,1]^A$ (resp. $S(D) = [0,1]^D$). When G is understood from the context, we write W and S.

Definition 9.2 (cf. Definition 2.2)**.** Given an argumentation graph $G = (A, D)$, a **grading** on A is a function $x\colon A \to [0,1]$. The space of all grading on A is denoted $X(A)$, or simply X if G is understood from the context.

Definition 9.3 (cf. Definition 2.3)**.** A **weighted argumentation graph with varying strengths of attacks** is a quadruple (A, D, w, s) where $G = (A, D)$ is an argumentation graph and w, s are weighting and scaling functions.

Intuitively, a scaling $s\colon D \to [0,1]$ is a way to assign strengths to attacks. We then take into account strengths (scaling) of attacks when calculating their effect on the grading of an argument.

In this context, a weighted gradual semantics Σ is a function which assigns to each weighted argumentation graph with varying strength of attacks $F = (A, D, w, s)$, a grading $x \in X = [0,1]^A$ of the arguments which is called the acceptability degree of F with respect to Σ and is denoted $\Sigma(F)$.

As we did in Section 4, we will reserve the word “scheme” to describe functions from $W \times S$ to some set Ω where W and S are the weighting and scaling spaces (on A and D) of some argumentation graph $G = (A, D)$.

Definition 9.4 (cf. Definition 4.1)**.** A **grading scheme** for an argumentation graph $G = (A, D)$ is a function

$$\sigma\colon W \times S \to X.$$

The set of all grading schemes for G is denoted by

$$\mathrm{GrScheme}(G) = \mathrm{Func}(W \times S, X).$$

Analogously to Section 4, an alternative definition for weighted gradual semantics is

Definition 9.5 (cf. Definition 4.4)**.** A **weighted gradual semantics** is a function

$$\Sigma\colon \mathcal{AF} \longrightarrow \bigcup_{G \in \mathcal{AF}} \mathrm{GrScheme}(G)$$

where for every $G \in \mathcal{AF}$, the grading scheme $\Sigma(G)$ belings to the component G in the codomain.

Recall from Definition 5.5 the notion of grading bases for $G = (A, D)$. These are functions $\varphi\colon A \to \mathcal{BISH}(A)$ where $\mathcal{BISH}(A)$ is the collection of functions $\psi\colon X \to [0, \infty)$ described in Definition 5.1. Grading bases are the functions which aggregate attacks on some argument i in G into some non-negative number whose value is used in order to update the grade of i. Mimicking the procedure in Section 6, we introduce the following definition.

Definition 9.6 (cf. Definition 6.1)**.** Let $G = (A, D)$ be an argumentation graph with corresponding spaces W, S of weightings and scalings. A **basing scheme** is a function

$$b\colon W \times S \to \mathrm{GrBase}(A).$$

Fix a basing scheme b for $G = (A, D)$. Any $(w, s) \in W \times S$ give rise to a grading base $\varphi = b(w, s)$, which by Definition 5.7 and Proposition 5.8 gives rise to a stable grading transition function $T_{\varphi,w}$ which, in turn, has a unique fixed point $x = \mathrm{fix}(T_{\varphi,w})$. This justifies the following definition.

Definition 9.7 (cf. Definition 6.2)**.** Let $G = (A, D)$ be an argumentation graph and b a basing scheme for G. Let W, S, X denote respectively the spaces of weightings, scalings and gradings of G. Define a grading scheme $\sigma_b \colon W \times S \to X$ for G as follows

$$\sigma_b(w, s) = \text{fix}(T_{b(s,w),w}).$$

Thus, for any $G \in \mathcal{AF}$ we obtain a function

$$\tilde{\sigma} \colon \text{BaseScheme}(G) \xrightarrow{b \mapsto \sigma_b} \text{GrScheme}(G)$$

Notice that since $T_{\varphi,w}$ is a stable grading transition function, by definition σ_b is calculated by limits. More precisely,

$$\sigma_b(w, s) = \lim_{k \to \infty} y_k$$

where y_k is a sequence in X which is defined recursively as follows. Choose $y_0 \in X$ arbitrarily, and for any $k \geq 0$

$$y_{k+1} = T_{b(w,s),w}(y_k).$$

Thus, for any $i \in A$

$$y_{k+1}(i) = \frac{w(i)}{1 + b(w, s)_i(y_k)}.$$

Following the strategy in Section 7, we redefine the notion of argumentation bases as follows.

Definition 9.8 (cf. Definition 7.1)**.** An **argumentation base** is a function

$$\beta \colon \mathcal{AF} \to \bigcup_{G \in \mathcal{AF}} \text{BaseScheme}(G)$$

such that for any $G \in \mathcal{AF}$ the basing scheme $\Sigma(G)$ belongs to the component of G in the codomain.

Definition 9.9 (cf. Definition 7.2)**.** Let β be an argumentation base. The composition

$$\Sigma_\beta \;:\; \mathcal{AF} \xrightarrow{\beta} \bigcup_{G \in \mathcal{AF}} \text{BaseScheme}(G) \xrightarrow{\bigcup_G \tilde{\sigma}} \bigcup_{G \in \mathcal{AF}} \text{GrScheme}(G)$$

is called the **weighted *based* gradual semantics** with base β and is denoted Σ_β.

Summarising, we see that in order to define a weighted gradual semantics it suffices to design a rule that for each argumentation graph $G = (A, D)$ assigns a basing scheme

$$\beta(G) \colon W \times S \to \text{GrBase}(A) \subseteq \text{Func}(X, [0, \infty))^A.$$

Construction 9.10 (cf. Construction 8.1)**.** Choose some $1 \leq p < \infty$. Define for any $G = (A, D)$ and any $w \in W$ and $s \in S$

$$\beta(G)(s, w)_i(x) = \left(\sum_{j \in \mathtt{Att}_G(i)} (s(j, i) \cdot x(j))^p \right)^{1/p} = \left\| \text{Res}^D_{\mathtt{Att}_G(i) \times \{i\}}(s) \cdot \text{Res}^A_{\mathtt{Att}_G(i)}(x) \right\|_p.$$

Intuitively, we use the L^p-norm in order to calculate the aggregate attacks at an argument $i \in A$ as we did in Construction 8.1 in order to define $\beta_{L^p\mathbb{B}}$, except that the grade of each attacker j of i is multiplied by the scaling $s(j, i)$ of the attack. We obtain a weighted gradual semantics Σ_β on weighted (and scaled) argumentation graphs G.

Observe that the semantics $\Sigma_{L^p\mathbb{B}}$ (and $\Sigma_{\mathbb{C}L^p\mathbb{B}}$) from Construction 8.1 are special cases of the construction in this section because we may view the basing schemes $\beta_{L^p\mathbb{B}}(G)$ (and $\beta_{\mathbb{C}L^p\mathbb{B}}(G)$) from Construction 8.1 as functions

$$\beta_{L^p\mathbb{B}}(G)\colon W \times S \to \mathrm{Func}(X, [0,\infty))^A$$

$$\beta_{\mathbb{C}L^p\mathbb{B}}(G)\colon W \times S \to \mathrm{Func}(X, [0,\infty))^A$$

which do not depend on S (formally, compose $\beta_{L^p\mathbb{B}}$ and $\beta_{\mathbb{C}L^p\mathbb{B}}$ from Construction 8.1 with the projection $\pi\colon W \times S \to W$).

The above construction is analogous to $\beta_{L^p\mathbb{B}}$ within the previous section, and thus defines a similar family of semantics but which also has varying strengths on attacks. Amgoud and Doder (Amgoud and Doder 2019) previously introduced a family of weighted gradual semantics which considers varied-strength attacks. They define gradual semantics in the same way we do and construct a family of gradual semantics by means of a concept they call "evaluation method" (EM), which consists of a triple of functions (f, g, h) whose utility and its connection to our work we describe next.

The function $h\colon [0,1] \times [0,1] \to [0,1]$ is used to "correct" the grade of an attacker according to the strength of the attack. That is, if an argument j attacks i with strength $s(j,i)$ then the attack is computed using the grade $h(s(j,i), x(j))$ in place of $x(j)$.

The function $g\colon \cup_{n=0}^{\infty} [0,1]^n \to [0,\infty)$ is the function that aggregates attacks on arguments with n attackers ($n \geq 0$) to some value in $[0,\infty)$. To avoid ambiguity the function g is required to be invariant to permutation of the coordinates $(x_1, \ldots, x_n)$ in $[0,1]^n$. The function g corresponds to our notion of grading bases (Definition 5.5). For example, the last condition in [Amgoud and Doder 2019, Definition 3.1] corresponds to the super homogeneity that is required from grading bases (see Definitions 5.1 and 5.5).

The function $f\colon [0,1] \times [0,\infty) \to [0,1]$ in an EM calculates the grade of an argument i from its weight w and the aggregate attacks z on it. In our paper (as per Definition 5.7 of the grading transition functions $T_{\varphi,w}$ or the formula in Theorem 7.6(b)), we only use the function

$$f_{\mathrm{frac}}(w, z) = \frac{w}{1+z}.$$

The use of this particular function can be motivated by the need for the weighted gradual semantics to satisfy the maximality (i.e. when $z = 0$, then the grading of the argument is w) and weakening (i.e., when $z > 0$, then the grading of the argument is less than w) principles among others (Amgoud, Doder, and Vesic 2022).

With $f = f_{\mathrm{frac}}$, the construction of Σ_β in this section (Definition 9.9) generalises the results of [Amgoud and Doder 2019, Section 3], specifically it generalises their Theorem 3.4. That is, any gradual semantics $S(M)$ obtained from an evaluation method $M = (f, g, h)$ with $f = f_{\mathrm{frac}}$ is equal to the gradual semantics Σ_β where the basing scheme β is defined as follows. For convenience, for any $s = (s_1, \ldots, s_n)$ and $x = (x_1, \ldots, x_n)$ we write

$$h_*(s, x) = (h(s_1, x_1), \ldots, h(s_n, x_n)).$$

Then for any $G = (A, D) \in \mathcal{AF}$ the basing scheme $\beta(G)\colon W \times S \to \mathrm{GrBase}(A)$ is defined for any $i \in A$ and any $x \in X$ by

$$\beta(G)(w, s)_i(x) = g|_{\mathtt{Att}_G(i)}(h_*(s|_{\mathtt{Att}_G(i)\times\{i\}}, x|_{\mathtt{Att}_G(i)})).$$

We see that all the gradual semantics $S(M)$ defined in [Amgoud and Doder 2019, Section 3] with evaluation methods for which $f = f_{\mathrm{frac}}$ are special cases of the based semantics we defined in this section. We point out that our formalism is a lot more flexible. For example, the formalism in (Amgoud and Doder 2019) allows to "correct" the grades of attackers in only one way using a fixed function $h(s, x)$ whereas our formalism is flexible enough to make the correction depend on the scales of other attacks or take into account more subtle combinatorial propertied of the graph $G = (A, D)$. Also, our formalism does not require the symmetry of the functions g [Amgoud and Doder 2019, Definition 2.2] nor the continuity of the functions h and g which is required in [Amgoud and Doder 2019, Definition 3.1]. We also point out that in many cases the semantics $S(M)$ associated with an evaluation method $M = (f, g, h)$ is equal to one associated with another evaluation method $M' = (f_{\text{frac}}, g', h')$, reducing to the construction we gave in this section. Having said that, not all EM's (f, g, h) have such reduction and in those cases our approach cannot capture these semantics.

10 Conclusions

In this paper we rephrase the notion of weighted gradual semantics by means of grading schemes (Definition 4.4, Section 4). This allowed us to formulate five problems, including the inverse problem of (Oren, Yun, et al. 2022) around such grading schemes. These five problems revolve around the existence and uniqueness of solutions, and the continuity of solutions for weighted gradual semantics, and thus encode highly desirable properties of any semantics.

We develop a very general framework to encode the information of attacks between arguments by means of a new concept which we call grading base and which, in turn, gives an approach to produce a large class of new weighted gradual semantics which are guaranteed to converge to unique final acceptability degrees. Under fairly general conditions, all the semantics in this class have positive answer to the five inverse-type problems we describe in Section 3. Importantly, this class includes the weighted h-categoriser, max-based and card-based semantics, as well as many others.

We also demonstrated that an analytical solution to the original inverse problem can be derived (as per Theorem 7.6) to compute an analytical description of the inverse problem, in contrast to the numeric approach used in (Oren, Yun, et al. 2022).

Building on the generality of our construction and the desirable properties of our problems, in Section 8 describes two new families of semantics (the L^p-based and card-L^p-based semantics) as well as a novel and very different new semantics, the weighted geometric mean based semantics, which ascribes to a different set of principles in comparison to the semantics described in (Amgoud, Doder, and Vesic 2022).

Our paper addresses several important theoretical gaps in argumentation theory, and facilitates computationally efficient solutions to the inverse problem. In the context of future work, the new semantics we propose in Section 8 are very interesting. One obvious avenue of such future work involves determining which of the properties described in (Amgoud, Doder, and Vesic 2022) this family of new semantics (rather than just the weighted geometric mean based semantics) comply with.

Our approach is so flexible that it can be generalised in an effortless way to yield a systematic framework to construct gradual semantics that take into account varying strengths of attacks, similar to the semantics introduced by Amgoud and Doder 2019. They described a general class of gradual semantics which contains evaluation method-based gradual semantics. These were shown to converge, and satisfy previously considered desirable properties (e.g., directionality, equivalence and weakening among others). While our semantics are also defined by limits (Theorem 7.6(b)), our focus in this paper is on the five inverse problems. We showed that the semantics we defined overlap with theirs and discussed the differences. By having positive solutions to the five problems we proposed, the family of semantics we introduce offers the possibility of facilitating expert knowledge elicitation, thereby reducing knowledge engineering effort. Thus, exploring the properties of these semantics, and how closely they mirror human reasoning, is an avenue of work we intend to pursue.

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A List of Symbols and Terms

$D_{\Sigma,G}$ space of acceptability degrees of G 8
D_σ space of acceptability degrees of σ 12
G underlying AF 4
$S(D)$ space of all scalings on D 25
T^k k-fold composition of T 6
$T_{\varphi,w}$ grading transition func. from a grading base φ and a weighting w 15
T grading transition function 6
$W(A)$ space of all weightings on A 4
$X(A)$ space of all gradings on A 4
F weighted AF 4
$\mathtt{Att}^*$ attackers with non-zero weight. 6
$\mathtt{Att}_G(a)$ attackers of an argument a in G 4
$\mathtt{Att}_G^u(i)$ attackers of i in support of u 20
$\mathcal{BISH}(A)$ the collection of all bounded, order preserving and super homogeneous functions on $X = [0,1]^A$ 14
$\mathcal{BITSH}(A)$ subspace of the tame super homogeneous bounded order preserving functions 14
BaseScheme(A) space of all basing schemes on A 16
GrBase(A) space of all grading bases on A 14
GrScheme(A) space of all grading schemes on A 12
CB_k recursively defined CB func. 6
HC_k recursively defined HC func. 7
MB_k recursively defined MB func. 6
$\Sigma(F)$ acceptability degrees for $F = (A, D, w)$ 5
Σ_β Weighted gradual semantics associated to base β 18
$\Sigma_{\mathbb{CB}}$ weighted card-based grad. sem. 6
$\Sigma_{\mathbb{HC}}$ weighted h-categorizer grad. sem. 7
$\Sigma_{\mathbb{MB}}$ weighted max-based grad. sem. 6
Σ gradual semantics 5
$\mathcal{WAF}$ all weighted argumentation frameworks 12
$\beta_{\mathbb{CL}^p\mathbb{B}}$ base for card L^p-based weighted gradual semantics 20
$\beta_{\mathbb{GM}}$ base for the geometric mean based weighted gradual semantics 23
$\beta_{L^p\mathbb{B}}$ base for L^p-based weighted gradual semantics 20
β argumentation base 18
ev_i evaluation map at $i \in A$ 14
$\mathrm{inv}_{\Sigma,G}$ computable inverse function 9
0 A constant function 12
$\mathcal{AF}$ all argumentation frameworks 12

Res_S^A Restriction to S 20
σ_b grading scheme associated to b 16
σ grading scheme 12
$\simeq$ equally preferred 7
$\mathrm{supp}(f)$ the support of f 16
$\tilde{\sigma}$ Mapping from basing scheme to grading scheme 16
$\trianglelefteq$ at most as preferred as 7
$\triangleright$ strictly preferred to 7
$\trianglerighteq$ at least as preferred as 7
$\wp(A)$ the power set of A 16
$\|u\|_p$ L^p-norm of u 19
$\|u\|$ Euclidean norm of u 9
b basing scheme 16
d metrics on $\mathbb{R}^A$ 9
s scaling function 25

basing scheme 16
bounded 14

constant on supports 17
continuous 17

dominated by weights 5

grading base 14
grading scheme 12

homeomorphism 9
homogeneous 14

inverse problem 9

locally constant on supports 18
locally continuous 18
locally tame 18

order preserving 13
order reversing 6

preserves supports 17
projective preference ordering problem 10

radiality problem 10
reflection problem 9

B Proof of the Results in Section 5

As part of the following proof, observe that any function $\varphi = \psi + c$, where ψ is a homogeneous function and $c \geq 0$ is a constant, is itself both super homogeneous and tame. Indeed, for any $x \in X$ and any $0 \leq t \leq 1$

$$\varphi(tx) = c + \psi(tx) = c + t\psi(x) \geq tc + t\psi(x) = t(c + \psi(x)) = t \cdot \varphi(x)$$
$$t \cdot \varphi(tx) = t(c + \psi(tx)) = tc + t^2\psi(x) \leq c + \psi(x) = \varphi(x).$$

Proof of Proposition 5.3. It is clear that constant functions and the projection functions $\pi_i \colon X \to [0, \infty)$ are bounded and order preserving. Projection functions are homogeneous by inspection. The result for evaluation maps follows from the observation above.

Constant functions $\psi(x) = c \geq 0$ are super homogeneous since for any $x \in X$ and any $0 \leq t \leq 1$

$$\psi(t \cdot x) = c \geq t \cdot c = t \cdot \psi(x).$$

They are clearly tame: $t\psi(tx) = tc \leq c = \psi(x)$. □

Proof of Proposition 5.4. All the functions in (i)–(iv) are easily checked to take values in $[0, \infty)$, to be bounded and order preserving because $\psi_1, \ldots, \psi_k$ are order preserving and since the functions $t \mapsto t^q$ are increasing for all $q > 0$. The function in (i), (iii) and (iv) are super homogeneous because any $x \in X$ and $0 \leq t \leq 1$

$$\psi(t \cdot x) = \sum_{i=1}^{k} \psi_i(t \cdot x) \geq \sum_{k=1}^{k} t \cdot \psi_i(x) = t \cdot \psi(x),$$
$$\psi(t \cdot x) = \sqrt[p]{\sum_{i=1}^{k} \psi_i(t \cdot x)^p} \geq \sqrt[p]{\sum_{i=1}^{k} (t \cdot \psi_i(x))^p} = t \cdot \sqrt[p]{\sum_{i=1}^{k} \psi_i(x)^p} = t \cdot \psi(x),$$
$$\psi(t \cdot x) = \sqrt[k]{\prod_{i=1}^{k} \psi_i(t \cdot x)} \geq \sqrt[k]{t^k \cdot \prod_{i=1}^{k} \psi_i(x)} = t \cdot \sqrt[k]{\prod_{i=1}^{k} \psi_i(t \cdot x)} = t \cdot \psi(x).$$

A similar calculation shows that if $\psi_1, \ldots, \psi_k$ are tame then so are the functions ψ above, i.e $t \cdot \psi(t \cdot x) \leq \psi(x)$.

The functions ψ in (ii) are homogeneous by inspection under the assumption that $\psi_1, \ldots, \psi_k$ are homogeneous. In particular they are super homogeneous and tame as per the observation above.

□

For the proof of our key result, Theorem 5.9, the reader is encouraged to think of $X = [0, 1]^A$ as the space of n-tuples $[0, 1]^n$ in $\mathbb{R}^n$ where $n = |A|$. Then the relation $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$ for $x, y \in X$ means that $x_i \leq y_i$ for all $1 \leq i \leq n$. The proof is correct as it stands, but the reader may find it easier to think in terms of n-tuples rather than functions $x \colon A \to [0, 1]$ (which are the same as tuples (x_i) indexed by $i \in A$ where $x_i \in [0, 1]$). Then the interval $[a, b]_X$ which is defined in the beginning of the proof of this theorem is simply the box in $\mathbb{R}^n$ of all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $a_i \leq x_i \leq b_i$.

Proof of Theorem 5.9. Recall that $X = [0, 1]^A$ is partially ordered by the relation $\preceq$ where for any $x, y \in X$ we have $x \preceq y$ if $x(i) \leq y(i)$ for all $i \in A$ (recall that x, y are functions $x, y \colon A \to [0, 1]$).

Notation: Given $a, b \in X$ such that $a \preceq b$, the **interval** between a and b in X is defined to be the following subset of X

$$[a, b]_X \stackrel{\text{def}}{=} \{x \in X \,:\, a \preceq x \preceq b\}.$$

Define a sequence $u^{(n)}$ in X by recursion by $u^{(0)} = 0$ and $u^{(n+1)} = f(u^{(n)})$ for every $n \geq 0$. Then $u^{(0)} \preceq u^{(1)}$ since $0 \in X$ is the minimum element in $(X, \preceq)$. Since f is order reversing, we get $u^{(0)} \preceq u^{(2)} \preceq u^{(1)}$. By induction one easily shows

(i) $u^{(2k)} \preceq u^{(2k-1)}$ for all $k \geq 1$,
(ii) $u^{(2k)} \preceq u^{(2k+2)}$ for all $k \geq 0$,
(iii) $u^{(2k+1)} \preceq u^{(2k-1)}$ for all $k \geq 1$.

Thus, the sequence $u^{(2k)}$ is increasing in the sense that $u_i^{(2k)}$ is an increasing sequence in $\mathbb{R}$ for every $i = 1, \dots, n$, and bounded above by $1 \in X$. Similarly, $u^{(2k-1)}$ is decreasing and bounded below by $0 \in X$. We may therefore define

$$u^{(\text{ev})} = \sup_k u^{(2k)} = \lim_{k\to\infty} u^{(2k)}$$
$$u^{(\text{odd})} = \inf_k u^{(2k-1)} = \lim_{k\to\infty} u^{(2k-1)}.$$

It follows from (i) that $u^{(\text{ev})} \preceq u^{(\text{odd})}$.

Our next goal is to show that $u^{(\text{ev})} = u^{(\text{odd})}$. For every $k \geq 1$, set:

$$\pi_k = \sup \{0 \leq t \leq 1 \,:\, t \cdot u^{(2k-1)} \preceq u^{(2k)}\}.$$

The set on the right is not empty since $0 \cdot u^{(2k-1)} = 0 \preceq u^{(2k)}$, so:

$$0 \leq \pi_k \leq 1.$$

It is clear from the construction that for every $k \geq 1$

$$\pi_k \cdot u^{(2k-1)} \preceq u^{(2k)}.$$

By applying hypotheses (a) and (b) to this inequality and then using (ii), for every $k \geq 1$

$$\begin{aligned} u^{(2k+1)} &= f(u^{(2k)}) \\ &\preceq f(\pi_k \cdot u^{(2k-1)}) \\ &\preceq \tfrac{1}{\pi_k + \alpha(1-\pi_k)} \cdot f(u^{(2k-1)}) \\ &= \tfrac{1}{\pi_k + \alpha(1-\pi_k)} \cdot u^{(2k)} \\ &\preceq \tfrac{1}{\pi_k + \alpha(1-\pi_k)} \cdot u^{(2k+2)}. \end{aligned}$$

Hence $(\pi_k + \alpha(1 - \pi_k)) \cdot u^{(2k+1)} \preceq u^{(2k+2)}$, so by the definition of π_{k+1}

$$\pi_k + \alpha(1 - \pi_k) \leq \pi_{k+1}.$$

It follows that $1 - \pi_{k+1} \leq (1 - \alpha)(1 - \pi_k)$ for all $k \geq 1$. Therefore, using induction on k,

$$1 - \pi_{k+1} \leq (1 - \pi_1)(1 - \alpha)^k \xrightarrow{k\to\infty} 0.$$

But $\pi_k \leq 1$ for all k, so $\lim_k \pi_k = 1$.

Consider some $\epsilon > 0$. Then $\pi_k > 1 - \epsilon$ for all $k \gg 0$. Equation (i) implies

$$(1 - \epsilon) \cdot u^{(2k-1)} \preceq \pi_k \cdot u^{(2k-1)} \preceq u^{(2k)} \preceq u^{(2k-1)}.$$

Letting $k \to \infty$, this implies $(1-\epsilon)u^{(\text{odd})} \preceq u^{(\text{ev})} \preceq u^{(\text{odd})}$. Since $\epsilon > 0$ was arbitrary,

$$u^{(\text{ev})} = u^{(\text{odd})}$$

as needed. We will denote $u^* := u^{(\text{ev})} = u^{(\text{odd})}$.

Write $f^n \colon X \longrightarrow X$ for the n-fold composition of f with itself, i.e $f^n = f \circ \cdots \circ f$. Since $0 \in X$ is minimal, $u^{(0)} \preceq x$ for any $x \in X$. Since f is order reversing, $u^{(0)} \preceq f(x) \preceq u^{(1)}$. It then follows by induction that for all $k \geq 1$

$$f^{2k-1}(X) \subseteq [u^{(2k-2)}, u^{(2k-1)}]_X$$
$$f^{2k}(X) \subseteq [u^{(2k)}, u^{(2k-1)}]_X.$$

It follows from Equation (ii) that

$$\begin{aligned} \bigcap_{n\geq 1} f^n(X) &= \bigcap_{k\geq 1} f^{2k-1}(X) \cap f^{2k}(X) \\ &\subseteq \bigcap_{k\geq 1} [u^{(2k-2)}, u^{(2k-1)}]_X \cap [u^{(2k)}, u^{(2k-1)}]_X \\ &= \bigcap_{k\geq 1} [u^{(2k)}, u^{(2k-1)}]_X \\ &= [u^{(\text{ev})}, u^{(\text{odd})}]_X = \{u^*\}. \end{aligned}$$

Notice that $u^* \in [u^{(2k)}, u^{(2k-1)}]_X$ for all $k \geq 1$. Since f is order reversing $f(u^*) \in [u^{(2k)}, u^{(2k+1)}]_X$ for all $k \geq 1$. Therefore $f(u^*) \in \{u^*\}$, so u^* is a fixed point of f.

If $y \in X$ is a fixed point of f then by induction $y \in f^n(X)$ for all n, so $y \in \cap_n f^n(X) = \{u^*\}$ and it follows that $y = u^*$. Therefore u^* is the unique fixed point of f.

Finally, choose some $x \in X$ and define a sequence by recursion $x^{(0)} = x$ and $x^{(n+1)} = f(x^{(n)})$. Then $u^{(0)} = 0 \preceq x$ and since f is order reversing, $u^{(0)} \preceq x^{(1)} \preceq u^{(1)}$. Then one proves by induction that $u^{(2k-2)} \preceq x^{(2k-1)} \preceq u^{(2k-1)}$ and $u^{(2k)} \preceq x^{(2k)} \preceq u^{(2k-1)}$ for all $k \geq 1$. By the squeeze rule, $\lim_n x^{(n)} = u^*$. □

Proof of Proposition 5.8. By definition the functions $\varphi_i \colon X \longrightarrow [0, \infty)$ are order preserving so if $x, y \in X$ are such that $x \preceq y$ then for any $i \in A$

$$T_{\varphi,w}(x)(i) = \frac{w(i)}{1 + \varphi_i(x)} \geq \frac{w(i)}{1 + \varphi_i(y)} = T_{\varphi,w}(y)(i).$$

Hence $T_{\varphi,w}(y) \preceq T_{\varphi,w}(x)$ and we have shown that $T_{\varphi,w}$ is order reversing. Hence it is a grading transition function (Definition 2.7).

It remains to prove that it is stable. To do this we will verify the conditions of Theorem 5.9. Condition (a) holds since $T_{\varphi,w}$ is a grading transition functions, in particular order reversing. We now prove that $T_{\varphi,w}$ satisfies condition (b).

For any $i \in A$, by assumption $\varphi_i \colon X \longrightarrow [0, \infty)$ is bounded, say by some $m_i < \infty$. Set

$$M = \max\{m_i : i \in A\}$$

and set

$$\alpha = \tfrac{1}{1+M}.$$

Notice that $0 < \alpha \leq 1$ and that for any $i \in A$ and any $x \in X$

$$\tfrac{1}{1+\varphi_i(x)} \geq \tfrac{1}{1+M} = \alpha.$$

Consider some $i \in A$. Since φ_i is super homogeneous, for any $x \in X$ and any $0 \leq t \leq 1$,

$$\begin{aligned} T_{\varphi,w}(t \cdot x)(i) &= \frac{w(i)}{1 + \varphi_i(t \cdot x)} \\ &\leq \frac{w(i)}{1 + t \cdot \varphi_i(x)} \\ &= \frac{w(i)}{(1-t) + t(1 + \varphi_i(x))} \\ &= \frac{1 + \varphi_i(x)}{(1-t) + t(1 + \varphi_i(x))} \cdot \frac{w(i)}{1 + \varphi_i(x)} \\ &= \frac{1}{(1-t)\frac{1}{1+\varphi_i(x)} + t} \cdot T_{\varphi,w}(x)(i) \\ &\leq \frac{1}{(1-t)\alpha + t} \cdot T_{\varphi,w}(x)(i). \end{aligned}$$

Thus, we have shown that for any $x \in X$ and any $0 \leq t \leq 1$

$$T_{\varphi,w}(t \cdot x) \preceq \tfrac{1}{(1-t)\alpha + t} \cdot T_{\varphi,w}.$$

This establishes condition (b) of Theorem 5.9 for $T_{\varphi,w}$, which in turn shows that $T_{\varphi,w}$ is a stable grading transition function.

Finally, fox any $x \in X$ and any $i \in A$

$$T_{\varphi,w}(x)(i) = \frac{w(i)}{1 + \varphi_i(x)} \leq w(i).$$

It follows that $T_{\varphi,w}(x) \preceq w$ for any $x \in X$. □

C Proof of the Results in Section 6

Proof of Proposition 6.3. Let $w \in W$ be arbitrary and set $x = \sigma_b(w)$. By definition, $\sigma_b(w) = \text{fix}(T_{b(w),w})$. Hence, by Proposition 5.8 $x = T_{b(w),w}(x) \preceq w$. □

Proof of Lemma 6.6. Set $x = \sigma_b(w)$. Set $\varphi = b(w)$. Recall that by definition $\varphi \in \text{GrBase}(A) = \mathcal{BISH}(A)^A$ with components $\varphi_i \colon X \longrightarrow [0, \infty)$. By definition $x \in X$ is the fixed point of $T_{b(w),w}$, that is $T_{\varphi,w}(x) = x$. Hence, for any $i \in A$

$$x(i) = T_{\varphi,w}(x)(i) = \frac{w(i)}{1 + \varphi_i(x)}.$$

Since $\varphi_i(x) \geq 0$, it follows that $x(i) = 0 \iff w(i) = 0$. In other words, $\text{supp}(x) = \text{supp}(w)$. □

Proof of Theorem 6.9. (a) Let $w \in W$ be arbitrary and set $x = \sigma_b(w)$. By Lemma 6.6, $\text{supp}(w) = \text{supp}(x)$. Since b is constant on supports, $b(w) = b(x)$, where we view x as an element of $W = [0, 1]^A$.

By Definition 6.2, x is the unique fixed point of $T_{b(w),w}$, that is $T_{b(w),w}(x) = x$. Then $x(i) = T_{b(w),w}(x)(i)$ for any $i \in A$, whence

$$x(i) = T_{b(w),w}(x)(i) = \frac{w(i)}{1 + b(w)_i(x)} = \frac{w(i)}{1 + b(x)_i(x)}.$$

It follows that for any $i \in A$

$$w(i) = x(i) \cdot (1 + b(x)_i(x)) = \text{inv}_b(x)(i).$$

We have shown that for any $w \in W$, if $x = \sigma_b(w)$ then

$$\text{inv}_b(x) = w.$$

Thus, the composition

$$W \xrightarrow{\sigma_b} D_{\sigma_b} \subseteq X \xrightarrow{\text{inv}_b} \mathbb{R}^A$$

is the identity on $W \subseteq \mathbb{R}^A$. In particular σ_b must be injective. By definition it is surjective on D_{σ_b}. So $\sigma_b \colon W \to D_{\sigma_b}$ is bijective with inverse $\text{inv}_b |_{D_{\sigma_b}}$.

Consider some $x \in X$. If $x \in D_{\sigma_b}$ then $x = \sigma_b(w)$ for some $w \in W$ and we have just shown above that in this case $\text{inv}_b(x) = w \in W$. Conversely, suppose that $\text{inv}_b(x) \in W$. Set $w = \text{inv}_b(x)$. By definition of inv_b, for any $i \in A$

$$w(i) = \text{inv}_b(x)(i) = x(i) \cdot (1 + b(x)_i(x)).$$

In particular $i \in \text{supp}(w) \iff i \in \text{supp}(x)$, i.e $\text{supp}(w) = \text{supp}(x)$. Since b is constant on supports, $b(x) = b(w)$. We deduce that for any $i \in A$

$$x(i) = \frac{w(i)}{1 + b(x)_i(x)} = \frac{w(i)}{1 + b(w)_i(x)} = T_{b(w),w}(x)(i).$$

It follows that $x \in X$ is a fixed point of $T_{b(w),w} \colon X \to X$. Since $T_{b(w),w}$ has a unique fixed point (by Proposition 5.8 and Definition 2.7), we get that $x = \text{fix}(T_{b(w),w}) = \sigma_b(w) \in D_{\sigma_b}$. We have shown that for any $x \in X$

$$x \in D_{\sigma_b} \iff \text{inv}_b(x) \in W$$

and in this case $x = \sigma_b(\text{inv}_b(x))$. This completes the proof of (a).

(b) Clearly, b is constant on supports (since it is constant). Denote the constant value of b by $\varphi \colon A \to \mathcal{BISH}(A)$. The function inv_b then has the form

$$\text{inv}_b(x)(i) = x(i) \cdot (1 + \varphi_i(x)).$$

It is continuous because by assumption the functions φ_i are continuous. Since $W = [0, 1]^A \subseteq \mathbb{R}^A$ is a closed subset, $\text{inv}_b^{-1}(W)$ is closed in $X = [0, 1]^A$ and therefore compact because $[0, 1]^A$ is compact. By part (a) $\text{inv}_b^{-1}(W) = D_{\sigma_b}$, so D_{σ_b} is compact. Then $\text{inv}_b \colon D_{\sigma_b} \to W$ is a homeomorphism because any bijective continuous function between compact metric spaces is a homeomorphism. Then its inverse $\sigma_b \colon W \to D_{\sigma_b}$ is a homeomorphism as well.

(c) Consider some $x \in X$. For any $0 < t \leq 1$ it is clear that

$$\text{supp}(t \cdot x) = \text{supp}(x).$$

Consider $x \in X = [0, 1]^A$ as an element of $W = [0, 1]^A$. By assumption b is constant on supports, so for any $0 < t \leq 1$,

$$b(x) = b(t \cdot x).$$

Furthermore, $b(x)_i \colon X \to [0, \infty)$ is bounded, say by some $M_i \geq 0$. By definition of inv_b, and since b is constant on supports, for any $i \in A$

$$\begin{aligned}
\lim_{t \to 0^+} \text{inv}_b(t \cdot x)(i) &= \lim_{t \to 0^+} (t \cdot x(i)) \cdot (1 + b(t \cdot x)_i(t \cdot x)) \\
&= \lim_{t \to 0^+} (t \cdot x(i)) \cdot (1 + b(x)_i(t \cdot x)) \\
&\leq \lim_{t \to 0^+} (t \cdot x(i)) \cdot (1 + M_i) \\
&= 0.
\end{aligned}$$

Thus, there exists some $s > 0$ such that $\text{inv}_b(t \cdot x)(i) < 1$ for all $i \in A$ and all $0 < t < s$. In particular,

$$\text{inv}_b(t \cdot x) \in W.$$

Therefore $t \cdot x \in D_{\sigma_b}$ by part (a).

(d) First, observe that $\mathbf{0} \in D_{\sigma_b}$ because if $w = \mathbf{0}$ then $T_{b(w),w} \colon X \to X$ is the zero map (see Definition 5.7 with $w = \mathbf{0}$) so $\text{fix}(T_{b(\mathbf{0}),\mathbf{0}}) = \mathbf{0}$.

Consider some $x \in D_{\sigma_b}$ and some $0 \leq t \leq 1$. We need to show that $t \cdot x \in D_{\sigma_b}$. Since $\mathbf{0} \in D_{\sigma_b}$ we may assume that $t > 0$. Hence, $\mathrm{supp}(t \cdot x) = \mathrm{supp}(x)$. We regard x as an element of $W = [0,1]^A$. Consider some $i \in A$. Since by assumption $b(x)_i$ is tame,

$$\begin{aligned} \mathrm{inv}_b(t \cdot x)(i) &= t \cdot x(i) \cdot (1 + b(t \cdot x)_i(t \cdot x)) \\ &= t \cdot x(i) \cdot (1 + b(x)_i(t \cdot x)) \\ &= x(i) \cdot (t + t \cdot \psi_i(t \cdot x)) \\ &\leq x(i) \cdot (1 + \psi_i(x)) \\ &= \mathrm{inv}_b(x)(i). \end{aligned}$$

By part (a) $\mathrm{inv}_b(x) \in W$, so $0 \leq \mathrm{inv}_b(t \cdot x)(i) \leq 1$ for all $i \in A$. In particular $\mathrm{inv}_b(t \cdot x) \in W$. Another application of part (a) shows that $t \cdot x \in D_{\sigma_b}$. □

Proof of Proposition 6.10. Consider $x \in D_{\sigma_{b'}}$. By Theorem 6.9(a) and its notation

$$\mathrm{inv}_{b'}(x) \in W.$$

Set $w = \mathrm{inv}_{b'}(x)$. Thus, for all $i \in A$

$$0 \leq w(i) \leq 1.$$

By Lemma 6.6 $\mathrm{supp}(x) = \mathrm{supp}(w)$. By the definition of inv_b and $\mathrm{inv}_{b'}$ in Theorem 6.9(a) and by the hypothesis

$$\begin{aligned} \mathrm{inv}_b(x)(i) &= x(i) \cdot (1 + b(x)_i(x)) \\ &= x(i) \cdot (1 + b(w)_i(x)) \\ &\leq x(i) \cdot (1 + b'(w)_i(x)) \\ &= x(i) \cdot (1 + b'(x)_i(x)) \\ &= \mathrm{inv}_{b'}(x)(i) \\ &= w(i) \\ &\leq 1. \end{aligned}$$

Therefore, $\mathrm{inv}_b(x) \in W$, hence $x \in D_{\sigma_b}$ by Theorem 6.9(a). This completes the proof. □

D Proof of the Results in Section 7

Proof of Theorem 7.6. We use the reformulation of weighted gradual semantics in Definition 4.4. Consider an argumentation graph $G = (A, D)$. Let X, W denote the gradings weightings spaces of G. Set $b = \beta(G) \in \mathrm{BaseScheme}(A)$.

By the definition of Σ_β, for any $w \in W$ the acceptability degree of $F = (G, w)$ is

$$\Sigma_\beta(G)(w) = \sigma_{\beta(G)}(w) = \sigma_b(w).$$

By Proposition 6.3 $\sigma_b(w) \preceq w$. This proves part (a).

By Definition 6.2 and Proposition 5.8 and the definition of stability of grading transition functions (Definition 2.7)

$$\sigma_b(w) = \mathrm{fix}(T_{b(w),w}) = \lim_{k \to \infty} (T_{b(w),w})^k(y_0)$$

where $y_0 \in X$ is chosen arbitrarily. Set

$$y_k = (T_{b(w),w})^k(y_0).$$

Then $y_{k+1} = T_{b(w),w}(y_k)$, hence by Definition 5.7 of $T_{b(w),w}$, for any $i \in A$

$$y_{k+1}(i) = \frac{w(i)}{1 + b(w)_i(y_k)}$$

This proves part (b).

Assume that β is locally constant on supports. Parts (c) and (e) of the theorem follow by applying Theorem 6.9(a), (c), (d) to $b = \beta(G)$. Part (d) of this theorem follows from Theorem 6.9(b).

□

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