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Control by Adding or Deleting Edges in Graph-Restricted Weighted Voting Games

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Abstract

Graph-restricted weighted voting games generalize weighted voting games, a well-studied class of succinct simple games, by embedding them into a communication structure: a graph whose vertices are the players some of which are connected by edges. In such games, only sufficiently connected coalitions are taken into consideration for calculating the players' power indices. Focusing on the probabilistic Penrose–Banzhaf index (which Dubey and Shapley proposed in 1979 as an alternative to the normalized Penrose–Banzhaf index) and the Shapley–Shubik index, we study control of these games by an agent who can add edges to or delete edges from the given graph. We determine upper and lower bounds on how much such control actions can change a distinguished player's power and we study the computational complexity of the related problems.

1. Introduction

Weighted voting games (see, e.g., Chalkiadakis & Wooldridge, 2016; Bullinger, Elkind, & Rothe, 2024) naturally model settings in which, e.g., the members of some committee have certain voting weights, and some quota needs to be reached for a decision to pass. Their advantage is that they provide a compact representation of simple cooperative games (i.e., games where each coalition formed among players can either win or lose, see Section 2 for formal definitions). To measure the influence of players in simple games (e.g., in weighted voting games), power indices like the Shapley–Shubik index (Shapley & Shubik, 1954) and the probabilistic Penrose–Banzhaf index (Dubey & Shapley, 1979) are used, the latter being based on the normalized Penrose–Banzhaf index originally proposed by Penrose (1946) and later reinvented by Banzhaf III (1965). Following the observation that voting weight does not equal political influence, these power indices formalize the notion of "king makers" by identifying influence with the frequency by which a committee member may be the decisive vote in a coalition. A relatively recent example for the difference between voting weight and political power occurred in the 36th Israeli government that was formed in 2021: Out of 61 members of parliament, coming from seven parties, Naftali Bennett, the leader of a party with only seven members, succeeded in being the prime minister showing that his political power was exceeding that of his relative voting weight. Another application can be an analysis of control in corporate structures. Such an analysis has been made, e.g., by Gambarelli and Owen (1994) and by Crama and Leruth (2007, 2013).

We focus on *graph-restricted* weighted voting games, which generalize weighted voting games by assuming some communication structure among the players. Such a structure is an additional decisive element of whether a coalition wins or loses: In these games, even if the players of a coalition achieve the given quota, this might not be enough for this coalition to win—it must also satisfy the requirements represented by the structure. The two power indices in games restricted by graphs are also known as the *Myerson value* (Myerson, 1977) and the *restricted Banzhaf index* (Owen, 1986), respectively. Myerson (1977) introduced *"graph-restricted games,"* i.e., cooperative games with undirected graphs that describe which sets of players, together, can form a coalition. Using the model of Myerson, Napel, Nohn, and Alonso-Meijide (2012) defined graph-restricted weighted voting games. Skibski, Michalak, Sakurai, and Yokoo (2015) studied problems related to the power indices in these games in terms of their computational complexity: They determined the complexity of computing them and designed a pseudo-polynomial algorithm for calculating them.

The restriction by a graph can help to better reflect real-life situations, since merely taking into account the players' weights in forming winning coalitions can be too big a simplification. A graph structure can model relations that cannot be represented simply by the weights of the players. For example, some players could play the role of mediators among other players, which might be needed to create a winning coalition. A graph structure could also express some set of skills that are needed for a winning coalition to form.

Israeli politics may serve as an example of the relevance of topological structures in the context of weighted voting games: The 36th Israeli government that was mentioned above contained quite a diverse set of political parties, spanning from the political left through the political center to the political right. It would be quite unreasonable to assume that a coalition of only political left and right could have been formed, due to the political divide between the left and right; however, the inclusion of the political center may have helped in bridging the ideological gaps between them, thus making the coalition structurally connected and therefore viable. Let us have a closer look at the following concrete example of the 2021 general elections for the Israeli parliament (the *Knesset*).

Example 1. There are 120 seats in the Knesset, and the results of the 2021 elections were as follows: Yesh Atid (17 seats); Blue and White (8); Labor (7); Yisrael Beiteinu (7); Yamina (7); New Hope (6); Meretz (6); Joint List (6); Likud (30); Ra'am (4); Shas (9); United Torah Judaism (7); and Religious Zionist (6).

As the electoral system in Israel is parliamentary, these election results translate naturally to a weighted voting game (where the weight of each party equals the number of its seats in the Knesset). The political negotiations following the election resulted in a coalition that contains the following parties: Yesh Atid (17); Blue and White (8); Labor (7); Yisrael Beiteinu (7); Yamina (7); New Hope (6); Meretz (6); and Ra'am (4).

First, as an anecdotal example of the importance of power indices, note that back then the Israeli prime minister was the head of the Yamina party, with only 7 seats in the parliament (compared to, e.g., Likud with 30 and Yesh Atid with 17).

Second, note that it is a quite diverse government, whose parties can be partially described as follows:[1] Meretz (left); Ra'am (left); Yesh Atid (center); Blue and White (center); Labor (center); Yisrael Beiteinu (right); Yamina (right); New Hope (right).

This situation can be naturally embedded in a graph: Imagine a path graph spanning from left to right. Then it is quite reasonable to assume that a coalition with, say, only Meretz (left) and Yamina (right) is not really feasible, as their stands are too far apart; however, as reality shows, a coalition containing Meretz and Yamina but also further centrist parties (e.g., Yesh Atid and Blue and White) makes this coalition feasible indeed.

Generally speaking, one can imagine a graph whose vertices represent the parties and any two vertices of which are connected by an edge if and only if these two parties are really happy with being in the same coalition, i.e., a missing edge between two parties does not mean they totally refuse to sit together in the same coalition (due to ideological gaps and/or personal disputes) but their willingness to cooperate diminishes with growing distance. Only if they are in different components of the graph (i.e., not connected), they refuse to work together. On a political left-right spectrum, one can also imagine a graph with three cliques: one for the left, one for the center, and one for the right parties; and perhaps with some edges between the left and the center clique and between the center and the right clique.

This naturally raises the question of how the influence (or power) of a given player may change, depending on which topological structure is used. Also, how might the power distribution change if one of the parties declared—as its political strategy—that it is not going to form a coalition with some other party it was previously cooperating with? Or, what happens if three parties make some kind of agreement to cooperate, even though they have never done so before (and no one would expect them to do it now without this agreement)? In summary, the feasibility of coalitions in parliamentary systems sometimes depends on the ideological compatibility or mutual agreements between the parties, which can be naturally represented as edges in a graph. Modification to this graph then affects the power of the various actors that are represented by the vertices of the graph.

Other settings in which the problems we study here are relevant include, e.g., corporate decision-making: Consider a shareholder setting containing shareholders voting within a corporation. Here, the communication graph might represent formal or informal alliances among shareholders. Controlling these relationships (e.g., through agreements or declaring conflicts) can significantly impact decision outcomes and power distribution. This can be naturally modeled as structural control of a graph-restricted weighted voting game.

Yet another example are blockchain governance settings: In certain blockchain settings, the network graph of the digital community, representing trust or delegation relationships, evolves dynamically as trust is established or revoked and sets of vertices form or dissolve alliances. For example, vertices may delegate voting power to trusted peers, create temporary coalitions for shared proposals, or restructure connections to counteract malicious actors. Here, control of the trust relations may affect the feasible coalitions and thus the corresponding power of the actors in the community.

1. This description roughly corresponds to economic issues, conflict-related topics, and religious stand; note that this is indeed a very simplistic view of these parties as, in fact, the map of Israeli politics is usually drawn with at least two dimensions (Schofield, 2007).

We model these settings by graph-restricted weighted voting games and study the computational problems of control by adding or deleting edges in the underlying communication structure. In particular, we consider two classical power indices, namely the Shapley–Shubik (SSI) and the probabilistic Penrose–Banzhaf index (PBI). For each of these power indices, we study the complexity of controlling the power of players via an alteration of the graph. In particular, we consider the problems of changing (either increasing or decreasing) or maintaining the power index of a distinguished player (defined either as the Shapley–Shubik or the probabilistic Penrose–Banzhaf power index) via controlling the communication graph either by adding some edges (in Section 3) or by removing them (in Section 4).

One may wonder where the graph structure comes from that is to be controlled by various actors—these may be the players participating in the game as well as outside actors having an incentive to modify the relations among players (i.e., the edges in the graph).[2] There is no "true" graph structure; rather, we consider this as a dynamic process: At the time being, there is some "current" graph, which is given as part of the input to our decision problems. During this process, actors who are not satisfied with the current situation and wish to, say, increase or decrease the power of a given player in the game, face the task of solving these problems, and our goal is to determine how hard this task is for them.

Our hardness results (i.e., the lower bounds for the related computational problems) are summarized in Table 1 (for control by adding edges) and Table 2 (for control by deleting edges), which also contain information about which lower bound is shown in which theorem. "ONDP" in these tables refers to the restriction where *only edges not incident to the distinguished player* (whose power index is to be changed or maintained) can either be added or deleted, and "ODP" refers to the restriction where *only edges incident to this player* can either be added or deleted (see Section 2 for formal details). The P results in these tables are trivial, as Theorem 2 (respectively, Theorem 8) in particular shows that it is impossible to decrease (respectively, increase) the power of a player by adding (respectively, deleting) edges directly incident to this player. For each of the other problems, its computational upper bound (i.e., its membership in some complexity class) will be analyzed in Section 5. In Section 6, we study the related problem of power index comparison in graph-restricted weighted voting games before we conclude in Section 7.

Our work complements previous work on how to change a distinguished player's power by manipulative actions in weighted voting games. For instance, Aziz, Bachrach, Elkind, and Paterson (2011) studied the impact of merging or splitting players—the latter a.k.a. "false-name manipulation." Rey and Rothe (2014) improved their results by raising their NP-hardness lower bounds to PP-hardness, inspired by the PP-hardness results of Faliszewski and Hemaspaandra (2009) who studied the complexity of comparing the power of a player in two weighted voting games. Zuckerman, Faliszewski, Bachrach, and Elkind (2012) studied how changing the quota in weighted voting games can have an impact on the players' power. Most closely related to our model is that of Rey and Rothe (2018), who introduced and studied control problems by adding players to or by deleting them from a weighted voting game. Their results have been improved by Kaczmarek and Rothe (2024b, 2024c, 2024a).

2. In the application scenarios listed above, these actors could be leaders of parties participating in a parliamentary election (but also foreign actors who wish to influence the election outcome in this country); they could be shareholders voting within a corporation (but also members of competing corporations who strategically aim to influence the election outcome at the general assembly of this corporation); etc.

Goal		ONDP Restriction	ODP Restriction	General Problem
Decrease	**PBI**	PP-hard (Thm. 4)	P (Thm. 2)	PP-hard (Thm. 4)
	SSI	PP-hard (Thm. 4)	P (Thm. 2)	PP-hard (Thm. 4)
Increase	**PBI**	PP-hard (Thm. 5)	NP-hard (Thm. 7)	NP-hard (Thm. 7)
	SSI	PP-hard (Thm. 6)	NP-hard (Thm. 7)	NP-hard (Thm. 7)
Maintain	**PBI**	coNP-hard (Thm. 7)	coNP-hard (Thm. 7)	coNP-hard (Thm. 7)
	SSI	coNP-hard (Thm. 7)	coNP-hard (Thm. 7)	coNP-hard (Thm. 7)

Table 1: Overview of complexity results for control by adding edges

Goal		ONDP Restriction	ODP Restriction	General Problem
Decrease	**PBI**	PP-hard (Thm. 9)	NP-hard (Thm. 12)	PP-hard (Thm. 9)
	SSI	PP-hard (Thm. 9)	NP-hard (Thm. 12)	PP-hard (Thm. 9)
Increase	**PBI**	PP-hard (Thm. 9)	P (Thm. 8)	PP-hard (Thm. 9)
	SSI	PP-hard (Thm. 9)	P (Thm. 8)	PP-hard (Thm. 9)
Maintain	**PBI**	coNP-hard (Thm. 10)	coNP-hard (Thm. 12)	coNP-hard (Thm. 10)
	SSI	coNP-hard (Thm. 11)	coNP-hard (Thm. 12)	coNP-hard (Thm. 11)

Table 2: Overview of complexity results for control by deleting edges

Unlike all these papers, we do not study the impact of controlling the player set in weighted voting games but we study the impact of controlling the underlying communication structure among the players in graph-restricted weighted voting games.

More generally speaking, we essentially consider graph-based weighted voting settings where the underlying topological structure—and not only the players' weights—can affect power. In this setting, we provide a complete taxonomy of the computational complexity of problems related to a natural way of control, namely altering the underlying communication structure so as to influence the power of the players.

2. Preliminaries

In this section, we discuss the needed notions from cooperative game theory, including weighted voting games and their generalization to graph-restricted weighted voting games. Then we provide some preliminaries on computational complexity, including several computational problems that will be used for our hardness reductions later on. We also assume the reader to be familiar with the basic concepts of graph theory such as a vertex, an undirected edge, a path, a connected component (i.e., a connected subgraph that is not part of any larger connected subgraph), an isolated vertex, a leaf or an induced subgraph. Basic knowledge about graph theory can be found, e.g., in the book by Diestel (2017).

2.1 Some Graph-Theoretical Notions

For an undirected graph $G = (N, E)$, we denote the *set of nonedges of* G by

$$\overline{E} = \Big\{\{x, y\} \mid x, y \in N \wedge x \neq y \wedge \{x, y\} \notin E\Big\},$$

i.e., $\overline{G} = (N, \overline{E})$ is the *complementary graph of* G. Further, for a subset $E' \subseteq \overline{E}$ of nonedges of G, we denote by $G_{\cup E'} = (N, E \cup E')$ the graph that results from G by adding the elements of E' as new edges to it. Similarly, for a subset $E'' \subseteq E$ of edges of G, we denote by $G_{\setminus E''} = (N, E \setminus E'')$ the graph that results from G by deleting the edges of E'' from it. For a vertex $i \in N$, let us define the *set of edges that are incident to* i by

$$E_i = \Big\{\{i, x\} \mid x \in N \setminus \{i\} \wedge \{i, x\} \in E\Big\}$$

and the *set of edges that can be added to be incident to* i by

$$\overline{E}_i = \Big\{\{i, x\} \mid x \in N \setminus \{i\} \wedge \{i, x\} \notin E\Big\}.$$

Finally, let $\mathcal{N}(i) = \{j \in N \mid \{i, j\} \in E\}$ denote the *neighborhood of vertex* i *in graph* G, and let

$$\mathcal{N}(S) = \Big(\bigcup_{i \in S} \mathcal{N}(i)\Big) \setminus S$$

be the *set of neighbors of a subset* $S \subseteq N$ *of vertices*.

2.2 Some Background on Cooperative Game Theory

We now provide the needed background of cooperative game theory. Let $N = \{1, \ldots, n\}$ denote a set of players. A *coalitional game* is a pair (N, v), where the *characteristic function* $v : 2^N \to \mathbb{R}_{\geq 0}$ assigns a nonnegative real value to each coalition (i.e., subset) of players with the assumption that $v(\emptyset) = 0$; it is said to be *simple* if it is *monotonic* (i.e., $v(A) \leq v(B)$ whenever $A \subseteq B$) and $v(C) \in \{0, 1\}$ for each $C \subseteq N$ (where $v(C) = 1$ means that coalition C *wins*, and $v(C) = 0$ means that C *loses*).

Definition 1. A *weighted voting game* $\mathcal{G} = (w_1, \ldots, w_n; q)$ is a simple coalitional game with players $N = \{1, \ldots, n\}$ that consists of a quota $q \in \mathbb{N}$ (i.e., a given threshold) and nonnegative integer weights, where w_i is the weight of player $i \in N$. For each coalition $S \subseteq N$, letting $w_S = \sum_{i \in S} w_i$, S *wins* if $w_S \geq q$, and *loses* otherwise:

$$v(S) = \begin{cases} 1 & \text{if } w_S \geq q, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we call player i *pivotal for coalition* S if

$$v(S \cup \{i\}) - v(S) = 1.$$

One of the most significant pieces of information about players is their importance in simple games, usually measured by so-called *power indices*, which take into consideration how many coalitions a player is pivotal for (i.e., how many coalitions S a player i can make win: $v(S \cup \{i\}) - v(S) = 1$ means that without player i coalition S loses, but will win as soon as i joins). We study two of the most popular power indices: the *Shapley–Shubik power index* introduced by Shapley and Shubik (1954), and the *probabilistic Penrose–Banzhaf power index* that Dubey and Shapley (1979) introduced as an alternative to the original *normalized Penrose–Banzhaf index* (Penrose, 1946; Banzhaf III, 1965). The intuition behind the probabilistic Penrose–Banzhaf index is given above, whereas the Shapley–Shubik index is based on possible orderings of the players joining a coalition where the last player added makes it winning. These two indices are defined as follows.

Definition 2. Let $n = |N|$ be the number of players in a simple game $\mathcal{G} = (N, v)$ and $i \in N$. The *probabilistic Penrose–Banzhaf power index of player i in $\mathcal{G}$* is defined by

$$\beta(\mathcal{G}, i) = \frac{\sum_{S \subseteq N \setminus \{i\}} (v(S \cup \{i\}) - v(S))}{2^{n-1}}.$$

The *Shapley–Shubik power index of player i in $\mathcal{G}$* is defined by

$$\varphi(\mathcal{G}, i) = \frac{\sum_{S \subseteq N \setminus \{i\}} |S|!(n-1-|S|)!(v(S \cup \{i\}) - v(S))}{n!}.$$

Graph-restricted weighted voting games generalize weighted voting games by assuming some communication structure among the players. Such games were first studied by Myerson (1977) and later on, e.g., by Napel et al. (2012) and Skibski et al. (2015). A formal definition is as follows.

Definition 3. A *graph-restricted weighted voting game* $(\mathcal{G}, G)$ consists of a weighted voting game $\mathcal{G} = (w_1, \ldots, w_n; q)$ with players $N = \{1, \ldots, n\}$, quota $q > \frac{1}{2} w_N$, and a graph $G = (N, E)$, where the characteristic function is defined by

$$v(S) = \begin{cases} 1 & \text{if there exists } S' \subseteq S \text{ with } w_{S'} \geq q \text{ and } G[S'] \\ & \quad \text{is a connected subgraph of } G \text{ induced by } S', \\ 0 & \text{otherwise.} \end{cases}$$

Slightly abusing notation, we will also say that a set $S \subseteq N$ of vertices is *connected* to mean that the induced subgraph $G[S]$ is connected.

Graph-restricted weighted voting games generalize weighted voting games, which are the special cases with a complete graph as their communication structure. In this situation, whether a coalition wins or loses is determined only by its total weight. However, if we limit the possibilities in communication among players, a coalition's weight alone is not enough. Before we define appropriate power indices in graph-restricted weighted voting games, let us present a few useful notions referring to coalitions in the sense of graph restrictions.

Definition 4. Let $(\mathcal{G}, G)$ be a graph-restricted weighted voting game with players N and graph $G = (N, E)$. The *set of all winning connected coalitions* is defined as

$$\mathcal{WC} = \{S \subseteq N \mid w_S \geq q \text{ and } S \text{ is connected}\}$$

and the *set of winning connected coalitions with player i* is denoted by $\mathcal{WC}_i$. For $S \subseteq N$, we denote a set of *maximal connected subsets of S in G* as S/G. The *set of all pivotal winning connected coalitions of player i* is defined as

$$\mathcal{PWC}_i = \{S \in \mathcal{WC}_i \mid ((S \setminus \{i\})/G) \cap \mathcal{WC} = \emptyset\}.$$

Skibski et al. (2015) provided the following general formulas for the probabilistic Penrose–Banzhaf power index and the Shapley–Shubik power index in graph-restricted weighted voting games.

Theorem 1. *Let $(\mathcal{G}, G)$ be a graph-restricted weighted voting game with players N. The probabilistic Penrose–Banzhaf index of player $i \in N$ in $(\mathcal{G}, G)$ satisfies*

$$\beta((\mathcal{G}, G), i) = \sum_{S \in \mathcal{PWC}_i} \frac{1}{2^{|S|+|\mathcal{N}(S)|-1}}.$$

The Shapley–Shubik index of player i in $(\mathcal{G}, G)$ satisfies

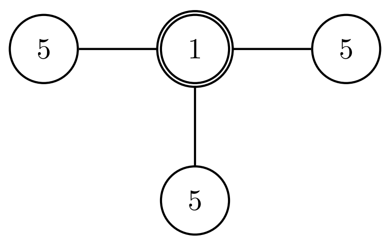
$$\varphi((\mathcal{G}, G), i) = \sum_{S \in \mathcal{PWC}_i} \frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!}.$$

To see the difference in power caused by communication structures, let us have a look at the following example.

Example 2. Let $\mathcal{G} = (1, 5, 5, 5; 9)$ be a weighted voting game and let us consider the player with weight 1. It is easy to see that this player is a dummy player, i.e., is not pivotal for any coalitions, and we have

$$\beta(\mathcal{G}, 1) = \varphi(\mathcal{G}, 1) = 0.$$

Now let us extend the game by a communication graph G (to make the presentation simpler, the vertices are labeled by the players' weights, not their names):



Now, it is impossible to form a winning coalition without player 1. The player is pivotal for the coalitions $\{2, 3\}$, $\{3, 4\}$, $\{2, 4\}$, and $\{2, 3, 4\}$. In the graph-restricted weighted voting game $(\mathcal{G}, G)$, players 1 has the following probabilistic Penrose–Banzhaf and Shapley–Shubik power index:

$$\beta((\mathcal{G}, G), 1) = \varphi((\mathcal{G}, G), 1) = 1/2.$$

Obviously, the underlying communication structure can significantly shift the power in a game.

To make the communication graphs easier to understand, as in the graph from Example 2 we will label the vertices by the players' weights instead of their names, and our distinguished player will be indicated by a double circle node in all figures throughout our paper. Moreover, when defining a set of players, we will write them in the same order as their weights are given in a game.

2.3 Some Background on Computational Complexity

We assume familiarity with the basic concepts of computational complexity theory, such as the well-known complexity classes P (*deterministic polynomial time*), NP (*nondeterministic polynomial time*), coNP (the class of problems that are complements of NP sets), and PP (*probabilistic polynomial time*, due to Gill, 1977, being a generalization of both NP and coNP, i.e., $\mathrm{P} \subseteq \mathrm{NP} \cap \mathrm{coNP} \subseteq \mathrm{NP} \cup \mathrm{coNP} \subseteq \mathrm{PP}$).

$\mathrm{NP}^{\mathrm{PP}}$ is the class of problems that can be solved by an NP oracle Turing machine accessing a PP oracle. For example, Littman, Goldsmith, and Mundhenk (1998) proved that various problems related to probabilistic planning are $\mathrm{NP}^{\mathrm{PP}}$-complete.

We will use the notions of completeness and hardness for a complexity class based on the polynomial-time many-one reducibility: A problem X (polynomial-time many-one) reduces to a problem Y (for short, $X \leq_{\mathrm{m}}^{\mathrm{p}} Y$) if there exists a polynomial-time computable function ρ such that for each input x, $x \in X$ if and only if $\rho(x) \in Y$; Y is hard for a complexity class $\mathcal{C}$ if $C \leq_{\mathrm{m}}^{\mathrm{p}} Y$ for each $C \in \mathcal{C}$; and Y is complete for $\mathcal{C}$ if Y is $\mathcal{C}$-hard and $Y \in \mathcal{C}$. Since $\mathrm{NP} \cup \mathrm{coNP} \subseteq \mathrm{PP}$, proving PP-hardness is more challenging than proving only NP- or coNP-hardness. For more background on complexity theory, we refer to some of the common text books (Garey & Johnson, 1979; Papadimitriou, 1995; Rothe, 2005).

Below we define some well-known NP-complete problems (Garey & Johnson, 1979; Karp, 1972) that will be used later on to provide reductions in our hardness proofs.

Exact-Cover-by-Three-Sets (X3C)	
Given:	A set $\mathcal{B}$, $\lvert\mathcal{B}\rvert = 3k$ for $k \in \mathbb{N}$, and a family of its three-element subsets $\mathcal{S}$.
Question:	Does there exist a subfamily $\mathcal{S}^*$ of $\mathcal{S}$ such that each element from $\mathcal{B}$ is contained in exactly one set in $\mathcal{S}^*$?

Partition	
Given:	A set $A = \{1, \ldots, n\}$ and a function $a : A \to \mathbb{N} \setminus \{0\}$, $i \mapsto a_i$, such that $\sum_{i=1}^{n} a_i$ is even.
Question:	Does there exist a partition of A into two subsets of equal weight, that is, does there exist a subset $A' \subseteq A$ such that $\sum_{i \in A'} a_i = \sum_{i \in A \setminus A'} a_i$?

SubsetSum	
Given:	A set $A = \{1, \ldots, n\}$, a function $a : A \to \mathbb{N} \setminus \{0\}$, $i \mapsto a_i$, and an integer q.
Question:	Does there exist a subset $A' \subseteq A$ such that $\sum_{i \in A'} a_i = q$?

As is common, we will write an instance of Partition simply as a sequence $(a_1, \ldots, a_n)$ of positive integers such that $\sum_{i=1}^{n} a_i$ is even, and an instance of SubsetSum similarly as $((a_1, \ldots, a_n), q)$.

Valiant (1979) introduced #P as the class of functions that give the number of solutions of NP problems; therefore, #P is also known as the "counting version of NP": For every NP problem X, $\#X$ denotes the function that maps each instance of X to the number of its solutions. For example, #SAT maps each boolean formula to the number of its satisfying assignments, and #SubsetSum maps each instance $((a_1, \ldots, a_n), q)$ of SubsetSum to the number of subsets $A' \subseteq \{1, \ldots, n\}$ satisfying that $\sum_{i \in A'} a_i = q$. Clearly, any NP problem

X is trivially subsumed by its counting version $\#X$ because if we can efficiently count the number of solutions of an instance x, we can immediately tell whether x is a yes- or a no-instance of X: $x \in X$ exactly if the number of solutions of x is positive.

Deng and Papadimitriou (1994) showed that computing the Shapley–Shubik index of a player in a given weighted voting game is complete for #P via *functional* polynomial-time many-one reductions defined formally as follows.

Definition 5. Let f and g be two functions mapping from Σ^* to $\mathbb{N}$. We say f *(polynomial-time many-one) reduces to* g if there exist two polynomial-time computable functions, $\psi : \mathbb{N} \to \mathbb{N}$ and $\rho : \Sigma^* \to \Sigma^*$, such that for each $x \in \Sigma^*$, $f(x) = \psi(g(\rho(x)))$.

Next, let us define the *parsimonious reduction*.

Definition 6. Let f and g be two functions mapping from Σ^* to $\mathbb{N}$. f *parsimoniously reduces to* g if there exists a polynomial-time computable function ρ such that for each $x \in \Sigma^*$, $f(x) = g(\rho(x))$; and g is *(parsimoniously) hard for* #P if every $f \in \#\mathrm{P}$ (parsimoniously) reduces to g, and g is *(parsimoniously) complete for* #P if $g \in \#\mathrm{P}$ and g is (parsimoniously) hard for #P.

Prasad and Kelly (1990) proved that computing the probabilistic Penrose–Banzhaf index is parsimoniously complete for #P. Skibski et al. (2015) observed that computing the Shapley–Shubik index and the probabilistic Penrose–Banzhaf index in *graph-restricted* weighted voting games is #P-complete as well.[3] For all these functional problems, there exist pseudo-polynomial-time algorithms due to Matsui and Matsui (2001) and Skibski et al. (2015).

#P and PP, even though the former is a class of functions and the latter a class of decision problems, are closely related by the well-known result that $\mathrm{P}^{\mathrm{PP}} = \mathrm{P}^{\#\mathrm{P}}$.[4] We also

3. One may wonder whether the hardness of the problems formalizing control by adding or deleting edges, which we will show in Sections 3 and 4, might simply be an artifact of the #P-hardness of computing these power indices. However, hardness for *functional* problems (a.k.a. *search* problems, such as computing power indices) is fundamentally different than hardness for *decision* problems, which we study. Let us quote from the—very recommendable—blog of Lance Fortnow on computational complexity (`https://blog.computationalcomplexity.org/2019/01/search-versus-decision.html`). About search versus decision, he writes:

> *"Decision is always at least as easy as search: If you have a solution you know there is one. What about the other direction? We can't actually prove search is hard without separating* P *and* NP*, but we have our conjectures.*
> *Sometimes both are easy. We can easily find the maximum weighted matching.*
> *Sometimes decision is easy and search is supposedly hard: Composite Numbers. The search version is factoring.*
> *Sometimes decision is trivial (i.e. they always exist) and search is still hard. Nash Equilibria. Ramsey Graphs."*

This means that our hardness results, even if they may be caused in part by the hardness of computing power indices, are not simply implied by them but have to be proven formally.

4. Quoting his blog on computational complexity again (`https://blog.computationalcomplexity.org/2002/09/complexity-class-of-week-pp.html`), Lance Fortnow states that *"I have never found the paper that first proves this equivalence,"* and neither did we. However, a proof can be found, e.g., in the work of Balcázar, Book, and Schöning (1986, Theorem A3) who denote $\mathrm{P}^{\#\mathrm{P}}$ by D#P. In a nutshell, the inclusion $\mathrm{P}^{\mathrm{PP}} \subseteq \mathrm{P}^{\#\mathrm{P}}$ is straightforward, and for proving the inclusion $\mathrm{P}^{\#\mathrm{P}} \subseteq \mathrm{P}^{\mathrm{PP}}$, use the PP oracle to perform a binary search.

use in our hardness proofs the following problems that were shown to be PP-complete by Rey and Rothe (2014):

Compare-#SubsetSum-RR	
Given:	A sequence $(a_1, \ldots, a_n)$ of positive integers, where $\alpha = \sum_{i=1}^{n} a_i$.
Question:	Is it true that $$\#\textsc{SubsetSum}((a_1, \ldots, a_n), \frac{\alpha}{2} - 2) > \#\textsc{SubsetSum}((a_1, \ldots, a_n), \frac{\alpha}{2} - 1)?$$

The problem Compare-#SubsetSum-ЯЯ is defined analogously, except that the inequality is inverted in the question:

Compare-#SubsetSum-ЯЯ	
Given:	A sequence $(a_1, \ldots, a_n)$ of positive integers, where $\alpha = \sum_{i=1}^{n} a_i$.
Question:	Is it true that $$\#\textsc{SubsetSum}((a_1, \ldots, a_n), \frac{\alpha}{2} - 2) < \#\textsc{SubsetSum}((a_1, \ldots, a_n), \frac{\alpha}{2} - 1)?$$

Finally, we will use the following lemma in some of our proofs. The result regarding X3C instances in it is due to Faliszewski and Hemaspaandra (2009) and Rey and Rothe (2014) observed that it can also be transferred to instances of SubsetSum since #X3C parsimoniously reduces to #SubsetSum.

Lemma 1. *Every* X3C *instance* $(B', \mathcal{S}')$ *can be transformed into an* X3C *instance* $(B, \mathcal{S})$, *where* $|B| = 3k$ *and* $|\mathcal{S}| = n$, *such that* $\frac{k}{n} = \frac{2}{3}$ *without changing the number of solutions. Consequently, we can assume that the size of each solution in a* SubsetSum *instance is* $\frac{2n}{3}$, *that is, each subsequence summing up to the given quota contains the same number of elements. Furthermore, it can be assumed that each solution of* SubsetSum *counted in both* #SubsetSum *answers in the question field of either* Compare-#SubsetSum-RR *or* Compare-#SubsetSum-ЯЯ *has the same number of elements, namely* $\frac{n+2}{3}$.

3. Control by Adding Edges

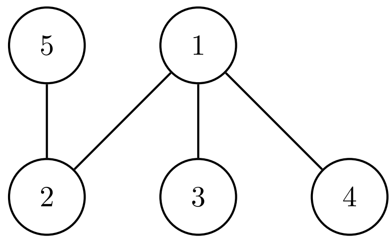
In this section, we study the effects of control by adding edges to a communication structure in a graph-restricted weighted voting game. For such control attacks, we first determine upper and lower bounds for the probabilistic Penrose–Banzhaf and the Shapley–Shubik power index of a given player. Then we define the decision problems capturing control by adding edges, discuss some restrictive scenarios versus the general case for them, and finally determine computational lower bounds for these problems.

3.1 Change of Power by Adding Edges to a Communication Structure

We begin our technical treatment by considering the impact of adding new edges to the communication structure of a given graph-restricted weighted voting game. By this structural change to the game, we allow some players to communicate with each other for whom this was impossible before. How does this influence the probabilistic Penrose–Banzhaf and the Shapley–Shubik power index of a given player?

Before we present our results, let us look at this type of control in the game of Example 3 to better understand the possible changes of the power indices by adding edges.

Example 3. Let $(\mathcal{G}, G)$ be a graph-restricted weighted voting game with $\mathcal{G} = (1, 2, 3, 4, 5; 8)$ and the following communication structure $G = (N, E)$:



All winning connected coalitions in this game are

$$\mathcal{WC} = \{\{1,3,4\}, \{1,2,5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, N\},$$

and we have the following pivotal winning connected coalitions for the single players:

$$\begin{aligned}
&\mathcal{PWC}_1 = \mathcal{WC}, \\
&\mathcal{PWC}_2 = \{\{1,2,5\}, \{1,2,3,5\}, \{1,2,4,5\}\}, \\
&\mathcal{PWC}_3 = \{\{1,3,4\}, \{1,2,3,4\}\}, \\
&\mathcal{PWC}_4 = \{\{1,3,4\}, \{1,2,3,4\}\}, \text{ and} \\
&\mathcal{PWC}_5 = \{\{1,2,5\}, \{1,2,3,5\}, \{1,2,4,5\}\}.
\end{aligned}$$

For all $S \in \mathcal{WC} \setminus \{\{1,3,4\}\}$, we have

$$|S| + |\mathcal{N}(S)| = 5$$

and for the remaining coalition, we have

$$|\{1,3,4\}| + |\mathcal{N}(\{1,3,4\})| = 4.$$

Let us now have a closer look at the indices for, say, players 2 and 3: Using Theorem 1, their probabilistic Penrose–Banzhaf indices are

$$\begin{aligned}
&\beta((\mathcal{G}, G), 2) = 3 \cdot \frac{1}{2^{5-1}} = \frac{3}{16} \quad \text{and} \\
&\beta((\mathcal{G}, G), 3) = \frac{1}{2^{4-1}} + \frac{1}{2^{5-1}} = \frac{2}{16} + \frac{1}{16} = \frac{3}{16},
\end{aligned}$$

while their Shapley–Shubik indices are

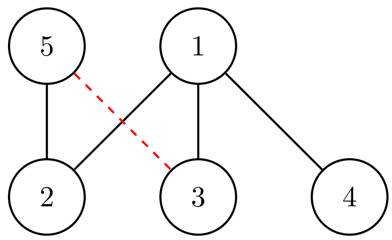
$$\begin{aligned}
&\varphi((\mathcal{G}, G), 2) = \frac{(3-1)!2!}{5!} + 2 \cdot \frac{(4-1)!1!}{5!} = \frac{2!2! + 2 \cdot 3!1!}{5!} = \frac{16}{120} = \frac{2}{15} \quad \text{and} \\
&\varphi((\mathcal{G}, G), 3) = \frac{(3-1)!1!}{4!} + \frac{(4-1)!1!}{5!} = \frac{5 \cdot 2!1! + 3!1!}{5!} = \frac{16}{120} = \frac{2}{15}.
\end{aligned}$$

Altogether, we obtain the following probabilistic Penrose–Banzhaf and Shapley–Shubik power indices for all the players:

$$\beta((\mathcal{G},G),1) = \frac{7}{16} \quad \text{and} \quad \beta((\mathcal{G},G),i) = \frac{3}{16} \quad \text{for } i \in \{2,\ldots,5\};$$
$$\varphi((\mathcal{G},G),1) = \frac{7}{15} \quad \text{and} \quad \varphi((\mathcal{G},G),i) = \frac{2}{15} \quad \text{for } i \in \{2,\ldots,5\}.$$

Thus, for both indices, 1 (even though having the smallest weight) is the most powerful player and the others have the same power, even though their weights are different; also, player 2 has a higher degree than each of the players 3, 4, and 5 in G.

Adding the new edge $e = \{3,5\}$ between players 3 and 5:



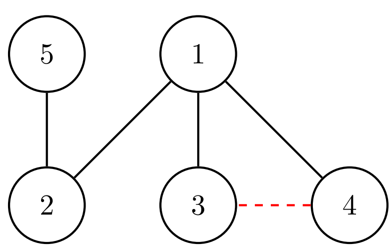
we obtain the new game, $(\mathcal{G}, G_{\cup\{e\}})$, in which the connected winning coalitions are

$$\mathcal{WC} \cup \{\{3,5\},\{1,3,5\},\{2,3,5\}\},$$

By this change, 3 and 5 have become the most powerful players for both power indices, whereas the power of the previously strongest player 1 has decreased:

$$\beta((\mathcal{G},G_{\cup\{e\}}),1) = \frac{1}{4}, \quad \beta((\mathcal{G},G_{\cup\{e\}}),2) = \beta((\mathcal{G},G_{\cup\{e\}}),4) = \frac{1}{8}, \text{ and}$$
$$\beta((\mathcal{G},G_{\cup\{e\}}),3) = \beta((\mathcal{G},G_{\cup\{e\}}),5) = \frac{1}{2};$$
$$\varphi((\mathcal{G},G_{\cup\{e\}}),1) = \frac{1}{6}, \quad \varphi((\mathcal{G},G_{\cup\{e\}}),2) = \varphi((\mathcal{G},G_{\cup\{e\}}),4) = \frac{1}{12}, \text{ and}$$
$$\varphi((\mathcal{G},G_{\cup\{e\}}),3) = \varphi((\mathcal{G},G_{\cup\{e\}}),5) = \frac{1}{3}.$$

On the other hand, it is not hard to see that adding the edge $\{3,4\}$ between players 3 and 4 to G instead:



does not change the set of all winning connected coalitions and both the Penrose–Banzhaf and the Shapley–Shubik power index of the players remain unchanged.

We investigate to what extent the probabilistic Penrose–Banzhaf and the Shapley–Shubik power index of a player i can change by connecting *this* player i with other players in a communication graph. Theorem 2 presents upper and lower bounds for how much i's power can change in this case, depending on the number of edges added to i. In particular, i can become pivotal (but does not have to) for more coalitions than before, yet it is impossible for i to stop being pivotal for any coalition i used to be pivotal for before the control action. Therefore, i's power cannot decrease by providing i with more connections.

Theorem 2. *Let $(\mathcal{G}, G)$ with $G = (N, E)$ be a graph-restricted weighted voting game and let $E^i \subseteq \overline{E}_i$, $|E^i| = m$, be a set of nonedges between a given player i and other players in G. For a power index $\gamma \in \{\beta, \varphi\}$ and $\mathcal{N}(i)$ being the set of i's neighbors in G, let*

$$\mathrm{diff}_\gamma(\mathcal{G}, G, G_{\cup E^i}, i) = \gamma((\mathcal{G}, G), i) - \gamma((\mathcal{G}, G_{\cup E^i}), i).$$

By adding the elements of E^i as new edges to G, thus creating the new game $(\mathcal{G}, G_{\cup E^i})$, the old and the new Penrose–Banzhaf index and the old and the new Shapley–Shubik index of player i can differ as follows:

$$-1 + 2^{-(|\mathcal{N}(i)|+m)} \leq \mathrm{diff}_\beta(\mathcal{G}, G, G_{\cup E^i}, i) \leq 0;$$
$$-1 + \frac{1}{|\mathcal{N}(i)| + m + 1} \leq \mathrm{diff}_\varphi(\mathcal{G}, G, G_{\cup E^i}, i) \leq 0.$$

Proof. The upper bounds are implied by the property of *"global monotonicity with respect to added communication possibilities"* that Napel et al. (2012) have shown to hold for both these power indices.

To analyze the lower bounds, let us consider the formulas from Theorem 1. Specifically, to prove the lower bound for the probabilistic Penrose–Banzhaf index, if $\mathcal{PWC}_i = \emptyset$ (i.e., in particular, if $v(\{i\}) = 0$), it is impossible for player i to be pivotal for all coalitions in the new game $(\mathcal{G}, G_{\cup E^i})$, and therefore, for $\mathcal{PWC}'_i$ being the set of all pivotal winning connected coalitions of player i and $\mathcal{N}'(S)$ being the neighborhood of S after adding the new edges from E^i, we have

$$\mathrm{diff}_\beta(\mathcal{G}, G, G_{\cup E^i}, i) \geq -\sum_{S \in \mathcal{PWC}'_i} \frac{1}{2^{|S|+|\mathcal{N}'(S)|-1}} \geq -1 + 2^{-(|\mathcal{N}(i)|+m)}.$$

Note that in the case of $v(\{i\}) = 1$, i is pivotal for all possible coalitions because any graph-restricted weighted voting game is superadditive, so its indices are equal to 1 and they cannot change after the addition of new edges; hence, the coalition S with $|S| = 1$—having $|\mathcal{N}(i)| + m$ neighbors in the new graph—is not an element of $\mathcal{PWC}'_i$ where the difference between the power indices is the largest.

Similarly, to prove the lower bound for Shapley–Shubik index, we now have

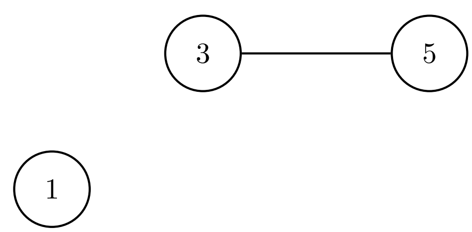
$$\begin{aligned}\mathrm{diff}_\varphi(\mathcal{G}, G, G_{\cup E^i}, i) &\geq -\sum_{S \in \mathcal{PWC}'_i} \frac{(|S|-1)!|\mathcal{N}'(S)|!}{(|S|+|\mathcal{N}'(S)|)!} \\ &\geq -1 + \frac{0!(|\mathcal{N}(i)|+m)!}{(|\mathcal{N}(i)|+m+1)!} = -1 + \frac{1}{|\mathcal{N}(i)|+m+1}.\end{aligned}$$

This completes the proof. □

However, if we add new edges in another part of the communication structure, connecting players other than the distinguished player i with each other, all situations are possible to happen: i's power can increase, decrease, or be maintained by such a control action, as can be seen in Example 3.

Let us now have a look at two small games showing that the intervals from Theorem 2 are tight.

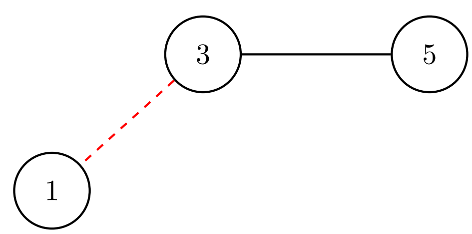
Example 4. Let $(\mathcal{G}, G)$ be a graph-restricted weighted voting game with player set $\{a, b, c\}$, $\mathcal{G} = (3, 5, 1; 6)$, and the following graph G (we will again label the vertices by the corresponding players' weights):



Let us analyze player a with weight 3: $\mathcal{PWC}_a = \{\{a, b\}\}$, so a's power indices are

$$\beta((\mathcal{G}, G), a) = \varphi((\mathcal{G}, G), a) = \frac{1}{2}.$$

If we add edge $\{a, c\}$:



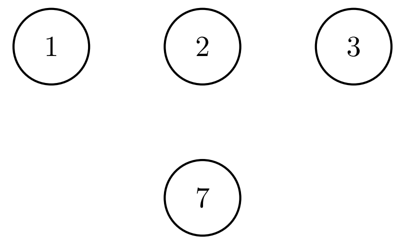
then a is pivotal for the same coalitions in the new game, i.e., for $\{b\}$ and $\{b, c\}$, so

$$\begin{aligned} \beta((\mathcal{G}, G_{\cup\{\{a,c\}\}}), a) &= \frac{2}{2^2} = \frac{1}{2} = \beta((\mathcal{G}, G), a) \quad \text{and} \\ \varphi((\mathcal{G}, G_{\cup\{\{a,c\}\}}), a) &= \frac{1!1! + 2!0!}{3!} = \frac{1+2}{6} = \frac{1}{2} = \varphi((\mathcal{G}, G), a), \end{aligned}$$

and therefore, the upper bounds for both power indices are tight:

$$\text{diff}_\beta(\mathcal{G}, G, G_{\cup\{\{a,c\}\}}, a) = \text{diff}_\varphi(\mathcal{G}, G, G_{\cup\{\{a,c\}\}}, a) = 0.$$

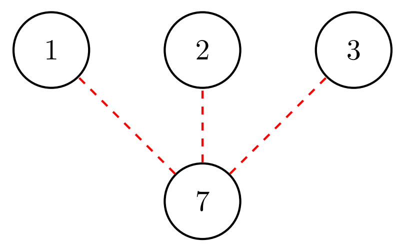
To see that also the lower bounds for both power indices are tight, let $(\mathcal{H}, H)$ be a graph-restricted weighted voting game with player set $\{a, b, c, d\}$, $\mathcal{H} = (7, 1, 2, 3; 8)$, and the following graph H (again labeling the vertices by the corresponding players' weights):



Note that player a with weight 7 is not pivotal for any coalition in $(\mathcal{H}, H)$, so

$$\beta((\mathcal{H}, H), a) = \varphi((\mathcal{H}, H), a) = 0.$$

Now, let $m = 3$ be the number of edges to be added, and let $E^a = \overline{E}_a = \{\{a,b\},\{a,c\},\{a,d\}\}$, so $H_{\cup E^a}$ looks as follows:



Since a becomes pivotal for $\{b\}, \{c\}, \{d\}, \{b,c\}, \{b,d\}, \{c,d\}$, and $\{b,c,d\}$, we have

$$\begin{aligned}\beta((\mathcal{H}, H_{\cup E^a}), a) &= \frac{7}{2^3} = \frac{7}{8} \quad \text{and}\\ \varphi((\mathcal{H}, H_{\cup E^a}), a) &= 3\frac{1!2!}{4!} + 3\frac{2!1!}{4!} + \frac{3!0!}{4!} = \frac{3 \cdot 3!}{4!} = \frac{3}{4},\end{aligned}$$

and therefore, the lower bounds from Theorem 2 are tight for both power indices as well:

$$\begin{aligned}\text{diff}_{\beta}(\mathcal{H}, H, H_{\cup E^a}, a) &= -\frac{7}{8} = -1 + 2^{-3} \quad \text{and}\\ \text{diff}_{\varphi}(\mathcal{H}, H, H_{\cup E^a}, a) &= -\frac{3}{4} = -1 + \frac{1}{3+1}.\end{aligned}$$

3.2 Defining the Decision Problems of Control by Adding Edges

We now focus on the question of how hard it is to decide whether adding edges to a communication structure underlying a graph-restricted weighted voting game can increase, decrease, or maintain a distinguished player's power in the game. For a given power index PI and for the goal of *increasing* the power, this problem is defined as follows:

Control-by-Adding-Edges-to-Increase-PI

Given: A graph-restricted weighted voting game $(\mathcal{G}, G)$ with players $N = \{1, \ldots, n\}$, a communication structure $G = (N, E)$, $|E| < \binom{n}{2}$, a distinguished player $p \in N$, and a nonnegative integer k.

Question: Is it sufficient to add k or fewer edges $E' \subseteq \overline{E}$, $E' \neq \emptyset$, to G to obtain a new graph-restricted weighted voting game, $(\mathcal{G}, G_{\cup E'})$, for which it holds that

$$\text{PI}((\mathcal{G}, G_{\cup E'}), p) > \text{PI}((\mathcal{G}, G), p)?$$

Analogously, we define the decision problems for *decreasing* and *maintaining*[5] a distinguished player's power in a given game by replacing ">" in the question by, respectively, "<" and "=" and denote these problems by Control-by-Adding-Edges-to-Decrease-PI and Control-by-Adding-Edges-to-Maintain-PI.

As we have seen in Section 3.1, it does make a difference whether the added new edges are incident to the distinguished player p or not. For example, it could be possible to control only connections of p because we may need p's cooperation. On the other hand, in another application scenario, we may not want to attract too much attention to the distinguished player, so may want to add new edges only among some other players—especially if the control action might be considered illegal or might result in severe consequences if caught (like removing the player from the game entirely), or in the case when the distinguished player refuses any change of power (for example, if we want some—political or other—party to be less powerful, its members may not be willing to consent). Therefore, in addition to considering the above-defined general control scenario in communication graphs, we are going to also analyze two restrictive scenarios where we are allowed to add edges that are either incident *only to the distinguished player* p or *only not incident to* p. Formally, for the goal of increasing the distinguished player p's power in a given game, we define the former problem by requiring $E' \subseteq \overline{E}_p$ in the above problem definition and denote it by Control-by-Adding-Edges-ODP-to-Increase-PI, where ODP stands for *"only to distinguished player."* Similarly, in the latter case we require $E' \subseteq \overline{E} \setminus \overline{E}_p$ in the above problem definition and denote this problem by Control-by-Adding-Edges-ONDP-to-Increase-PI, where ONDP stands for *"only not to distinguished player."*

The analogous problems for *decreasing* and *maintaining* the distinguished player's power in a given game by adding edges incident only to p or only not incident to p are again defined by changing the relation sign in the question accordingly. Further, in such problem names we will replace PI by PBI to refer to the probabilistic Penrose–Banzhaf index and by SSI to refer to the Shapley–Shubik index.

3.3 Discussion of the Restrictive Scenarios versus the General Case

In the case of adding edges, if we can increase a distinguished player's probabilistic Penrose–Banzhaf or Shapley–Shubik power index by adding k edges to a communication graph which are not incident to the distinguished player, we can also increase them by adding no more than k edges directly to this player instead.

Lemma 2. *Let $(\mathcal{G}, G)$ be a graph-restricted weighted voting game with player set N and graph $G = (N, E)$. Let $i \in N$ be the distinguished player. If i's probabilistic Penrose–Banzhaf or Shapley–Shubik power index increases after the addition of k edges from $\overline{E} \setminus \overline{E}_i$ to G, then it will also increase after adding some $\ell \leq k$ edges from $\overline{E}_i$ instead.*

5. The control goal of maintaining the power index of a player may need some more motivating explanation. Note that by requiring $E' \neq \emptyset$ in the problem definition, we assume that there is a change in the graph—at least one edge has to be added (or, for the problems considered in Section 4, has to be deleted). Imagine a situation where it is known that the change in the graph has to happen (maybe with the goal of changing some other player's power), but we may have a say in *how* that happens, i.e., *which* edges are to be added or deleted. Now, we may want "our" player to maintain his or her power.

Proof. In this proof, "index" refers to either the probabilistic Penrose–Banzhaf or the Shapley–Shubik power index. Assume that i's index increases after adding k edges from $\overline{E} \setminus \overline{E}_i$ to G. Considering the formulas from Definition 2, an index will increase when i becomes pivotal for some coalitions for which it was not pivotal before the control action—the summand for each coalition is the same except the part making it positive or equal to zero; hence, the only possible change is to make a summand to be greater than 0. So, if the index increased, the players incident with the added edges created coalitions for which i became pivotal. Let $S \subseteq N \setminus \{i\}$ be such a coalition: i became pivotal for S after adding $k' \leq k$ edges to $G[S]$, i.e., $S \cup \{i\}$ was losing before the control action, and i with the players incident with some of the added edges are contained in the winning connected component of $S \cup \{i\}$ after the control action. First note that adding an edge between two players connected with i through a path does not change the connected components of $S \cup \{i\}$ and therefore, it does not change the set into a winning coalition and we do not have to consider this case. So, for each edge added to $G[S]$, either

- exactly one of the vertices incident with the added edge had to be connected with i through a path in $G[S \cup \{i\}]$ before the control action (otherwise, i would not be a part of the component that became winning), so if we connect the other player directly to i, i would still become pivotal for the coalition—the same component would be also connected and winning although the communication graph would look differently, and without i, the player will not be a part of this connected component, as he or she is in the first scenario, i.e., S will still be losing, or
- none of the players connected with each other had been connected with i through paths in $G[S \cup \{i\}]$, so more edges have to be added to make $S \cup \{i\}$ a winning coalition—one of the added edges has to be incident to some player j which had been connected through a path with i, so we get the situation from the previous case, we exchange the edge with one incident to i, and we repeat the analysis for the resulting coalition.

Summing up, if we add some $\ell \leq k$ edges only incident to i instead of k edges from $\overline{E} \setminus \overline{E}_i$ to G, we can make i pivotal for at least one coalition for which it was not pivotal before, and at the same time, by Theorem 2, i will be pivotal for the same coalitions as before. Therefore, to increase an index, it is enough to consider adding edges directly to the distinguished player. □

That is, if we want to increase the distinguished player's probabilistic Penrose–Banzhaf or Shapley–Shubik power indices by adding edges, it is enough to consider adding only edges incident to the distinguished player. As a consequence, we obtain the following result.

Theorem 3. *For* PI $\in$ {PBI, SSI}, Control-by-Adding-Edges-to-Increase-PI *and* Control-by-Adding-Edges-ODP-to-Increase-PI *are identical problems.*

Proof. We have to show that for any instance $((\mathcal{G}, G), p, k)$ (with $G = (N, E)$) to either problem, it holds that

$$((\mathcal{G}, G), p, k) \in \text{Control-by-Adding-Edges-to-Increase-PI}$$
$$\iff ((\mathcal{G}, G), p, k) \in \text{Control-by-Adding-Edges-ODP-to-Increase-PI}.$$

The implication from right to left is obvious: If there exists a set of at most k edges that we can add to player p such that p's power index increases due to the addition, this immediately also gives a solution to the general problem Control-by-Adding-Edges-to-Increase-PI.

From left to right, let $((\mathcal{G}, G), p, k) \in$ Control-by-Adding-Edges-to-Increase-PI. Suppose that there exists a set of edges $F_1 \cup F_2$ such that $F_1 \subseteq \overline{E}_p$, $F_2 \subseteq \overline{E} \setminus \overline{E}_p$, $|F_1 \cup F_2| \leq k$, and p's index in $(\mathcal{G}, G_{\cup F_1 \cup F_2})$ is greater than p's index in $(\mathcal{G}, G)$.[6] For a contradiction, let us assume there exists no solution for the instance containing edges only from $\overline{E}_p$, i.e., $((\mathcal{G}, G), p, k) \notin$ Control-by-Adding-Edges-ODP-to-Increase-PI. By Lemma 2, if there exists no solution consisting only of edges incident to p, there exists no solution consisting of edges only from $\overline{E} \setminus \overline{E}_p$ either, i.e.,

$$F_1 \neq \emptyset \ \wedge \ F_2 \neq \emptyset.$$

Let us consider games of the form

$$(\mathcal{G}, G_{\cup F_1'})$$

for any $F_1' \subseteq F_1$. In all these games, the index of player p cannot be greater than p's index in the initial game (otherwise, we would have a contradiction with the assumption that there is no solution consisting only of edges incident to p), and by Theorem 2, the index of p remains unchanged in all the cases. Therefore, the index of p in $(\mathcal{G}, G_{\cup F_1})$ increases after adding the edges F_2 with $|F_2| = k' < k$. But then, again by Lemma 2, there exists a set of edges F_3, $|F_3| \leq k'$, $F_3 \subseteq \overline{E}_p \setminus F_1$, such that after adding the set F_3 instead of F_2, the index of p also increases. Let us consider set $F_1 \cup F_3$. It contains only edges from $\overline{E}_p$, it has at most k elements, and after adding them to G, player p's index increases, which means that $((\mathcal{G}, G), p, k)$ has a solution and so is in Control-by-Adding-Edges-ODP-to-Increase-PI, so we get a contradiction. Therefore, the existence of a solution in the general case implies the existence of a solution for the considered restrictive scenario. □

Interestingly, an analogue of Theorem 3 does not exist in the other cases. Below we present counterexamples for such situations.

Example 5. Let us consider adding edges to a communication graph. Again, we refer by "index" to either the probabilistic Penrose–Banzhaf or the Shapley–Shubik power index in this example.

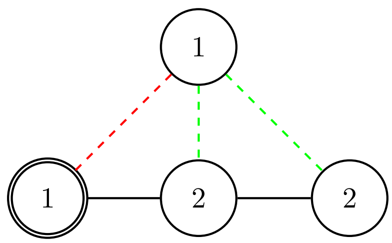
1. It is not true in general that if we can increase the index of the distinguished player p by adding k edges incident to p, then we can also increase p's index by adding up to k edges only among players other than p. By Theorem 3, Control-by-Adding-Edges-to-Increase-PI and Control-by-Adding-Edges-ONDP-to-Increase-PI thus are not identical problems for $\mathrm{PI} \in \{\mathrm{PBI}, \mathrm{SSI}\}$.

   To see this, consider a game where p is isolated from the remaining players, which together form a connected component. Let the quota be equal to the total sum of all players' weights, i.e., only the grand coalition—the coalition containing all players—wins. If we connect p with any other player, p's index increases from 0 to some positive value, but adding edges not incident to p will not change any player's index in the new game (since there will still not exist a winning coalition in the game).

6. Again, "index" refers to either the probabilistic Penrose–Banzhaf or the Shapley–Shubik power index.

2. It is not true in general that if we can maintain the index of the distinguished player p by adding $k \geq 1$ edges incident to p, then we can also maintain p's index by adding up to k edges (but at least one) only among players other than p.

   To see this, consider the game $(\mathcal{G}, G)$ with $\mathcal{G} = (1, 1, 2, 2; 5)$, distinguished player 1, and graph G, where players 1, 3, and 4 form a path in which player 1 is a leaf and where player 2 is an isolated vertex:

   Player 1 is pivotal for the coalitions $\{3, 4\}$ and $\{2, 3, 4\}$, so 1's indices are

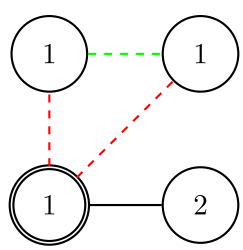
   $$\beta((\mathcal{G}, G), 1) = \frac{1}{4} \quad \text{and} \quad \varphi((\mathcal{G}, G), 1) = \frac{1}{3}.$$

   If we add edge $\{1, 2\}$ (the red dashed one), player 1 will stay pivotal for the same coalitions and 1's indices will not change. But if we add edge $\{2, 3\}$ or edge $\{2, 4\}$ or even both of them (i.e., the green dashed edges), 1 will stop being pivotal for $\{2, 3, 4\}$ and will not become pivotal for any other coalition, so 1's indices will decrease to

   $$\beta((\mathcal{G}, G_{\cup\{\{2,3\}\}}), 1) = \frac{1}{8} \quad \text{and} \quad \varphi((\mathcal{G}, G_{\cup\{\{2,3\}\}}), 1) = \frac{1}{12}.$$

3. It is not true in general that if we can maintain the index of the distinguished player p by adding $k \geq 1$ edges not incident to p, then we can also maintain p's index by only adding up to k (but at least one) edges incident to p.

   To see this, consider the game $(\mathcal{G}, G)$ with $\mathcal{G} = (1, 1, 1, 2; 4)$, distinguished player 1, and graph G, where players 1 and 4 are connected, and players 2 and 3 are isolated vertices:

   There are no winning coalitions in this game, so both indices of player 1 are equal to 0. If we add edge $\{2, 3\}$ (the green dashed one) the indices will still be equal to 0. But if we connect player 1 with any of the isolated players, 1 will become pivotal for at least one coalition, i.e., 1's indices will increase to some positive value in each case.

It follows from the last two counterexamples above that Control-by-Adding-Edges-to-Maintain-PI, Control-by-Adding-Edges-ODP-to-Maintain-PI, and Control-by-Adding-Edges-ONDP-to-Maintain-PI are pairwise distinct problems.

3.4 Complexity of Control by Adding Edges

We now turn to studying the computational lower bounds of control by adding edges. Starting with the goal of decreasing a given player's power, we establish PP-hardness for control by adding edges with respect to both power indices, using the same reduction for both problems.

Theorem 4. *For* PI $\in$ {PBI, SSI}*,* Control-by-Adding-Edges-ONDP-to-Decrease-PI *and* Control-by-Adding-Edges-to-Decrease-PI *are* PP*-hard.*

Proof. We prove that all four problems are PP-hard by reducing from the PP-hard problem Compare-#SubsetSum-RR; in fact, this will be essentially one and the same reduction for each of them. Let $A = (a_1, \ldots, a_n)$ with $\alpha = \sum_{i=1}^{n} a_i$ be a given instance of Compare-#SubsetSum-RR. Define

$$\xi_1 = \#\textsc{SubsetSum}\left(A, \frac{\alpha}{2} - 1\right) \quad \text{and}$$
$$\xi_2 = \#\textsc{SubsetSum}\left(A, \frac{\alpha}{2} - 2\right)$$

and construct an instance of our control problem (for either the Penrose–Banzhaf or the Shapley–Shubik index) as follows. Choose r, q, and ℓ such that

$$\begin{aligned} 10^r &> 2 + 10\sum_{j=1}^{n} a_j, \\ q &= 2\sum_{j=1}^{n} a_j \cdot 10^r + 1, \text{ and} \\ 10^\ell &> 2q + 2 + 2\sum_{j=1}^{n} (a_j \cdot 10^r + 10 \cdot a_j) + 1. \end{aligned}$$

Consider the following graph-restricted weighted voting game $(\mathcal{G}, G)$:

$$\begin{aligned} \mathcal{G} = \Big(1, 10^\ell, 1, 1,\ & q - \left(\frac{\alpha}{2} - 2\right) \cdot 10^r - 1, a_1 \cdot 10^r, \ldots, a_n \cdot 10^r, \\ & q - \left(\frac{\alpha}{2} - 1\right) \cdot 10 - 2, 10 \cdot a_1, \ldots, 10 \cdot a_n; 10^\ell + q + 1\Big) \end{aligned}$$

with $2n+6$ players, quota $10^\ell+q+1$, distinguished player 1, and the following communication graph $G = (N, E)$: All players but player 4 (with weight 1) form a complete subgraph, whereas player 4 is an isolated vertex. Thus we can only add edges between player 4 and the large complete subgraph. Finally, let the addition limit be $k = 1$.

Let us analyze for which coalitions of $N \setminus \{1\}$ the distinguished player 1 is pivotal in $(\mathcal{G}, G)$. Player 1 is pivotal for coalitions with weight $10^\ell + q$ which do not contain the isolated player 4, and for coalitions created from them by adding player 4 (although their weight is $10^\ell + q + 1$, the players from their larger connected part still have a total weight of only $10^\ell + q$ and thus lose, but adding 1 makes them win). Player 1 cannot be pivotal for any coalition with weight smaller than $10^\ell + q$ because the coalition loses with and without player 1. Any coalition of weight $10^\ell + q$ containing player 4 will still lose when 1 joins because the players from its larger connected part have a total weight of $10^\ell + q - 1$, and with player 1 of $10^\ell + q$. Finally, any other coalition with weight $10^\ell + q + 1$ or greater has a connected component of weight at least $10^\ell + q + 1$ and wins with and without player 1.

Only player 2 can make a coalition achieve a weight not smaller than 10^ℓ, so 2 has to be in all coalitions for which 1 is pivotal. Therefore, we will focus now on players connected with each other whose total weight is q. Also, note that players 5 and $n + 6$ (with weights $q - \left(\frac{\alpha}{2} - 2\right) \cdot 10^r - 1$ and $q - \left(\frac{\alpha}{2} - 1\right) \cdot 10 - 2$, respectively) together have a total weight larger than q. Therefore, a coalition for which player 1 is pivotal must contain exactly one of them, since all other players (excluding, of course, player 2) together have a total weight smaller than q.

First, let us consider the probabilistic Penrose–Banzhaf power index. Player 1's Penrose–Banzhaf index in this game is

$$\beta((\mathcal{G}, G), 1) = \frac{\xi_2}{2^{2n+4}}.$$

We will show that

$$(\exists e \in \overline{E})\ [\beta((\mathcal{G}, G_{\cup\{e\}}), 1) - \beta((\mathcal{G}, G), 1) < 0] \iff \xi_1 < \xi_2. \tag{1}$$

From right to left, assume that $\xi_1 < \xi_2$. After adding an edge e between players 3 and 4, player 1 stops being pivotal for coalitions with weight $10^\ell + q + 1$ containing player 4 (since they are connected now by e) but becomes pivotal for coalitions with weight $10^\ell + q$ formed by, among others, players 3, 4, and $n + 6$. Therefore, the probabilistic Penrose–Banzhaf index of the distinguished player will decrease to

$$\begin{aligned}\beta((\mathcal{G}, G_{\cup\{e\}}), 1) &= \frac{\xi_2}{2^{2n+5}} + \frac{\xi_1}{2^{2n+5}} \\ &< \frac{\xi_2 + \xi_2}{2^{2n+5}} \\ &= \frac{\xi_2}{2^{2n+4}} = \beta((\mathcal{G}, G), 1).\end{aligned}$$

From left to right, assume now that $\xi_1 \geq \xi_2$. Let e be an added edge (recall that e can only be added between player 4 and any of the other players), and consider the following cases.

Case 1: e connects player 4 with either player 1 or player 2. 1 becomes pivotal for ξ_2 coalitions with weight $10^\ell + q$ formed by, among others, player 4 and player 5 but without player 3, for ξ_1 coalitions with weight $10^\ell + q$ formed by, among others, players 3, 4, and $n + 6$, and it is still pivotal for ξ_2 coalitions with the same weight formed by, among others, player 3 and player 5 but without player 4. So,

$$\begin{aligned}\beta((\mathcal{G}, G_{\cup\{e\}}), 1) &\geq \frac{2\xi_2}{2^{2n+5}} + \frac{\xi_1}{2^{2n+5}} \\ &= \beta((\mathcal{G}, G), 1) + \frac{\xi_1}{2^{2n+5}} \\ &\geq \beta((\mathcal{G}, G), 1).\end{aligned}$$

Case 2: e connects player 4 with player 3. Then (analogously to the previous argument),

$$\beta((\mathcal{G}, G_{\cup\{e\}}), 1) = \frac{\xi_2}{2^{2n+5}} + \frac{\xi_1}{2^{2n+5}} \geq \frac{\xi_2 + \xi_2}{2^{2n+5}} = \beta((\mathcal{G}, G), 1).$$

Case 3: e connects player 4 with player 5. Then 1 stops being pivotal for the coalitions with weight $10^\ell + q + 1$ containing both players 3 and 4, but becomes pivotal for ξ_2 coalitions with weight $10^\ell + q$ formed, among others, by players 4 and 5 but without player 3, so

$$\beta((\mathcal{G}, G_{\cup\{e\}}), 1) = \frac{2\xi_2}{2^{2n+5}} = \beta((\mathcal{G}, G), 1).$$

Case 4: e connects player 4 with one of the players from $\{6, \dots, n+5\}$, say with player j. Let $\xi_2' \leq \xi_2$ be the number of coalitions containing j (and, obviously, player 5) for which our distinguished player 1 was pivotal before the addition. Then the distinguished player 1 stops being pivotal for coalitions of weight $10^\ell + q + 1$ containing both player 4 and player j because they are connected now (not through player 1) but 1 becomes pivotal for the coalitions of weight $10^\ell + q$ formed by, among others, players 4 and 5 but without player 3, so

$$\beta((\mathcal{G}, G_{\cup\{e\}}), 1) = \frac{\xi_2 - \xi_2'}{2^{2n+4}} + \frac{2\xi_2'}{2^{2n+5}} = \beta((\mathcal{G}, G), 1).$$

Case 5: e connects player 4 with player $n+6$. Then 1 still is pivotal for the same coalitions as before (because none of them contains player $n+6$, ergo the coalitions with player 4 still have two connected components) and additionally, 1 becomes pivotal for ξ_1 coalitions with weight $10^\ell + q$ formed by, among others, players 3, 4, and $n+6$, i.e.,

$$\begin{aligned}\beta((\mathcal{G}, G_{\cup\{e\}}), 1) &= \frac{\xi_2}{2^{2n+4}} + \frac{\xi_1}{2^{2n+5}} \\ &= \beta((\mathcal{G}, G), 1) + \frac{\xi_1}{2^{2n+5}} \\ &\geq \beta((\mathcal{G}, G), 1).\end{aligned}$$

Case 6: Finally, e connects player 4 with one of the players from $\{n+7, \dots, 2n+6\}$, say with player j. Let $\xi_1' \leq \xi_1$ be the number of coalitions containing j for which our distinguished player 1 is pivotal. As in the previous case, 1 stays pivotal for the same coalitions as before adding the edge, and 1 additionally becomes pivotal for the coalitions of weight $10^\ell + q$ containing, among others, players 3, 4, and j. Thus

$$\begin{aligned}\beta((\mathcal{G}, G_{\cup\{e\}}), 1) &= \frac{\xi_2}{2^{2n+4}} + \frac{\xi_1'}{2^{2n+5}} \\ &= \beta((\mathcal{G}, G), 1) + \frac{\xi_1'}{2^{2n+5}} \\ &\geq \beta((\mathcal{G}, G), 1).\end{aligned}$$

Therefore, in each case, the distinguished player's probabilistic Penrose–Banzhaf index does not decrease, which completes the proof of (1).

Turning now to the Shapley–Shubik power index, consider the above reduction between the problem Compare-#SubsetSum-RR and our control problem. By Lemma 1, we may assume that every set counted in ξ_1 and in ξ_2 has the same size t. Our distinguished player's Shapley–Shubik index in the constructed game $(\mathcal{G}, G)$ is

$$\varphi((\mathcal{G}, G), 1) = \xi_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!}.$$

Analogously to the case of the probabilistic Penrose–Banzhaf index, we will show that

$$(\exists e \in \overline{E})\ [\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) - \varphi((\mathcal{G}, G), 1) < 0] \iff \xi_1 < \xi_2. \tag{2}$$

From right to left, assume that $\xi_1 < \xi_2$. After adding an edge e between players 3 and 4, the Shapley–Shubik index of our distinguished player will decrease to

$$\begin{aligned}\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &< \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_2 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &= \xi_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &= \varphi((\mathcal{G}, G), 1).\end{aligned}$$

From left to right, assume now that $\xi_1 \geq \xi_2$. Let e be an added edge (which, recall, can only connect player 4 with any of the other players). Consider the following cases.

Case 1: e connects player 4 with either player 1 or player 2. Then

$$\begin{aligned}\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &\geq 2\xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\geq 2\xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_2 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\geq \varphi((\mathcal{G}, G), 1).\end{aligned}$$

Case 2: e connects player 4 with player 3. Then

$$\begin{aligned}\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\geq \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} + \xi_2 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &= \varphi((\mathcal{G}, G), 1).\end{aligned}$$

Case 3: e connects player 4 with player 5. Then

$$\begin{aligned}\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= 2\xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} \\ &= \xi_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} \\ &\quad + \frac{2n-t+2}{t+4} \xi_2 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\geq \varphi((\mathcal{G}, G), 1),\end{aligned}$$

since $\frac{2n-t+2}{t+4} \geq 1$ due to the fact that $t < n$ for $n > 1$.

Case 4: e connects player 4 with one of the players from $\{6, \ldots, n+5\}$, say with player j. Let $\xi'_2 \leq \xi_2$ be the number of coalitions containing j for which our distinguished player 1 was pivotal before the addition. Then

$$\begin{aligned}\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= (\xi_2 - \xi'_2) \frac{(t+3)!(2n-t+1)!}{(2n+5)!} + 2\xi'_2 \frac{(t+3)!(2n-t+2)!}{(2n+6)!} \\ &= (\xi_2 - \xi'_2) \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &\quad + 2\frac{2n-t+2}{2n+6} \xi'_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &= (\xi_2 - \xi'_2) \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &\quad + 2\left(1 - \frac{t+4}{2n+6}\right) \xi'_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &\geq (\xi_2 - \xi'_2) \frac{(t+3)!(2n-t+1)!}{(2n+5)!} + \xi'_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!} \\ &= \varphi((\mathcal{G}, G), 1).\end{aligned}$$

Case 5: e connects player 4 with player $n+6$. Then

$$\begin{aligned}\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= \xi_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!} + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &= \varphi((\mathcal{G}, G), 1) + \xi_1 \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\geq \varphi((\mathcal{G}, G), 1).\end{aligned}$$

Case 6: Finally, e connects player 4 with one of the players from $\{n+7,\dots,2n+6\}$, say with player j. Let $\xi_1' \leq \xi_1$ be the number of coalitions containing j for which our distinguished player 1 is pivotal. Then

$$\begin{aligned}\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= \xi_2 \frac{(t+3)!(2n-t+1)!}{(2n+5)!} + \xi_1' \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &= \varphi((\mathcal{G}, G), 1) + \xi_1' \frac{(t+4)!(2n-t+1)!}{(2n+6)!} \\ &\geq \varphi((\mathcal{G}, G), 1).\end{aligned}$$

Therefore, in each case, also the distinguished player's Shapley–Shubik index does not decrease, which completes the proof of (2) and of the theorem. □

Next, we turn to the goal of increasing a given player's power and start with the Penrose–Banzhaf index. Again, we show PP-hardness of the control problem when we are not allowed to connect any player with the distinguished player by adding edges.

Theorem 5. Control-by-Adding-Edges-ONDP-to-Increase-PBI *is* PP-*hard.*

Proof. Again, we reduce from the PP-hard problem Compare-#SubsetSum-RR to prove PP-hardness. Given an instance $A = (a_1, \dots, a_n)$ of Compare-#SubsetSum-RR, where $\alpha = \sum_{i=1}^{n} a_i$, define

$$\begin{aligned}\xi_1 &= \#\text{SubsetSum}\left(A, \frac{\alpha}{2} - 1\right) \quad \text{and} \\ \xi_2 &= \#\text{SubsetSum}\left(A, \frac{\alpha}{2} - 2\right)\end{aligned}$$

and construct an instance of our control problem as follows. Choose r_1, r_2, q, and ℓ such that

$$\begin{aligned}10^{r_2} &> n, \\ 10^{r_1} &> n + \sum_{j=1}^{n} \left(a_j \cdot 10^{r_2}\right), \\ q &= 2\sum_{j=1}^{n} a_j \cdot 10^{r_1} + 1, \text{ and} \\ 10^{\ell} &> 2q + n + \sum_{j=1}^{n} \left(a_j \cdot 10^{r_1} + a_j \cdot 10^{r_2}\right) + 1.\end{aligned}$$

Further, let $q_1 = q - \left(\frac{\alpha}{2} - 1\right) \cdot 10^{r_1}$, $q_2 = q - \left(\frac{\alpha}{2} - 2\right) \cdot 10^{r_2}$, $a_i' = a_i \cdot 10^{r_1}$, and $a_i'' = a_i \cdot 10^{r_2}$ for $i \in \{1, \dots, n\}$. Construct the following graph-restricted weighted voting game $(\mathcal{G}, G)$:

$$\mathcal{G} = \left(1, 10^{\ell}, q_1, a_1', \dots, a_n', q_2, a_1'', \dots, a_n'', \underbrace{1, \dots, 1}_{n}; 10^{\ell} + q + 1\right)$$

with $3n+4$ players in N, distinguished player 1, and the following communication graph $G = (N, E)$, as displayed in Figure 1:

- The players $1, 2, 3, 2n+5, \ldots, 3n+4$ with weights $1, 10^{\ell}, q_1, 1, \ldots, 1$ form a complete subgraph,
- the players $4, \ldots, n+3$ with weights $a'_1, \ldots, a'_n$ are connected with each other and with player 3 with weight q_1 and players $2n+5, \ldots, 3n+4$ (i.e., they are not connected only with players 1 and 2 in the component), and
- the players $n+4, \ldots, 2n+4$ with weights $q_2, a''_1, \ldots, a''_n$ form another complete component without any edges to the remaining players.

Call the first component X and the other component Y; the dashed rectangles in Figure 1 represent these connected components. So, the only edges that can possibly be added are edges between the components X and Y, and within X, between player 2 with weight 10^{ℓ} and any of the players $4, \ldots, n+3$ with weights $a'_1, \ldots, a'_n$.

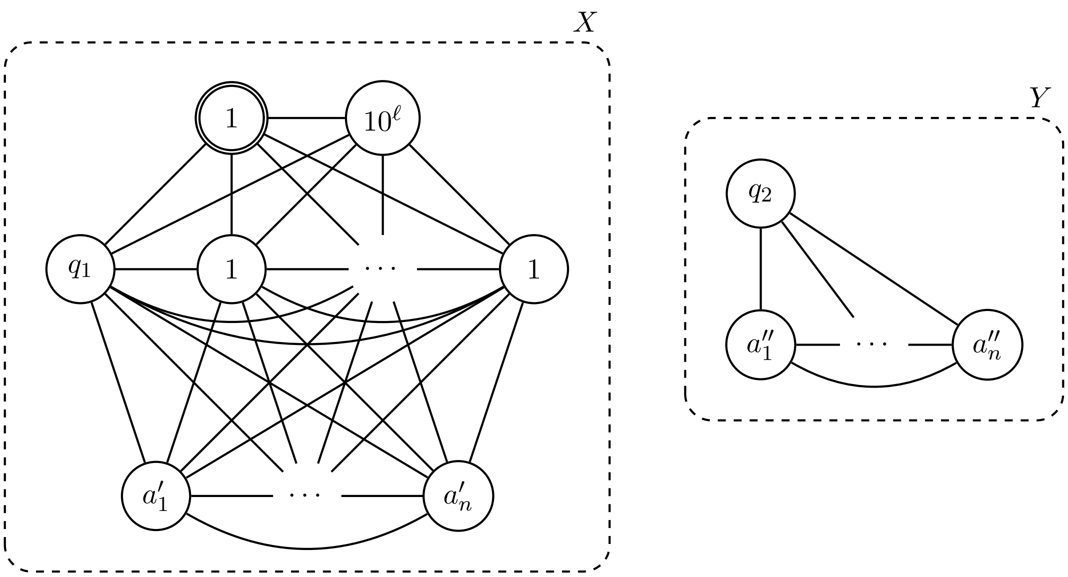


Figure 1: Communication structure of the game $(\mathcal{G}, G)$ from the proof of Theorem 5

First note that each winning coalition has to contain player 2, so we need to consider only those coalitions. In the winning connected component of a coalition for which player 1 is pivotal, there can be exactly one of the players with weights q_1 and q_2, because together they are heavier than q and all other players' total weight is smaller than q. Player 1 is pivotal for ξ_1 coalitions from component X with weight $10^{\ell}+q$ and for each coalition created from them by adding players from component Y (their connected components containing player 2 still have the weight $10^{\ell}+q$).

Player 1's probabilistic Penrose–Banzhaf index in this game is

$$\beta((\mathcal{G}, G), 1) = \frac{\xi_1}{2^{2n+2}}.$$

Let the addition limit be $k = 1$. We will show that

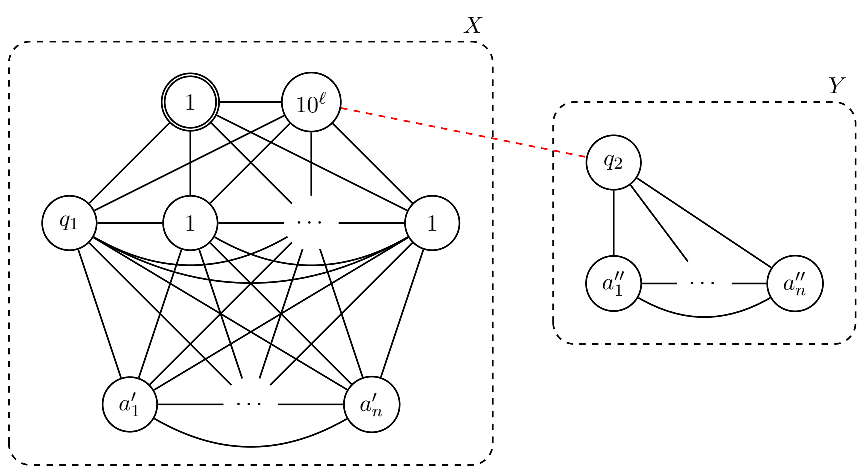
$$(\exists e \in \overline{E} \setminus \overline{E}_1)\ [\beta((\mathcal{G}, G_{\cup\{e\}}), 1) - \beta((\mathcal{G}, G), 1) > 0] \iff \xi_1 < \xi_2.$$

From right to left, assume that $\xi_1 < \xi_2$. Add an edge e between player 2 and, for example, player $n+4$ with weight $q_2 = q - \left(\frac{\alpha}{2} - 2\right) \cdot 10^{r_2}$. Then player 1 will stop being pivotal for the coalitions containing player $n+4$ because the coalitions' larger component will have weight greater than the quota. But 1 becomes pivotal for ξ_2 connected coalitions with weight exactly $10^\ell + q$ formed by, among others, player $n+4$ and the coalitions created from them by adding players $4, \ldots, n+3$. Therefore, player 1's Penrose–Banzhaf index will increase to

$$\begin{aligned}\beta((\mathcal{G}, G_{\cup\{e\}}), 1) &= \frac{\xi_1}{2^{2n+3}} + \frac{\xi_2}{2^{2n+3}} \\ &> \frac{\xi_1 + \xi_1}{2^{2n+3}} = \frac{\xi_1}{2^{2n+2}} = \beta((\mathcal{G}, G), 1).\end{aligned}$$

From left to right, assume now that $\xi_1 \geq \xi_2$. Adding any of the edges inside of the component X will not change the distinguished player's Penrose–Banzhaf index. Let us focus on edges connecting the components X and Y with each other. Let e be an added edge, and consider the following possibilities which players are connected by e (where the red dashed edges represent possible connections):

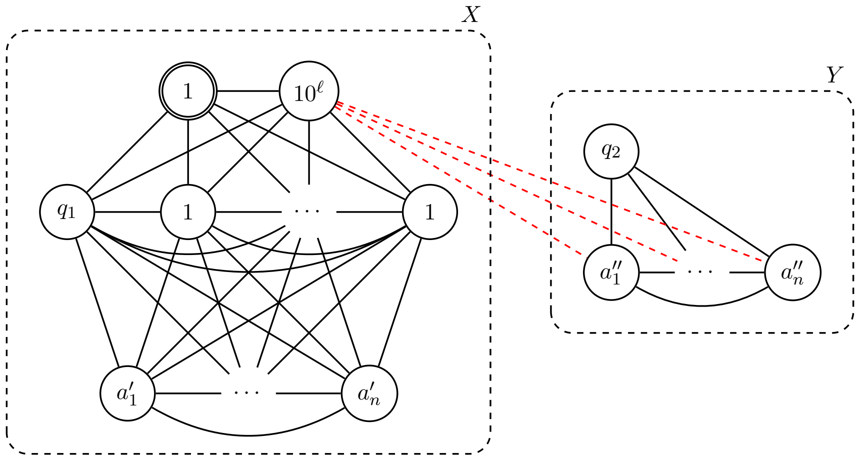
Case 1: e connects player 2 with player $n+4$:



Then, as in the previous implication, we have

$$\beta((\mathcal{G}, G_{\cup\{e\}}), 1) = \frac{\xi_1}{2^{2n+3}} + \frac{\xi_2}{2^{2n+3}} \leq \frac{\xi_1 + \xi_1}{2^{2n+3}} = \beta((\mathcal{G}, G), 1).$$

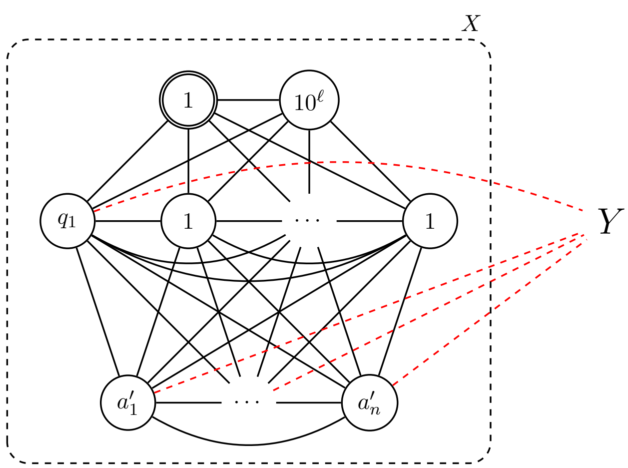
Case 2: e connects player 2 with one of the players from $Y \setminus \{n+4\}$, say i:



Let $\xi'_2 \leq \xi_2$ be the number of connected coalitions containing i for which player 1 is pivotal. Then 1 stops being pivotal for coalitions containing i, but 1 becomes pivotal for connected coalitions with weight $10^\ell + q$ formed by, among others, players $n+4$ and i and the coalitions created from them by adding players $4, \dots, n+3$, so

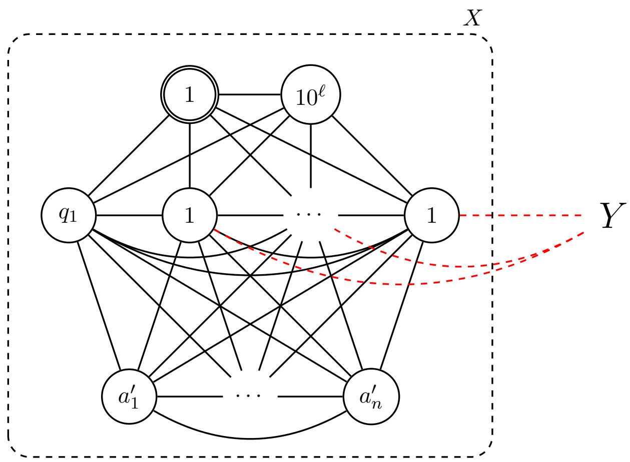
$$\begin{aligned}\beta((\mathcal{G}, G_{\cup\{e\}}), 1) &= \frac{\xi_1}{2^{2n+3}} + \frac{\xi'_2}{2^{2n+3}} \leq \frac{\xi_1 + \xi_2}{2^{2n+3}} \\ &\leq \frac{\xi_1 + \xi_1}{2^{2n+3}} = \beta((\mathcal{G}, G), 1).\end{aligned}$$

Case 3: e connects any player from Y with one of the players $3, \dots, n+3$:



This will only add a neighbor to those coalitions counted in the old probabilistic Penrose–Banzhaf index of player 1, i.e., 1 will stop being pivotal for the coalitions containing the player from Y, and will not make player 1 pivotal for any other coalition, so 1's new probabilistic Penrose–Banzhaf index will not be greater.

Case 4: e connects the component Y with one of the players $2n+5, \dots, 3n+4$:



In this case, the probabilistic Penrose–Banzhaf index of player 1 will not change, since the latter players are only neighbors of the sets counted in player 1's old probabilistic Penrose–Banzhaf index and they cannot be contained in any coalition for which 1 could be pivotal in a game with the given set of players and weights and the given quota.

Therefore, it is impossible for the distinguished player's Penrose–Banzhaf index to increase by adding any edge not incident to him or her. □

By slightly modifying the reduction provided in the previous proof, we obtain the same result for the Shapley–Shubik index.

Theorem 6. Control-by-Adding-Edges-ONDP-to-Increase-SSI *is* PP-*hard.*

Proof. Again, we show PP-hardness by reducing from Compare-#SubsetSum-RR. Given an instance $A = (a_1, \dots, a_n)$ of this problem, where $\alpha = \sum_{i=1}^{n} a_i$, we again define

$$\xi_1 = \#\text{SubsetSum}\left(A, \frac{\alpha}{2} - 1\right), \quad \text{and}$$
$$\xi_2 = \#\text{SubsetSum}\left(A, \frac{\alpha}{2} - 2\right)$$

and construct an instance of our control problem as follows. Let $z_1 = \cdots = z_n = 1$ and choose r_1, r_2, q, and ℓ such that for $v = n + 1$,

$$\begin{aligned}
10^{r_2} &> 2v, \\
10^{r_1} &> 2v + \sum_{j=1}^{n} a_j \cdot 10^{r_2}, \\
q &= 2\sum_{j=1}^{n} a_j \cdot 10^{r_1} + 1, \text{ and} \\
10^{\ell} &> 2q + 2v + \sum_{j=1}^{n} (a_j \cdot 10^{r_1} + a_j \cdot 10^{r_2}) + 1.
\end{aligned}$$

Further, let $q_1 = q - \left(\frac{\alpha}{2} - 1\right) \cdot 10^{r_1}$, $q_2 = q - \left(\frac{\alpha}{2} - 2\right) \cdot 10^{r_2} - v$, $a'_i = a_i \cdot 10^{r_1}$, and $a''_i = a_i \cdot 10^{r_2}$ for $i \in \{1, \ldots, n\}$. Construct the following graph-restricted weighted voting game $(\mathcal{G}, G)$:

$$\mathcal{G} = \left(1, 10^{\ell}, q_1, a'_1, \ldots, a'_n, q_2, v, a''_1, \ldots, a''_n, z_1, \ldots, z_n; 10^{\ell} + q + 1\right)$$

with $3n + 5$ players, distinguished player 1, and the following communication graph $G = (N, E)$, as displayed in Figure 2:

- The players $1, 2, 3, 2n+6, \ldots, 3n+5$ with weights $1, 10^{\ell}, q_1, z_1, \ldots, z_n$ form a complete subgraph, except that the two edges between player $3n+5$ with weight z_n and players 1 and 2 having weights 1 and 10^{ℓ} are missing,
- the players $4, \ldots, n + 3$ with weights $a'_1, \ldots, a'_n$ are connected with each other and with player 3 with weight q_1 and the players with weights $z_1, \ldots, z_n$, and
- the players $n + 4, \ldots, 2n + 5$ with weights $q_2, v, a''_1, \ldots, a''_n$ form another complete component without any edges with the remaining players.

Call the first component X and the other component Y; the dashed rectangles in Figure 2 represent these connected components. So, the only edges that can possibly be added are edges between the components X and Y, and within X, between player 2 with weight 10^{ℓ} and any of the players $4, \ldots, n + 3, 3n + 5$ with weights $a'_1, \ldots, a'_n, z_n$.

Using Lemma 1, we can assume that all solutions counted in ξ_1 and ξ_2 have the same size t. Player 1 is pivotal for coalitions from X having weight $10^{\ell} + q$ and for coalitions created from them by adding players from Y. Among the remaining possible coalitions formed by players in component X, they have weight either smaller than $10^{\ell} + q$, so they lose without and with the distinguished player, and the coalitions of weight greater than $10^{\ell}+q$ are all connected (all of them have to contain player 2 and player 3 which is connected with every player in X), so they win with and without player 1.

Player 1's Shapley–Shubik index in this game is

$$\varphi((\mathcal{G}, G), 1) = \xi_1 \frac{(t + 2)!(2n - t)!}{(2n + 3)!}.$$

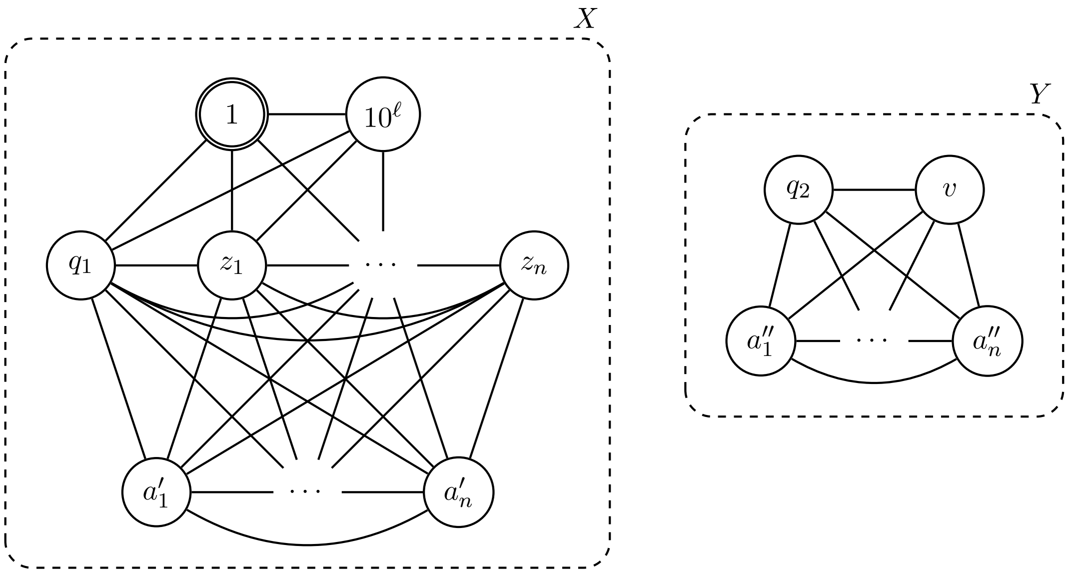


Figure 2: Communication structure of the game $(\mathcal{G}, G)$ from the proof of Theorem 6

Let the addition limit be $k = 1$. We will show that

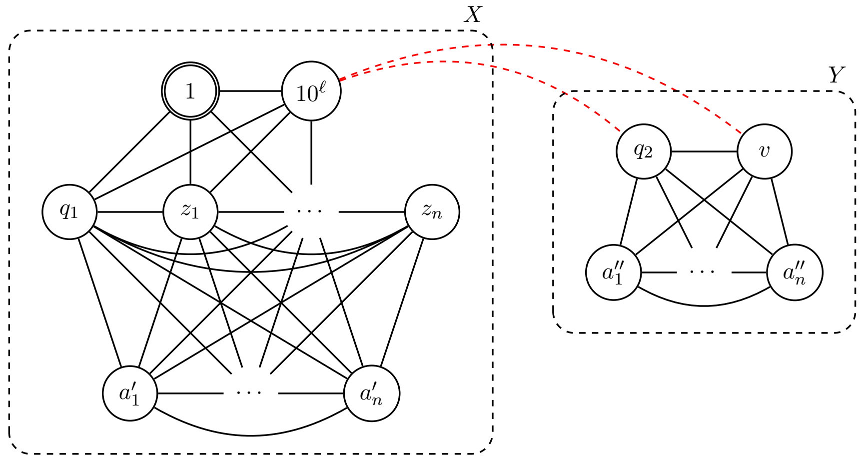
$$(\exists e \in \overline{E} \setminus \overline{E}_1)\ [\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) - \varphi((\mathcal{G}, G), 1) > 0] \iff \xi_1 < \xi_2.$$

From right to left, assume that $\xi_1 < \xi_2$. Add an edge e between player 2 and, for example, player $n+4$ with weight $q_2 = q - \left(\frac{\alpha}{2} - 2\right) \cdot 10^{r_2} - v$. Then player 1 stops being pivotal for the coalitions containing player $n+4$ but becomes pivotal for ξ_2 connected coalitions of weight $10^\ell + q$ formed by, among others, player $n+4$ and for coalitions created from these coalitions by adding players $4, \ldots, n+3$, and $3n+5$. Thus player 1's Shapley–Shubik index will increase to

$$\begin{aligned}
\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= \xi_1 \frac{(t+2)!(2n-t+1)!}{(2n+4)!} + \xi_2 \frac{(t+3)!(2n-t)!}{(2n+4)!} \\
&> \xi_1 \frac{(t+2)!(2n-t+1)!}{(2n+4)!} + \xi_1 \frac{(t+3)!(2n-t)!}{(2n+4)!} \\
&= \xi_1 \frac{(t+2)!(2n-t)!}{(2n+3)!} = \varphi((\mathcal{G}, G), 1).
\end{aligned}$$

From left to right, assume now that $\xi_1 \geq \xi_2$. Adding any of the edges inside of the component X will not change the distinguished player's Shapley–Shubik index. Let e be an edge added between the components X and Y. There are the following possibilities (where red dashed edges represent possible connections to be added in the discussed case):

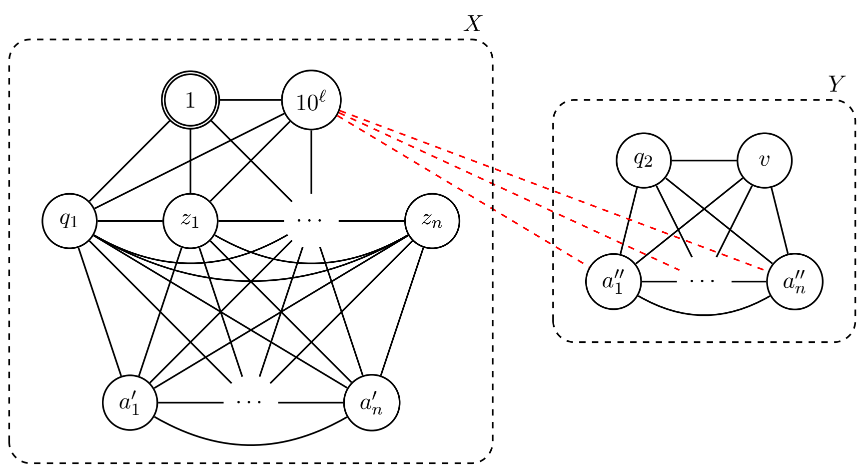
Case 1: e connects player 2 with either player $n+4$ or player $n+5$:



Then, analogously to the previous implication, we have

$$\begin{aligned}\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= \xi_1 \frac{(t+2)!(2n-t+1)!}{(2n+4)!} + \xi_2 \frac{(t+3)!(2n-t)!}{(2n+4)!} \\ &\leq \xi_1 \frac{(t+2)!(2n-t+1)!}{(2n+4)!} + \xi_1 \frac{(t+3)!(2n-t)!}{(2n+4)!} \\ &= \varphi((\mathcal{G}, G), 1).\end{aligned}$$

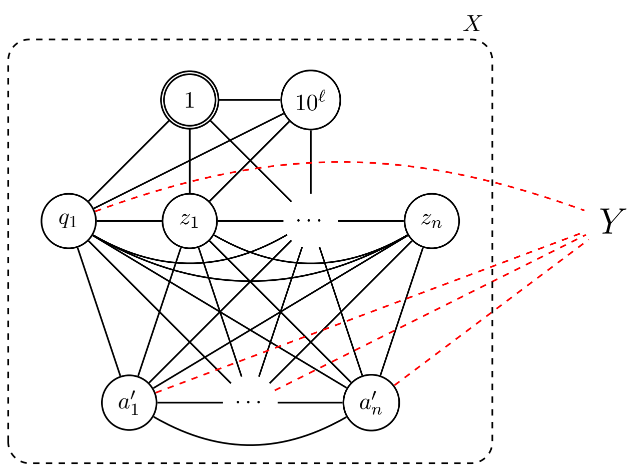
Case 2: e connects player 2 with one of the players from $Y \setminus \{n+4, n+5\}$, say i:



Let $\xi'_2 \leq \xi_2$ be the number of coalitions containing i for which player 1 is pivotal. Player 1 stops being pivotal for the coalitions containing player i but 1 becomes pivotal for ξ'_2 connected coalitions with weight $10^\ell + q$ formed, among others, by i and for the coalitions created from them by adding players 4, …, $n+3$, and $3n+5$, so

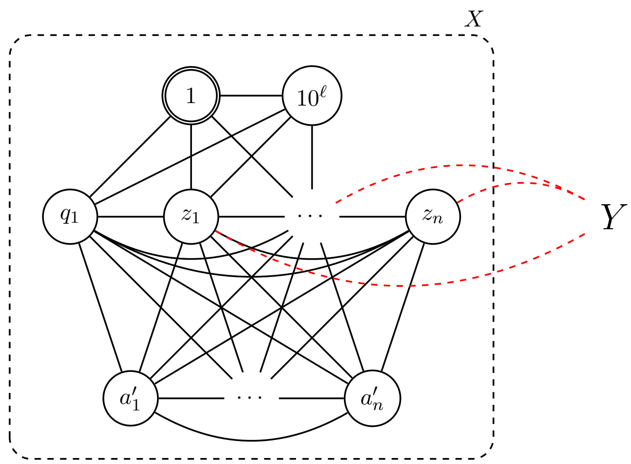
$$\begin{aligned}\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) &= \xi_1 \frac{(t+2)!(2n-t+1)!}{(2n+4)!} + \xi_2' \frac{(t+3)!(2n-t)!}{(2n+4)!} \\ &\le \xi_1 \frac{(t+2)!(2n-t+1)!}{(2n+4)!} + \xi_2 \frac{(t+3)!(2n-t)!}{(2n+4)!} \\ &\le \xi_1 \frac{(t+2)!(2n-t+1)!}{(2n+4)!} + \xi_1 \frac{(t+3)!(2n-t)!}{(2n+4)!} \\ &= \varphi((\mathcal{G}, G), 1).\end{aligned}$$

Case 3: e connects any player from Y with one of the players $3, \ldots, n+3$:



This will add a neighbor to those coalitions counted in the old Shapley–Shubik index of player 1, i.e., 1 will stop being pivotal for coalitions containing the player we connected with component X, and will not make player 1 pivotal for any other coalition, so 1's new Shapley–Shubik index will not be greater.

Case 4: e connects the component Y with one of the players $2n+6, \ldots, 3n+5$:



> In this case, the Shapley–Shubik index of player 1 will not change, since the latter players are only neighbors of the sets counted in player 1's old Shapley–Shubik index and they are not contained in any coalition for which 1 could be pivotal in a game with the given set of players and weights and the given quota: There cannot exist a connected coalition of weight exactly $10^\ell + q$ containing any of the players because such coalition has to contain either player 3 or $n+4$, then players from $\{4, \ldots, n+3\}$ or $\{n+6, \ldots, 2n+5\}$ accordingly because $n+5$ and $2n+6, \ldots, 3n+5$ together have too small a total weight. Then either we get a set of weight q or $q - v$ which can change to q only if we add player $n+5$.

Therefore, it is impossible for the distinguished player's Shapley–Shubik index to increase by adding any edge from $\overline{E} \setminus \overline{E}_1$. □

Finally, we cover the case of control by only adding edges incident to the distinguished player when our goal is to increase his or her power index.[7] For this case, we will show NP-hardness for both power indices we consider. By essentially the same reduction, we additionally show coNP-hardness for control by adding edges with the goal of maintaining the distinguished player's power with respect to both these indices, no matter whether these added edges each are incident to this player, each are not incident to him or her, or whether they can be added anywhere in the graph.

Theorem 7. *For* PI $\in$ {PBI, SSI}*,* Control-by-Adding-Edges-ODP-to-Increase-PI *is* NP*-hard and all three problems of control by adding edges to maintain* PI *are* coNP*-hard.*

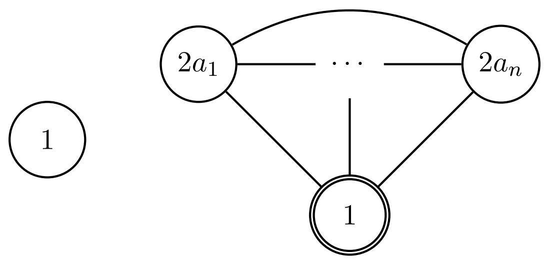
Proof. We provide a reduction from either Partition or its complement to our control problems to show either their NP-hardness or coNP-hardness. Let $(a_1, \ldots, a_n)$ be a given instance of Partition (respectively, of the complement of Partition), let $\alpha = \sum_{i=1}^{n} a_i$, and let $\xi = \#\text{Partition}(a_1, \ldots, a_n)$ be the number of its solutions.

7. Recall from Theorem 2 that this case is trivial for the goal of decreasing the distinguished player's power, i.e., it is never possible to decrease this power by adding edges incident to the distinguished player.

Construct the graph-restricted weighted voting game $(\mathcal{G}, G)$:

$$\mathcal{G} = (1, 2a_1, \ldots, 2a_n, 1; \alpha + 2)$$

with $n+2$ players, distinguished player 1, and the communication structure $G = (N, E)$, where all players but the $(n+2)$nd player (with weight 1) form a complete subgraph and the $(n+2)$nd player is an isolated vertex:



Therefore, only the edges in $\overline{E} = \Big\{\{n+2, i\} \mid i \in N \land i \neq n+2\Big\}$ can be added to G.

Set the addition limit to $k = 1$. We will prove that

$$\begin{aligned}
(\exists e \in \overline{E})\ [\beta((\mathcal{G}, G_{\cup\{e\}}), 1) - \beta((\mathcal{G}, G), 1) > 0] &\iff \xi > 0, \\
(\exists e \in \overline{E})\ [\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) - \varphi((\mathcal{G}, G), 1) > 0] &\iff \xi > 0, \\
(\exists e \in \overline{E})\ [\beta((\mathcal{G}, G_{\cup\{e\}}), 1) - \beta((\mathcal{G}, G), 1) = 0] &\iff \xi = 0, \text{ and} \\
(\exists e \in \overline{E})\ [\varphi((\mathcal{G}, G_{\cup\{e\}}), 1) - \varphi((\mathcal{G}, G), 1) = 0] &\iff \xi = 0.
\end{aligned}$$

Moreover, we will show that e can be any edge from $\overline{E}$, so the reduction works equally well for the general problem and for either of its restrictive scenarios—note, however, that for the ONDP restriction we even have shown PP-hardness in Theorem 5 for the Penrose–Banzhaf index and in Theorem 6 the Shapley–Shubik index.

Before player $n+2$ is connected to the larger component, player 1 can be pivotal only for the connected coalitions with total weight $\alpha+1$. Note that this value is an odd number. Therefore,

$$\beta((\mathcal{G}, G), 1) = \varphi((\mathcal{G}, G), 1) = 0.$$

After connecting player $n+2$, player 1 could additionally be pivotal for coalitions containing player $n+2$ with total weight either $\alpha+1$ (and then, the rest of the coalition has to have a total weight of α) or $\alpha+2$ if player $n+2$ is connected to the graph through player 1. However, the latter case is impossible to happen because the rest of the coalition would again have to have a total weight of $\alpha+1$.

Suppose that $\xi = 0$. Then, if we add any new edge, the indices of our distinguished player will remain unchanged:

- If we add an edge to any player which is not our distinguished player, then there exists no coalition containing player $n+2$ of weight $\alpha+1$ for which 1 can be pivotal due to $\xi = 0$, any coalition with smaller weight loses without and with player 1, and any coalition with greater weight wins with and without 1.

- If we connect players $n+2$ and 1, the distinguished player could be additionally pivotal for a coalition containing $n+2$ having weight $\alpha+2$, but as mentioned before, this situation is impossible to occur because all other players have even weights.

Suppose now that $\xi > 0$. If we add any new edge, the indices of our distinguished player will increase because it will be possible to create a coalition with an even weight containing player 1 and player $n+2$, and the players $2, \dots, n+1$ form ξ—so at least two—coalitions with weight $\alpha = 2 \cdot \frac{1}{2} \sum_{i=1}^{n} a_i$, and each of the players is contained in at least one such a coalition. □

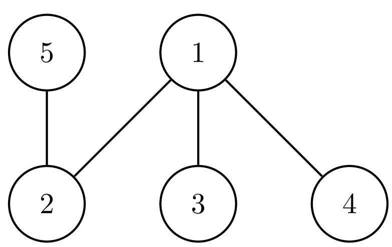
4. Control by Deleting Edges

In this section, we study the effects of control by deleting edges from a communication structure in a graph-restricted weighted voting game. Again, we first determine upper and lower bounds for the probabilistic Penrose–Banzhaf and the Shapley–Shubik power index of a given player when such control attacks are executed. Then we define the decision problems capturing control by deleting edges, again discussing some restrictive scenarios versus the general case for them, and we finally determine computational lower bounds for these problems.

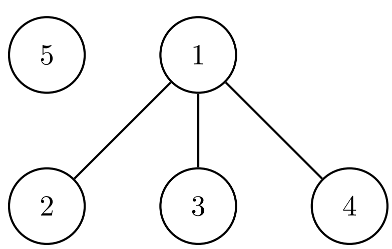
4.1 Change of Power by Deleting Edges from a Communication Structure

We now study how limiting the communication among players can change their power in a graph-restricted weighted voting game. Before presenting our results, we give an example that illustrates the impact of this structural change to the game.

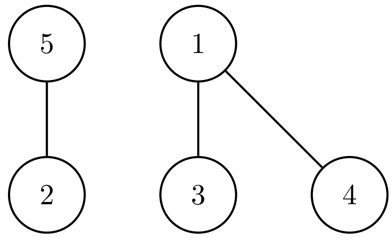
Example 6. Consider again the graph-restricted weighted voting game $(\mathcal{G}, G)$ from Example 3 with $\mathcal{G} = (1,2,3,4,5;8)$ and the following communication structure $G = (N, E)$:



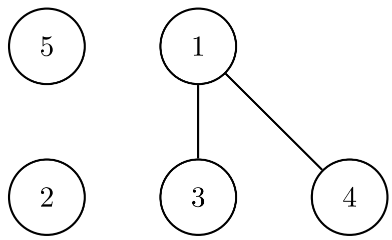
No matter whether we delete the edge $x = \{2,5\}$:



(having the set of all winning connected coalitions $\mathcal{WC} \setminus \{\{1,2,5\},\{1,2,3,5\},\{1,2,4,5\},N\}$) or the edge $y = \{1,2\}$:



(having the set of all winning connected coalitions $\mathcal{WC} \setminus \{\{1,2,5\},\{1,2,3,5\},\{1,2,3,4\},\{1,2,4,5\},N\}$) or both:



we will get the same probabilistic Penrose–Banzhaf and Shapley–Shubik power indices of the players, i.e., for $E' \in \{\{x\},\{y\},\{x,y\}\}$, we have:

$$\beta((\mathcal{G},G_{\setminus E'},1) = \beta((\mathcal{G},G_{\setminus E'},3) = \beta((\mathcal{G},G_{\setminus E'},4) = \frac{1}{4} \quad \text{and}$$
$$\beta((\mathcal{G},G_{\setminus E'},2) = \beta((\mathcal{G},G_{\setminus E'},5) = 0;$$
$$\varphi((\mathcal{G},G_{\setminus E'},1) = \varphi((\mathcal{G},G_{\setminus E'},3) = \varphi((\mathcal{G},G_{\setminus E'},4) = \frac{1}{3} \quad \text{and}$$
$$\varphi((\mathcal{G},G_{\setminus E'},2) = \varphi((\mathcal{G},G_{\setminus E'},5) = 0.$$

This illustrates that the power indices can change the same way (increase or decrease) for both the stronger and the weaker players, even after deleting one or more connections.

Our next result shows how the probabilistic Penrose–Banzhaf and the Shapley–Shubik power index of a player i can change by deleting edges connecting *this* player i with other players in a communication graph, i.e., after removing the possibility for i to communicate with some of his or her neighbors in the graph. Theorem 8 presents upper and lower bounds for how much i's power can change in this case. Analogously to the case of adding new edges to a communication structure, we see that i can remain pivotal either for the same coalitions as before (and thus maintains his or her power) or for fewer coalitions than before (and thus loses power), yet it is impossible for i to become pivotal for any coalition i used to be not pivotal for before the control attack. Therefore, i's power cannot increase by providing i with fewer connections. Only by deleting edges in some other part of the graph, i might become pivotal for new coalitions and increase his or her power.

Theorem 8. *Let $(\mathcal{G},G)$ with $G=(N,E)$ be a graph-restricted weighted voting game and let $E^i \subseteq E_i$ be a set of edges connecting a given player i with other players in G. For a power index $\gamma \in \{\beta,\varphi\}$, let*

$$\mathrm{diff}_\gamma(\mathcal{G},G,G_{\setminus E^i},i) = \gamma((\mathcal{G},G),i) - \gamma((\mathcal{G},G_{\setminus E^i}),i).$$

By deleting the edges E^i from G, thus creating the new game $(\mathcal{G}, G_{\setminus E^i})$, the old and the new power index $\gamma \in \{\beta, \varphi\}$ of player i can differ as follows:

$$0 \leq \mathrm{diff}_{\gamma}(\mathcal{G}, G, G_{\setminus E^i}, i) \leq \gamma((\mathcal{G}, G), i).$$

Proof. Just as the upper bounds presented in Theorem 2, the lower bounds here come from the property of the indices shown by Napel et al. (2012).

To analyze the upper bounds, let us consider the formulas from Theorem 1. Recall that $\mathcal{N}(S)$ denotes the set of the neighbors of a set S of players in the original game and let $\mathcal{N}'(S)$ denote the set of the neighbors of the players in S in the new game. Further, let $\mathcal{PWC}_i$ and $\mathcal{PWC}'_i$, respectively, denote the set of all pivotal winning connected coalitions of player i in the old and the new game. We upper-bound the difference between the probabilistic Penrose–Banzhaf power index of player i in the old and the new game as follows:

$$\begin{aligned}
&\mathrm{diff}_{\beta}(\mathcal{G}, G, G_{\setminus E^i}, i) \\
&= \beta((\mathcal{G}, G), i) - \beta((\mathcal{G}, G_{\setminus E^i}), i) \\
&= \sum_{S \in \mathcal{PWC}_i} \frac{1}{2^{|S|+|\mathcal{N}(S)|-1}} - \sum_{S \in \mathcal{PWC}'_i} \frac{1}{2^{|S|+|\mathcal{N}'(S)|-1}} \\
&= \sum_{S \in \mathcal{PWC}_i \setminus \mathcal{PWC}'_i} \frac{1}{2^{|S|+|\mathcal{N}(S)|-1}} + \sum_{S \in \mathcal{PWC}'_i} \frac{1}{2^{|S|+|\mathcal{N}(S)|-1}} - \sum_{S \in \mathcal{PWC}'_i} \frac{1}{2^{|S|+|\mathcal{N}'(S)|-1}} \\
&= \sum_{S \in \mathcal{PWC}_i \setminus \mathcal{PWC}'_i} \frac{1}{2^{|S|+|\mathcal{N}(S)|-1}} - \sum_{S \in \mathcal{PWC}'_i} \left(\frac{1}{2^{|S|+|\mathcal{N}'(S)|-1}} - \frac{1}{2^{|S|+|\mathcal{N}(S)|-1}} \right) \\
&\leq \sum_{S \in \mathcal{PWC}_i} \frac{1}{2^{|S|+|\mathcal{N}(S)|-1}} = \beta((\mathcal{G}, G), i),
\end{aligned}$$

where $\mathcal{PWC}'_i \subseteq \mathcal{PWC}_i$ and $\mathcal{N}'(S) \subseteq \mathcal{N}(S)$ (because deleting edges cannot make the winning component—if there exists any—larger in terms of number of edges, and it cannot change the connections without changing their number).

Similarly, the difference between the Shapley–Shubik power index of player i in the old and the new game can be upper-bounded as follows:

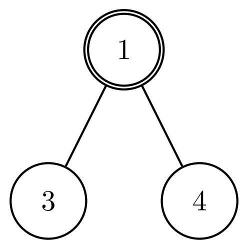
$$\begin{aligned}
\mathrm{diff}_{\varphi}(\mathcal{G}, G, G_{\setminus E^i}, i) &= \varphi((\mathcal{G}, G), i) - \varphi((\mathcal{G}, G_{\setminus E^i}), i) \\
&= \sum_{S \in \mathcal{PWC}_i} \frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!} - \sum_{S \in \mathcal{PWC}'_i} \frac{(|S|-1)!|\mathcal{N}'(S)|!}{(|S|+|\mathcal{N}'(S)|)!} \\
&= \sum_{S \in \mathcal{PWC}_i \setminus \mathcal{PWC}'_i} \frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!} + \sum_{S \in \mathcal{PWC}'_i} \frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!} \\
&\quad - \sum_{S \in \mathcal{PWC}'_i} \frac{(|S|-1)!|\mathcal{N}'(S)|!}{(|S|+|\mathcal{N}'(S)|)!}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{S \in \mathcal{PWC}_i \setminus \mathcal{PWC}'_i} \frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!} \\
&\quad - \sum_{S \in \mathcal{PWC}'_i} \left(\frac{(|S|-1)!|\mathcal{N}'(S)|!}{(|S|+|\mathcal{N}'(S)|)!} - \frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!} \right) \\
&\leq \sum_{S \in \mathcal{PWC}_i} \frac{(|S|-1)!|\mathcal{N}(S)|!}{(|S|+|\mathcal{N}(S)|)!} = \varphi((\mathcal{G}, G), i).
\end{aligned}
$$

This completes the proof. □

We now present a few examples showing that the intervals given in Theorem 8 are tight.

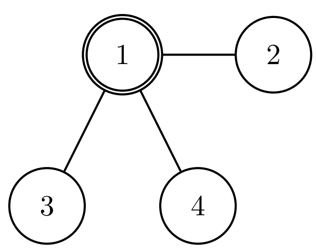
Example 7. Let $(\mathcal{G}, G)$ be a graph-restricted weighted voting game with $\mathcal{G} = (1, 3, 4; 5)$, player set $\{1, 2, 3\}$, and the following communication graph G:



Let us focus on player 1 with weight 1 and both power indices being greater than 0. Deleting both edges will isolate 1 from the rest of the players and 1's power indices will thus decrease to 0, which shows that the upper bounds given in Theorem 8 are tight for $\gamma \in \{\beta, \varphi\}$:

$$\mathrm{diff}_\gamma(\mathcal{G}, G, G_{\setminus\{\{1,2\},\{1,3\}\}}, 1) = \gamma((\mathcal{G}, G), 1).$$

Let $(\mathcal{H}, H)$ be another graph-restricted weighted voting game with $\mathcal{H} = (1, 2, 3, 4; 8)$, player set $\{1, 2, 3, 4\}$, and the following communication graph H:



Let us focus on player 1, who is pivotal for coalitions $\{3, 4\}$ and $\{2, 3, 4\}$, so 1's power indices are

$$\beta((\mathcal{H}, H), 1) = \frac{2}{2^3} = \frac{1}{4} \quad \text{and} \quad \varphi((\mathcal{H}, H), 1) = \frac{2!1!}{4!} + \frac{3!0!}{4!} = \frac{1}{3}.$$

If we remove the edge $\{1, 2\}$, both indices will remain unchanged, which shows that the lower bounds given in Theorem 8 are tight for $\gamma \in \{\beta, \varphi\}$:

$$\mathrm{diff}_\gamma(\mathcal{H}, H, H_{\setminus\{\{1,2\}\}}, 1) = 0.$$

4.2 Defining the Decision Problems of Control by Deleting Edges

Similarly as before, we define the problem of control by deleting edges to *increase* the power index PI of a given player in given graph-restricted weighted voting game as follows:

Control-by-Deleting-Edges-to-Increase-PI	
Given:	A graph-restricted weighted voting game $(\mathcal{G}, G)$ with players $N = \{1, \ldots, n\}$, a communication structure $G = (N, E)$, a distinguished player $p \in N$, and a positive integer $k \leq \|E\|$.
Question:	Can at most k edges $E' \subseteq E$, $E' \neq \emptyset$, be deleted from G such that for the new game $(\mathcal{G}, G_{\setminus E'})$, it holds that $$\mathrm{PI}((\mathcal{G}, G_{\setminus E'}), p) > \mathrm{PI}((\mathcal{G}, G), p)?$$

Again, we analogously define this problem for the goals of *decreasing* and *maintaining* a distinguished player's power in a given game by replacing ">" in the question by, respectively, "<" and "=" and denote these problems by Control-by-Deleting-Edges-to-Decrease-PI and Control-by-Deleting-Edges-to-Maintain-PI. For each of these problems, we also consider their restrictive variants where we are allowed to delete edges incident *only to the distinguished player p*, denoted by Control-by-Deleting-Edges-ODP-to-Increase-PI, or incident *only not incident to p*, denoted by Control-by-Deleting-Edges-ONDP-to-Increase-PI.

4.3 Discussion of the Restrictive Scenarios versus the General Case

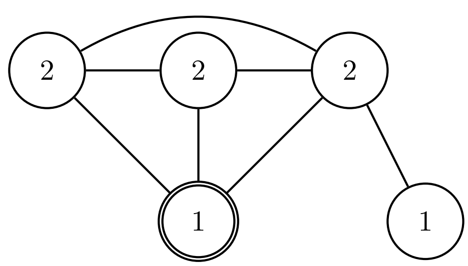
Recall from Theorem 3 that the problems Control-by-Adding-Edges-to-Increase-PI and Control-by-Adding-Edges-ODP-to-Increase-PI are identical for both PI = PBI and PI = SSI. One might be tempted to suspect that, analogously to Theorem 3, Control-by-Deleting-Edges-to-Decrease-PI and Control-by-Deleting-Edges-ODP-to-Decrease-PI were identical problems as well for PI $\in$ {PBI, SSI}. However, such an analogue of Theorem 3 is not known to hold. Let us give some explanation as to why attempting to prove this analogue fails. Intuitively, by adding edges to a communication graph or deleting edges from it, we only change the connectedness of a coalition. Adding edges (not incident to our distinguished player p) may make p pivotal for a coalition that was not winning when p joins before this control action because this coalition was not "sufficiently connected" in the graph, i.e., some of the players who after the addition have more connections were connected neither with p nor with the component connected with p, but now (after this control action) there is a connected component that makes p pivotal for such a coalition. However, *directly* connecting these players with p instead can give the same result of making p pivotal for this coalition, and essentially that is what we use in the proofs of Lemma 2 and Theorem 3. By contrast, when we delete edges incident to p, we may not be able to make a coalition (for which p was pivotal before the control action) "sufficiently unconnected" and that way to make it losing now—despite having a large enough total weight. However, somewhere else in the graph, there can be fewer connections which would achieve this effect within our deletion limit (and vice versa, it can sometimes be possible to decrease the power indices by removing edges incident to p but impossible to do so by removing other edges). In Case 1 of upcoming Example 8, we see an explicit example of this behavior where the distinguished player $p = 1$ has sufficiently many edges to ensure that all coalitions counted for p's power indices always are connected, no matter how many edges incident to p we delete.

In Example 5 from Section 3.3, we have seen counterexamples showing that an analogue of Theorem 3 does not exist in certain situations when we add edges to a communication graph. We now focus on deleting edges from a communication graph when discussing the restrictive scenarios versus the general case for our problems.

Example 8. As in Example 5, we refer by "index" to either the probabilistic Penrose–Banzhaf or the Shapley–Shubik power index in this example.

1. It is not true that if we can decrease the indices by removing k edges that are not incident to a distinguished player, we can also do this by only deleting up to k edges incident to him or her.

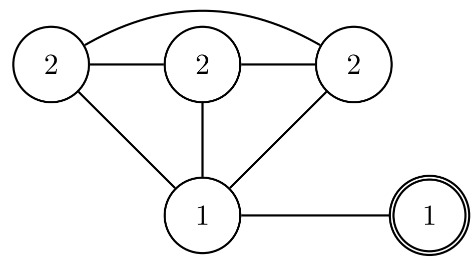
   Consider the game $(\mathcal{G}, G)$ with $\mathcal{G} = (1, 2, 2, 2, 1; 8)$, distinguished player 1, and graph G, where the first four players form a complete subgraph and player 5 is connected only with player 4:

   Assume that we can remove only one edge from the graph. Player 1 is pivotal for the coalition $\{2, 3, 4, 5\}$, so 1's indices are greater than 0. If we delete the edge $\{4, 5\}$, there will be no winning coalition in the game and 1's indices will decrease to 0. But removing any edge incident to the distinguished player 1 will not change the indices because 1 will still be pivotal for the same coalition.

2. It is not true that if we can decrease the indices by removing k edges incident to a distinguished player, we can also do this by only deleting up to k edges which are not incident to him or her.

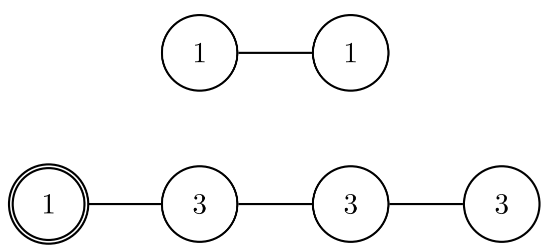
   Consider the game $(\mathcal{G}, G)$ with $\mathcal{G} = (1, 2, 2, 2, 1; 8)$, distinguished player 1, and graph G, where the first four players form a complete subgraph and player 5 is only connected with player 1:



Player 1 is pivotal for the coalition $\{2, 3, 4, 5\}$, so 1's indices are greater than 0. If we delete the edge $\{1, 5\}$, we will isolate the distinguished player and 1's indices will decrease to 0. But removing any other edge will not change the set of coalitions for which 1 is pivotal.

3. It is not true that if we can maintain the indices by removing $k \geq 1$ edges that are not incident to a distinguished player, we can also do this by only deleting up to k (but at least one) edges incident to him or her.

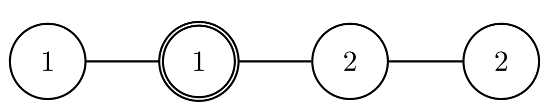
   Consider the game $(\mathcal{G}, G)$ with $\mathcal{G} = (1, 3, 3, 3, 1, 1; 10)$, distinguished player 1, and graph G, where the first four players form a path in which player 1 is a leaf and players 5 and 6 are only connected with each other:



   Player 1 is pivotal for the coalitions $\{2, 3, 4\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 4, 6\}$, and $\{2, 3, 4, 5, 6\}$, i.e., 1's indices are greater than 0. If we remove the edge $\{5, 6\}$, 1 will be pivotal for the same coalitions, so 1's indices will not change. But if we delete the edge $\{1, 2\}$, 1's indices will decrease to 0 because there will be no winning coalition anymore in the new game.

4. It is not true that if we can maintain the indices by removing $k \geq 1$ edges incident to a distinguished player, we can also do this by only deleting up to k (but at least one) edges that are not incident to him or her.

   Consider the game $(\mathcal{G}, G)$ with $\mathcal{G} = (1, 1, 2, 2; 5)$, distinguished player 1, and graph G, where the players form a path in which player 1 is sitting between players 2 and 3 (so 1 is not a leaf) and player 4 is connected with player 3:



Player 1 is pivotal for the coalitions $\{3,4\}$ and $\{2,3,4\}$, so 1's indices are

$$\beta((\mathcal{G},G),1) = \frac{1}{4} \quad \text{and} \quad \varphi((\mathcal{G},G),1) = \frac{1}{3}.$$

If we remove the edge $\{1,2\}$, player 1 will stay pivotal for the same coalitions and 1's indices will not change. But if we delete the edge $\{3,4\}$, there is no winning coalition in the new game anymore, so the indices of the distinguished player decrease to 0.

4.4 Complexity of Control by Deleting Edges

For control by deleting edges to increase or to decrease a given player's power, we can also show PP-hardness. Note that these problems are PP-hard to solve even if we delete only a single edge from the given communication graph.

Theorem 9. *For* PI $\in$ {PBI, SSI}*,* Control-by-Deleting-Edges-ONDP-to-Increase-PI*,* Control-by-Deleting-Edges-ONDP-to-Decrease-PI*,* Control-by-Deleting-Edges-to-Increase-PI*, and* Control-by-Deleting-Edges-to-Decrease-PI *are* PP*-hard.*

Proof. We show PP-hardness of Control-by-Deleting-Edges-ONDP-to-Decrease-PI and Control-by-Deleting-Edges-to-Decrease-PI by means of a reduction from Compare-#SubsetSum-RR. Note that PP-hardness of Control-by-Deleting-Edges-ONDP-to-Increase-PI and Control-by-Deleting-Edges-to-Increase-PI can be proven analogously with the same reduction when starting from the problem Compare-#SubsetSum-ЯЯ (recall the definition of both these problems from Section 2.3).

Let $A = (a_1, \ldots, a_n)$ be a given instance of Compare-#SubsetSum-RR with $\alpha = \sum_{i=1}^{n} a_i$ and $\max_{1 \le i \le n} a_i < \frac{\alpha}{2} - 1$. Let

$$\xi_1 = \#\text{SubsetSum}\left(A, \frac{\alpha}{2} - 1\right) \quad \text{and}$$
$$\xi_2 = \#\text{SubsetSum}\left(A, \frac{\alpha}{2} - 2\right)$$

and construct the control problem instance consisting of a graph-restricted weighted voting game

$$\mathcal{G} = \left(1, a_1, \ldots, a_n, 2\alpha, 1; \frac{5\alpha}{2}\right)$$

with $n+3$ players, distinguished player 1, and the communication graph $G = (N, E)$ shown in Figure 3: All players except the weight-1 player $n+3$ form a complete subgraph and player $n+3$ is connected with the weight-2α player $n+2$ by an edge called x.

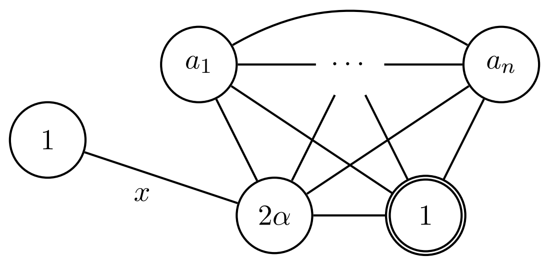


Figure 3: Communication structure of the game $(\mathcal{G}, G)$ from the proof of Theorem 9

First note that to achieve the quota, a coalition has to contain player $n+2$ because all other players together have a total weight of only $\alpha + 2 < 2\alpha + 1 \leq \frac{5}{2}\alpha$. Therefore, player 1 is pivotal for all coalitions with weight $\frac{5}{2}\alpha - 1$, since any coalition with a smaller weight loses even with the distinguished player and any coalition with a weight at least the quota is connected without 1, so it wins with and without 1. Hence, the distinguished player is pivotal for the coalitions containing either

1. player $n+2$, player $n+3$, and the rest of its players have a total weight of $\frac{\alpha}{2} - 2$, or
2. player $n+2$, but without player $n+3$, and the rest of its players have a total weight of $\frac{\alpha}{2} - 1$.

Let t be the size of each solution according to Lemma 1. Player 1's probabilistic Penrose–Banzhaf index in this game is

$$\beta((\mathcal{G}, G), 1) = \frac{\xi_1 + \xi_2}{2^{n+2}},$$

and player 1's Shapley–Shubik index in it is

$$\varphi((\mathcal{G}, G), 1) = \xi_1 \frac{(t+1)!(n+1-t)!}{(n+3)!} + \xi_2 \frac{(t+2)!(n-t)!}{(n+3)!}.$$

Let the deletion limit be $k = 1$. We will first show that for the distinguished player's probabilistic Penrose–Banzhaf index, we have

$$(\exists e \in E)\ [\beta((\mathcal{G}, G_{\setminus\{e\}}), 1) - \beta((\mathcal{G}, G), 1) < 0] \iff \xi_1 < \xi_2.$$

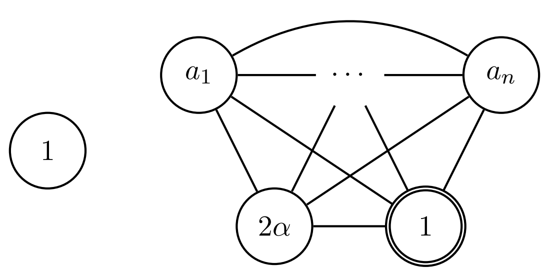


Figure 4: Communication structure of the game $(\mathcal{G}, G_{\setminus\{x\}})$ from the proof of Theorem 9

From right to left, assume that $\xi_1 < \xi_2$. Then, after deleting the edge x (as shown in Figure 4), player 1 will stop being pivotal for coalitions of weight $\frac{5}{2}\alpha - 1$ with player $n+3$, but it will become pivotal for coalitions with weight $\frac{5}{2}\alpha$ containing player $n+3$, i.e., having a connected component of weight $\frac{5}{2}\alpha - 1$, so the probabilistic Penrose–Banzhaf index of player 1 will decrease to

$$\beta((\mathcal{G}, G_{\setminus\{x\}}), 1) = \frac{\xi_1}{2^{n+1}} = \frac{2\xi_1}{2^{n+2}} < \frac{\xi_1 + \xi_2}{2^{n+2}} = \beta(\mathcal{G}, 1).$$

From left to right, assume that $\xi_1 \geq \xi_2$. We have

$$\beta((\mathcal{G}, G_{\setminus\{x\}}), 1) = \frac{\xi_1}{2^{n+1}} = \frac{2\xi_1}{2^{n+2}} \geq \frac{\xi_1 + \xi_2}{2^{n+2}} = \beta(\mathcal{G}, 1),$$

so the probabilistic Penrose–Banzhaf index of player 1 does not decrease by deleting x. If we remove any other edge, player 1's probabilistic Penrose–Banzhaf index will not change at all (note that each coalition for which 1 is pivotal contains at least three players).

Similarly, we now show that for the distinguished player's Shapley–Shubik index, we have

$$(\exists e \in E)\ \varphi((\mathcal{G}, G), 1) - \varphi((\mathcal{G}, G_{\setminus\{e\}}), 1) > 0 \iff \xi_1 < \xi_2.$$

From right to left, assume that $\xi_1 < \xi_2$. After deleting the edge x, the Shapley–Shubik index of the distinguished player will decrease to

$$\begin{aligned}
\varphi((\mathcal{G}, G_{\setminus\{x\}}), 1) &= \xi_1 \frac{(t+1)!(n-t)!}{(n+2)!} \\
&= \xi_1 \frac{(t+1)!(n+1-t)!}{(n+3)!} \frac{n+3}{n+1-t} \\
&= \xi_1 \frac{(t+1)!(n+1-t)!}{(n+3)!} + \xi_1 \frac{(t+2)!(n-t)!}{(n+3)!} \\
&< \varphi((\mathcal{G}, G), 1).
\end{aligned}$$

From left to right, assume that $\xi_1 \geq \xi_2$. We have

$$\begin{aligned}
\varphi((\mathcal{G}, G_{\setminus\{x\}}), 1) &= \xi_1 \frac{(t+1)!(n+1-t)!}{(n+3)!} + \xi_1 \frac{(t+2)!(n-t)!}{(n+3)!} \\
&\geq \varphi((\mathcal{G}, G), 1),
\end{aligned}$$

so the distinguished player's Shapley–Shubik index does not decrease. If we remove any other edge, the distinguished player's Shapley–Shubik index will not change at all. □

Next, we turn to the goal of maintaining the distinguished player's power in a graph-restricted weighted voting game, and we will show coNP-hardness in all three scenarios. We start with the probabilistic Penrose–Banzhaf power index and both the general case and the restriction where only edges not incident to the distinguished player can be deleted.

Theorem 10. *The problems* Control-by-Deleting-Edges-ONDP-to-Maintain-PBI *and* Control-by-Deleting-Edges-to-Maintain-PBI *are* coNP*-hard.*

Proof. We show coNP-hardness of this control problem by means of a reduction from the complement of the Partition problem. Let $(a_1, \dots, a_n)$ be a Partition instance with $n > 1$, let $\alpha = \sum_{i=1}^{n} a_i$, and let $\xi = \#\text{Partition}((a_1, \dots, a_n))$ denote the number of its solutions.

Construct the control problem instance by defining a graph-restricted weighted voting game $(\mathcal{G}, G)$ with

$$\mathcal{G} = (1, 4a_1, \dots, 4a_n, 2, 1, 2\alpha - 2, 2\alpha - 2, 5\alpha; 9\alpha + 1)$$

consisting of $n+6$ players, distinguished player 1 (with weight 1), deletion limit $k = 1$, and the communication structure $G = (N, E)$ shown in Figure 5.

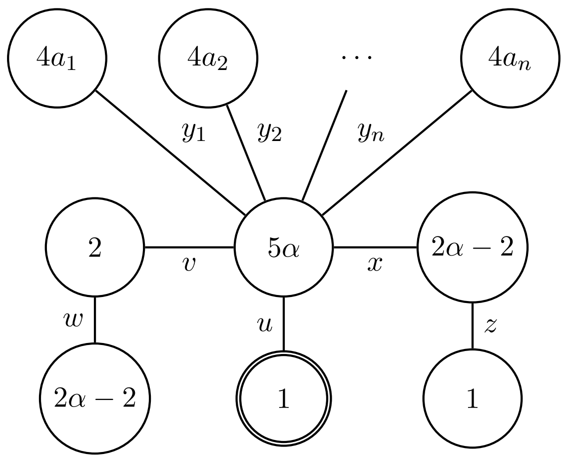


Figure 5: Communication structure of the game $(\mathcal{G}, G)$ from the proof of Theorem 10

Note that all players but $n+6$ (with weight 5α) together have a total weight smaller than the quota; therefore, all winning coalitions have to contain player $n+6$. We also need at least one player from $\{2, \dots, n+1\}$ (the i-th of which has weight $4a_i$), because all other remaining players (without the distinguished player, i.e., all players in $\{n+2, \dots, n+6\}$), have a total weight of $9\alpha - 1$. Next, note that player 1 can be pivotal only for connected coalitions with weight exactly 9α (since the connectedness of any coalition is independent from our player) and for these coalitions with additional players unconnected with them. There are only two possibilities for not connected coalitions containing player $n + 6$: (a) coalitions containing player $n + 3$ (with weight 1) but not containing the other player incident with edge z (with weight $2\alpha - 2$), and (b) coalitions formed by, among others, the weight-$(2\alpha - 2)$ player incident with edge w but not containing the other player incident with this edge (with weight 2). Hence, player 1 could be pivotal for the coalitions containing player $n + 6$ and

- all players from $\{2, \dots, n + 1\}$, and none, either one, or both of the weight-$(2\alpha - 2)$ player incident with edge w and the weight-1 player incident with edge z; note that the two latter players are not connected with the rest of the coalition;
- player $n + 2$ (with weight 2), one of the players with weight $2\alpha - 2$, and some players from $\{2, \dots, n+1\}$ with a total weight of 2α; if the player incident with w is contained

in the coalition, player 1 can be pivotal also for the coalition created from it by adding weight-1 player $n+3$.

We now show that

$$(\exists e \in E)\ [\beta((\mathcal{G}, G_{\setminus\{e\}}), 1) - \beta((\mathcal{G}, G), 1) = 0] \iff \xi = 0.$$

From right to left, assume that $\xi = 0$. Then the only connected coalition having weight 9α is $\{2, \ldots, n+1, n+6\}$. As mentioned earlier, player 1 is pivotal for this coalition and is also pivotal for the coalitions created from it by adding either one or both of the weight-$(2\alpha - 2)$ player incident with edge w and the weight-1 player incident with edge z (i.e., player $n+3$). It follows that

$$\beta((\mathcal{G}, G), 1) = \frac{1}{2^{n+3}}.$$

If we delete either the edge w or the edge z, the probabilistic Penrose–Banzhaf index will not change—player 1 will stay pivotal for exactly the same coalitions as before and will not become pivotal for any other coalition; therefore, it is possible to maintain the probabilistic Penrose–Banzhaf power index if there is no solution of Partition$((a_1, \ldots, a_n))$.

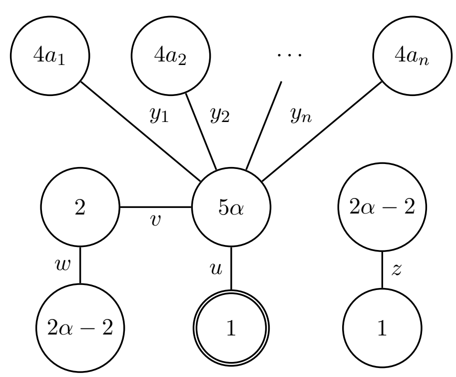
From left to right, assume now that $\xi > 0$. Then

$$\beta((\mathcal{G}, G), 1) = \frac{1}{2^{n+3}} + \frac{\xi}{2^{n+5}} + \frac{\xi}{2^{n+4}} = \frac{4+3\xi}{2^{n+5}}.$$

Consider the following cases.

Case 1: If we remove edge u, the probabilistic Penrose–Banzhaf index will decrease to 0 because we will isolate our distinguished player.

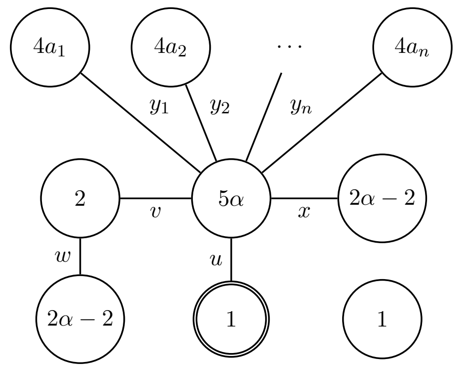
Case 2: If we delete the edge x:



the probabilistic Penrose–Banzhaf index of player 1 will increase:

$$\beta((\mathcal{G}, G_{\setminus\{x\}}), 1) = \frac{1}{2^{n+2}} + \frac{\xi}{2^{n+3}} = \frac{2+\xi}{2^{n+3}} = \frac{8+4\xi}{2^{n+5}} > \beta((\mathcal{G}, G), 1).$$

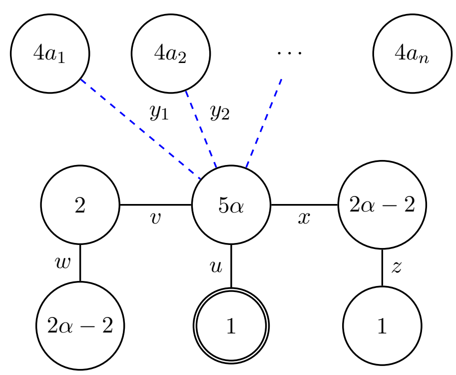
Case 3: If we delete the edge z:



the probabilistic Penrose–Banzhaf index of player 1 will also increase:

$$\beta((\mathcal{G}, G_{\setminus\{z\}}), 1) = \frac{1}{2^{n+3}} + \frac{2\xi}{2^{n+4}} = \frac{4+4\xi}{2^{n+5}} > \beta((\mathcal{G}, G), 1).$$

Case 4: If we delete any of the y_i-edges, $1 \leq i \leq n$, say y_n:

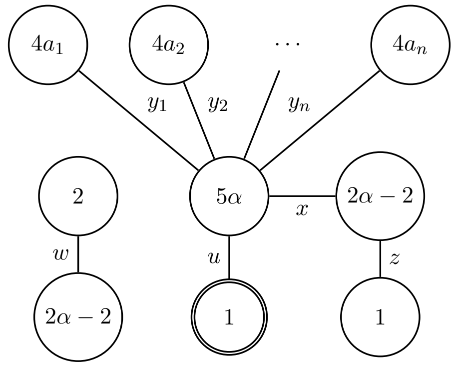


where the blue dashed edges represent other possible edges for removing, the probabilistic Penrose–Banzhaf index of player 1 will decrease:[8]

$$\beta((\mathcal{G}, G_{\setminus\{y_i\}}), 1) = \frac{\xi}{2} \cdot \frac{1}{2^{n+4}} + \frac{\xi}{2} \cdot \frac{1}{2^{n+3}} = \frac{3\xi}{2^{n+5}} < \beta((\mathcal{G}, G), 1).$$

Case 5: If we delete the edge v:

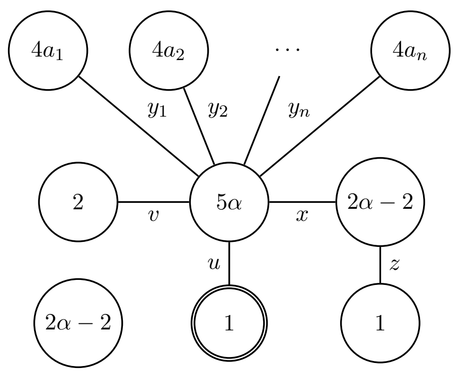
8. Note that for each solution $A' \subseteq A = \{1, \ldots, n\}$ of Partition$((a_1, \ldots, a_n))$, its complement $\overline{A'} = A \setminus A'$ is also a solution. Hence, the number of solutions is always an even number and each element a_i is contained in exactly half of them—either in A' or its complement $\overline{A'}$.



the probabilistic Penrose–Banzhaf index of player 1 will decrease, too—the player with weight 2 is not connected with player $n+6$, so player 1 is pivotal only for $\{2,\ldots,n+1\} \cup \{n+6\}$ among the connected coalitions:

$$\beta((\mathcal{G}, G_{\setminus\{v\}}), 1) = \frac{1}{2^{n+2}} = \frac{8}{2^{n+5}} \leq \frac{4+2\xi}{2^{n+5}} < \beta((\mathcal{G}, G), 1).$$

Case 6: Finally, if we delete the edge w:



the probabilistic Penrose–Banzhaf index of player 1 will decrease as well:

$$\beta((\mathcal{G}, G_{\setminus\{w\}}), 1) = \frac{1}{2^{n+3}} + \frac{\xi}{2^{n+4}} = \frac{2+\xi}{2^{n+4}} = \frac{4+2\xi}{2^{n+5}} < \beta((\mathcal{G}, G), 1).$$

Therefore, control by deleting edges to maintain a distinguished player's probabilistic Penrose–Banzhaf index is coNP-hard. □

Next, we consider the same problems for the Shapley–Shubik power index and obtain again coNP-hardness by another reduction the proof of correctness of which requires a significantly more detailed argumentation.

Theorem 11. *The problems* Control-by-Deleting-Edges-ONDP-to-Maintain-SSI *and* Control-by-Deleting-Edges-to-Maintain-SSI *are* coNP*-hard.*

Proof. We prove coNP-hardness of our control problems by providing a reduction from the complement of the SubsetSum problem, which is well-known to be NP-complete. Let $((a_1, \ldots, a_n), q)$ be an instance of SubsetSum, where we may assume, without loss of generality, that $n \geq 9$. Let $\alpha = \sum_{i=1}^{n} a_i$ and let $\xi = \#\text{SubsetSum}((a_1, \ldots, a_n), q)$ be the number of its solutions; from Sperner's theorem (Sperner, 1928),[9] we know that $\xi \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Using Lemma 1, we can assume that each solution has the same size, namely $\frac{2}{3}n$, and therefore, n is divisible by 3. From this SubsetSum instance $((a_1, \ldots, a_n), q)$, we construct our control problem instance as follows.

Set the values y_1, y_2, and z as follows:

$$\begin{aligned} y_1 &= n + 1, \\ y_2 &= 2n + 2, \text{ and} \\ z &= 2 \cdot \left(\alpha \cdot 10^\ell + n + y_1 + y_2\right) + 1, \end{aligned}$$

where the positive integer ℓ is chosen such that

$$10^\ell > n + y_1 + y_2 = 4n + 3.$$

Further, let $a_i' = a_i \cdot 10^\ell$ and $a_i'' = z - a_i \cdot 10^\ell - \lfloor \frac{n}{2} \rfloor$ for $i \in \{1, \ldots, n\}$, and let $b_1 = z - q \cdot 10^\ell - y_1$ and $b_2 = z - q \cdot 10^\ell - y_2$. Define the following graph-restricted weighted voting game $(\mathcal{G}, G)$:

$$\mathcal{G} = \left(1, a_1', \ldots, a_n', a_1'', \ldots, a_n'', \underbrace{1, \ldots, 1}_{n}, b_1, y_1, b_2, y_2, 10^t; 10^t + z + 1\right)$$

with $3n + 6$ players the last of which is heavier than the total weight of all other players (i.e., if w_i is the weight of player i, then t is chosen such that $w_{3n+6} = 10^t > \sum_{i=1}^{3n+5} w_i$). Our distinguished player is 1, and we define the communication structure $G = (N, E)$, as shown in Figure 6.

Let us analyze for which coalitions from $\{2, \ldots, 3n+6\}$ our distinguished player is pivotal in $(\mathcal{G}, G)$. Player 1 is pivotal for coalitions with weight at least $10^t + z$, and since player $3n + 6$'s weight is greater than the total sum of all other players' weights and is exactly 10^t, this player has to be in each of these coalitions. Note that any coalition with and without player 1 consists of the same number of connected components, i.e., any connected winning coalition from $\mathcal{WC}_1$ with weight greater than the quota is still connected without player 1 and has weight at least the quota. Moreover, since 1 is connected only with player $3n + 6$, deleting any edge will not make 1 the only element for any connected coalition keeping it in the form of one connected component, i.e., after the deletion, a coalition $S \subseteq N \setminus \{1\}$ such that $S \cup \{1\}$ is connected is connected itself—independently of which edge was removed. Therefore, player 1 is pivotal for the coalitions $S \in \mathcal{WC}_1$ with weight exactly $10^t + z + 1$ (i.e., $w_{S \setminus \{1\}} = 10^t + z$).

9. Note that all elements in the list $(a_1, \ldots, a_n)$ are positive integers. Therefore, calling the family of its sublists whose elements sum up to the same given value q a *Sperner family*, none of the lists in the Sperner family can be a strict sublist of any other list from this Sperner family. That is, for any of the lists in the Sperner family, we have that the elements of any of its strict sublists sum up to a value strictly smaller than q, i.e., such a strict sublist cannot belong to the Sperner family.

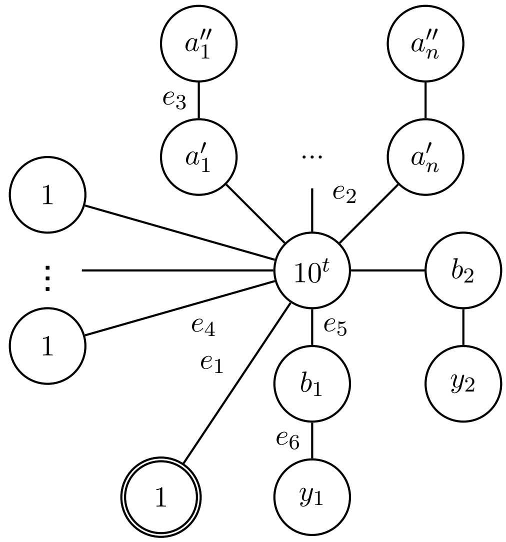


Figure 6: Communication structure of the game $(\mathcal{G}, G)$ from the proof of Theorem 11

Let us focus now on the summand z. Note that any two players with weights from $\{a''_1, \ldots, a''_n, b_1, b_2\}$ have a total weight larger than z, so no two such players can be in any connected coalition player 1 is pivotal for. At the same time, all other players (i.e., players with weights from $\{a'_1, \ldots, a'_n, y_1, y_2, 1\}$) have total weight smaller than z. Therefore, any connected coalition player 1 is pivotal for has to contain *exactly* one player of the former type (i.e., *exactly* one player with weight from $\{a''_1, \ldots, a''_n, b_1, b_2\}$).

Let $S \subseteq N \setminus \{1\}$ be a connected coalition for which player 1 is pivotal, i.e., $S \cup \{1\} \in \mathcal{PWC}_1$. Then:

(a) If S contains some player with weight a''_i, $i \in \{1, \ldots, n\}$, it has to contain the player connecting this weight-a''_i player with player $3n + 6$ and $\lfloor \frac{n}{2} \rfloor$ weight-1 players. For each such coalition S, $|S| = \lfloor \frac{n}{2} \rfloor + 3$.

(b) If S contains some player with weight b_i, $i \in \{1, 2\}$, it has to also contain the player with weight y_i and a subset of $\{2, \ldots, n + 1\}$ with total weight $q \cdot 10^\ell$. For each such coalition S, $|S| = \frac{2}{3}n + 3$.

Any other connected coalition (without the distinguished player) has total weight either smaller or larger than $10^t + z$: If the weight is smaller, 1 cannot be pivotal for it because the total weight is too small for a winning coalition, and if the weight is larger, i.e., at least the quota, 1 does not change anything since the coalition is already connected. Any other coalition with total weight exactly $10^t + z$ is not connected. For example, if there exist i and j with $i \neq j$ but $a'_i = a'_j$, then a coalition $C = \{1, i + 1, n + 1 + j, 3n + 6\}$ has a total weight of $10^t + z + 1$, but player $n + 1 + j$ is not connected with the others, so the coalition loses and 1 cannot be pivotal for $C \setminus \{1\}$.

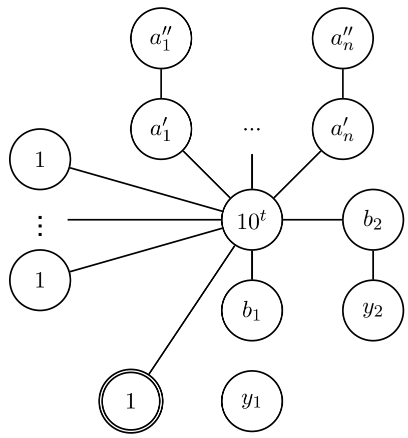
Let the deletion limit be $k = 1$. We will prove that

$$(\exists e \in E)\ [\varphi((\mathcal{G}, G_{\setminus\{e\}}), 1) - \varphi((\mathcal{G}, G), 1) = 0] \iff \xi = 0.$$

From right to left, assume that $\xi = 0$. Then player 1 is pivotal only for the coalitions described in case (a) above and

$$\varphi((\mathcal{G}, G), 1) = n\binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!},$$

and after removing, e.g., the edge e_6:



the Shapley–Shubik index of the distinguished player does not change.

From left to right, assume that $\xi > 0$. Then

$$\varphi((\mathcal{G}, G), 1) = n\binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} + 2\xi\frac{(\frac{2}{3}n + 3)!(2n+1)!}{(\frac{8}{3}n + 5)!}. \tag{3}$$

Before we analyze the change of player 1's Shapley–Shubik index after removing any edge from the communication graph, let us first compare the fractions in (3):

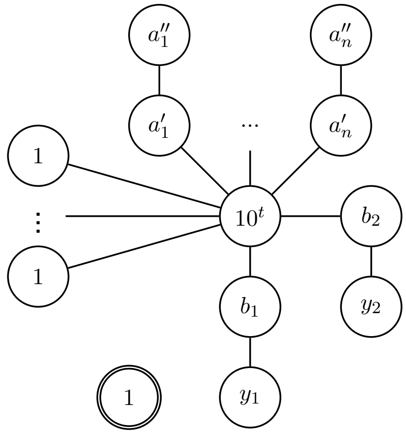
$$\begin{aligned}
& \frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} - \frac{(\frac{2}{3}n + 3)!(2n+1)!}{(\frac{8}{3}n + 5)!} \\
= & \frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!(\frac{8}{3}n + 5)!}{(2n+5)!(\frac{8}{3}n + 5)!} - \frac{(\frac{2}{3}n + 3)!(2n+1)!(2n+5)!}{(2n+5)!(\frac{8}{3}n + 5)!} \\
= & \frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!(2n+1)!}{(2n+5)!(\frac{8}{3}n + 5)!} \\
& \cdot \left((2n+2) \cdots \left(\frac{8}{3}n + 5\right) - \left(\left\lfloor \frac{n}{2} \right\rfloor + 4\right) \cdots \left(\frac{2}{3}n + 3\right) \left(\left\lceil \frac{n}{2} \right\rceil + n + 2\right) \cdots (2n+5) \right) \\
> & \ 0,
\end{aligned}$$

so

$$\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} > \frac{(\frac{2}{3}n + 3)!(2n+1)!}{(\frac{8}{3}n + 5)!}. \tag{4}$$

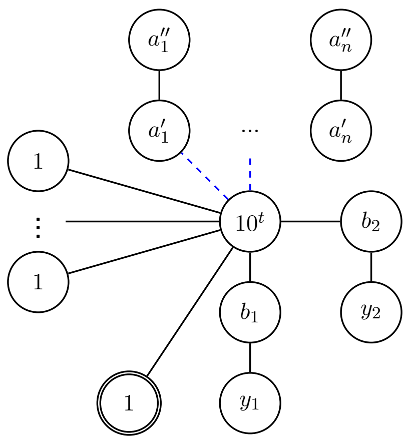
Now, consider the following cases (blue dashed edges are other possible edges that could be removed instead).

Case 1: If we delete the edge e_1:



the Shapley–Shubik index of player 1 will decrease to 0.

Case 2: If we remove an edge of type e_2 (i.e., one of the edges between player $3n+6$ and any player of weight a_i'), for instance:



then let $\xi_i = \#\textsc{SubsetSum}\,((a_1, \ldots, a_{i-1}, a_{i+1}, a_n), q)$ be the number of solutions of our SubsetSum instance without the i-th element. Note that $\xi_i \leq \xi$; no coalition containing the player with weight a_i' and player 1 is connected now (i.e., using Theorem 1, we do not count them in player 1's Shapley–Shubik index anymore), and each coalition for which player 1 remains pivotal has one neighbor less. Therefore, we have

$$
\begin{aligned}
&\varphi((\mathcal{G}, G_{\setminus\{e_2\}}), 1) \\
&= (n-1)\binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n)!}{(2n+4)!} + 2\xi_i \frac{(\frac{2}{3}n+3)!(2n)!}{(\frac{8}{3}n+4)!} \\
&= (n-1)\frac{2n+5}{\lceil \frac{n}{2} \rceil + n + 1} \cdot \binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} + 2\xi_i \frac{(\frac{2}{3}n+3)!(2n)!}{(\frac{8}{3}n+4)!}.
\end{aligned}
$$

First, we check by how much the first summand of the new Shapley–Shubik index of player 1 (i.e., the summand defined by the coalitions containing players with weights a''_j) is greater than the first summand of the old one in (3):

$$\begin{aligned}(n-1)\frac{2n+5}{\lceil \frac{n}{2}\rceil + n + 1} &= \frac{2n^2+3n-5}{\lceil \frac{n}{2}\rceil + n + 1}\\ &= \frac{n(\lceil \frac{n}{2}\rceil + n + 1) + n\lfloor \frac{n}{2}\rfloor + 2n - 5}{\lceil \frac{n}{2}\rceil + n + 1}\\ &= n + \frac{n\lfloor \frac{n}{2}\rfloor + 2n - 5}{\lceil \frac{n}{2}\rceil + n + 1},\end{aligned}$$

and therefore,

$$\begin{aligned}&(n-1)\binom{n}{\lfloor \frac{n}{2}\rfloor}\frac{(\lfloor \frac{n}{2}\rfloor + 3)!(\lceil \frac{n}{2}\rceil + n)!}{(2n+4)!} - n\binom{n}{\lfloor \frac{n}{2}\rfloor}\frac{(\lfloor \frac{n}{2}\rfloor + 3)!(\lceil \frac{n}{2}\rceil + n + 1)!}{(2n+5)!}\\ = &\frac{n\lfloor \frac{n}{2}\rfloor + 2n - 5}{\lceil \frac{n}{2}\rceil + n + 1}\binom{n}{\lfloor \frac{n}{2}\rfloor}\frac{(\lfloor \frac{n}{2}\rfloor + 3)!(\lceil \frac{n}{2}\rceil + n + 1)!}{(2n+5)!}.\end{aligned}$$

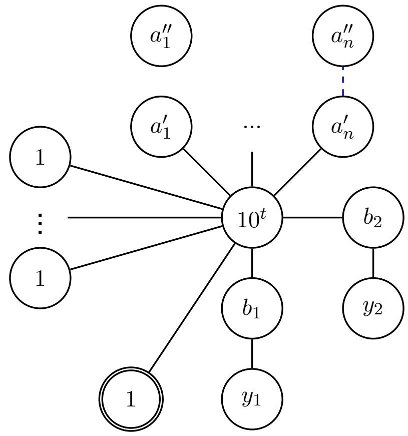
So, the value of the first summand of $\varphi((\mathcal{G}, G_{\setminus\{e_2\}}), 1)$ alone is greater than $\varphi((\mathcal{G}, G), 1)$ for $n \geq 6$ since:

$$\begin{aligned}&\frac{n\lfloor \frac{n}{2}\rfloor + 2n - 5}{\lceil \frac{n}{2}\rceil + n + 1}\binom{n}{\lfloor \frac{n}{2}\rfloor}\frac{(\lfloor \frac{n}{2}\rfloor + 3)!(\lceil \frac{n}{2}\rceil + n + 1)!}{(2n+5)!}\\ > \ &\frac{n\lfloor \frac{n}{2}\rfloor + 2n - 5}{\lceil \frac{n}{2}\rceil + n + 1}\binom{n}{\lfloor \frac{n}{2}\rfloor}\frac{(\frac{2}{3}n+3)!(2n)!}{(\frac{8}{3}n+5)!} \quad \text{by (4)}\\ \geq \ &\frac{n\lfloor \frac{n}{2}\rfloor + 2n - 5}{\lceil \frac{n}{2}\rceil + n + 1}\cdot \xi \frac{(\frac{2}{3}n+3)!(2n)!}{(\frac{8}{3}n+5)!} \quad \text{by Sperner's theorem}\\ > \ &2\xi \frac{(\frac{2}{3}n+3)!(2n)!}{(\frac{8}{3}n+5)!} \quad \text{for } n \geq 6.\end{aligned}$$

Consequently, the Shapley–Shubik index of player 1 increases independently of the value of ξ_i for $n \geq 6$, which concludes the current case as we have assumed that $n \geq 9$.[10]

Case 3: If we delete an edge of type e_3 (between a player of weight a'_i and a player of weight a''_i for some i), for instance:

10. We mention in passing that for $n = 3$ (which is the only positive $n < 6$ that is divisible by 3), we also have $\varphi((\mathcal{G}, G), 1) \neq \varphi((\mathcal{G}, G_{\setminus\{e_2\}}), 1)$ by straightforward calculations.



then in the resulting new game, no coalition containing the player with weight a_i'' and player 1 is connected, and in the coalitions counted in player 1's Shapley–Shubik index, the weight-a_i' player has one neighbor less. Therefore, we have

$$\begin{aligned}\varphi((\mathcal{G}, G_{\setminus\{e_3\}}), 1) \;=\; & (n-1)\binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} \\ & + 2\xi_i \frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!} + 2(\xi - \xi_i)\frac{(\frac{2}{3}n+3)!(2n)!}{(\frac{8}{3}n+4)!},\end{aligned}$$

where ξ_i is the number of subsequences without a_i, and $\xi - \xi_i$ is the number of subsequences containing a_i. Next,

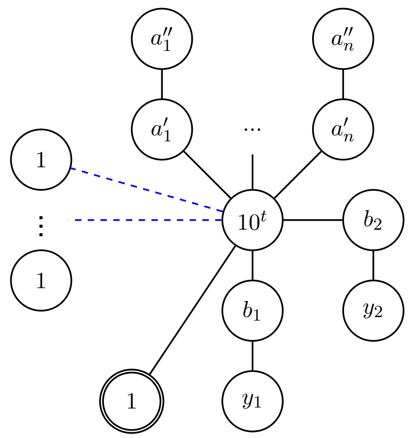
$$\begin{aligned}\varphi((\mathcal{G}, G_{\setminus\{e_3\}}), 1) = & (n-1)\binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} \\ & + 2\Big(\xi_i + \frac{\frac{8}{3}n+5}{2n+1}(\xi - \xi_i)\Big)\frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!} \\ = & \; n\binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} + 2\xi\frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!} \\ & + \frac{\frac{2}{3}n+4}{2n+1}(\xi - \xi_i)\frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!} \\ & - \binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!}.\end{aligned}$$

By (4), Sperner's Theorem, and since $\frac{\frac{2}{3}n+4}{2n+1} < 1$ for $n \geq 3$, we have

$$\frac{\frac{2}{3}n+4}{2n+1}(\xi - \xi_i)\frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!} < \binom{n}{\lfloor \frac{n}{2} \rfloor}\frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!}.$$

Therefore, the Shapley–Shubik index of player 1 decreases, i.e., does not remain the same.

Case 4: If we remove an edge of type e_4 (between the player of weight 10^t and a player of weight 1 that is not the distinguished player), for instance:



there are fewer possibilities to create a coalition containing a weight-a''_i player, $i \in \{1, \ldots, n\}$, for which player 1 is pivotal, and each coalition counted in the Shapley–Shubik index of player 1 in the new game has one neighbor less than in the old game. Therefore,

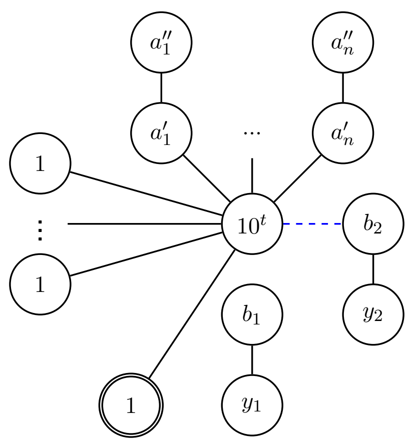
$$\begin{aligned}
\varphi((\mathcal{G}, G_{\setminus\{e_4\}}), 1) &= n\binom{n-1}{\lfloor \frac{n}{2} \rfloor} \frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n)!}{(2n+4)!} + 2\xi \frac{(\frac{2}{3}n+3)!(2n)!}{(\frac{8}{3}n+4)!} \\
&= n\binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{\lceil \frac{n}{2} \rceil}{n} \frac{2n+5}{\lceil \frac{n}{2} \rceil + n + 1} \frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} \\
&\quad + 2\xi \frac{\frac{8}{3}n+5}{2n+1} \frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!} \\
&= n\binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} + 2\xi \frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!} \\
&\quad - n\binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{n^2 - n\lceil \frac{n}{2} \rceil + n - 5\lceil \frac{n}{2} \rceil}{n^2 + n\lceil \frac{n}{2} \rceil + n} \frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} \\
&\quad + 2\xi \frac{\frac{2}{3}n+4}{2n+1} \frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!} \\
&= \varphi((\mathcal{G}, G), 1) \\
&\quad - \frac{n^2 - n\lceil \frac{n}{2} \rceil + n - 5\lceil \frac{n}{2} \rceil}{n + \lceil \frac{n}{2} \rceil + 1} \binom{n}{\lfloor \frac{n}{2} \rfloor} \frac{(\lfloor \frac{n}{2} \rfloor + 3)!(\lceil \frac{n}{2} \rceil + n + 1)!}{(2n+5)!} \\
&\quad + \frac{\frac{4}{3}n+8}{2n+1} \xi \frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!}.
\end{aligned}$$

For $n \geq 9$, we have

$$\begin{aligned}
&\frac{n^2 - n\lceil\frac{n}{2}\rceil + n - 5\lceil\frac{n}{2}\rceil}{n + \lceil\frac{n}{2}\rceil + 1} - \frac{\frac{4}{3}n + 8}{2n+1} \\
&= \frac{2n^3 - 2n^2\lceil\frac{n}{2}\rceil + \frac{5}{3}n^2 - \frac{37}{3}n\lceil\frac{n}{2}\rceil - \frac{25}{3}n - 13\lceil\frac{n}{2}\rceil - 8}{(2n+1)(n + \lceil\frac{n}{2}\rceil + 1)} \\
&\geq \frac{2n^3 - 2n^2\frac{n+1}{2} + \frac{5}{3}n^2 - \frac{37}{3}n\frac{n+1}{2} - \frac{25}{3}n - 13\frac{n+1}{2} - 8}{(2n+1)(n + \lceil\frac{n}{2}\rceil + 1)} \\
&= \frac{n^3 - \frac{33}{6}n^2 - 21n - \frac{29}{2}}{(2n+1)(n + \lceil\frac{n}{2}\rceil + 1)} > 0,
\end{aligned}$$

so the Shapley–Shubik index of player 1 will decrease for these n, which concludes the current case.[11]

Case 5: If we delete an edge of type e_5 (between player $3n+6$ and a player of weight either b_1 or b_2), for instance:



then there is no connected coalition containing these two players, and other coalitions counted in the Shapley–Shubik index of player 1 have one neighbor less, so

$$\begin{aligned}
\varphi((\mathcal{G}, G_{\setminus\{e_5\}}), 1) &= n\binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{(\lfloor\frac{n}{2}\rfloor + 3)!(\lceil\frac{n}{2}\rceil + n)!}{(2n+4)!} + \xi\frac{(\frac{2}{3}n+3)!(2n)!}{(\frac{8}{3}n+4)!} \\
&= n\binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{2n+5}{\lceil\frac{n}{2}\rceil + n + 1}\frac{(\lfloor\frac{n}{2}\rfloor + 3)!(\lceil\frac{n}{2}\rceil + n + 1)!}{(2n+5)!} \\
&\quad + \xi\frac{\frac{8}{3}n+5}{2n+1}\frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!}
\end{aligned}$$

11. Again, for $n = 3$ and $n = 6$ (the only positive $n < 9$ that are divisible by 3), we also have $\varphi((\mathcal{G}, G), 1) \neq \varphi((\mathcal{G}, G_{\setminus\{e_2\}}), 1)$ by straightforward calculations.

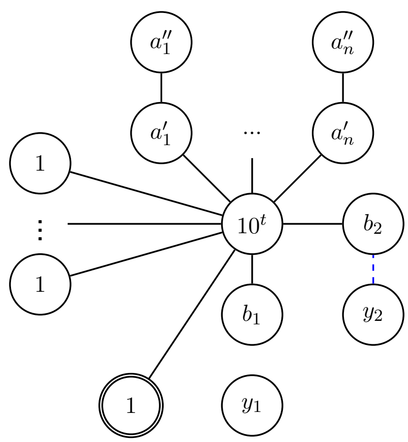
$$
\begin{aligned}
&= n\binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{(\lfloor\frac{n}{2}\rfloor+3)!(\lceil\frac{n}{2}\rceil+n+1)!}{(2n+5)!}+2\xi\frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!}\\
&\quad + n\binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{n-\lceil\frac{n}{2}\rceil+4}{\lceil\frac{n}{2}\rceil+n+1}\frac{(\lfloor\frac{n}{2}\rfloor+3)!(\lceil\frac{n}{2}\rceil+n+1)!}{(2n+5)!}\\
&\quad - \xi\frac{\frac{4}{3}n-3}{2n+1}\frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!}.
\end{aligned}
$$

Hence, by (4), Sperner's Theorem, and since

$$
n\frac{n-\lceil\frac{n}{2}\rceil+4}{\lceil\frac{n}{2}\rceil+n+1}\geq 1 \quad \text{and} \quad \frac{\frac{4}{3}n-3}{2n+1}<1,
$$

the Shapley–Shubik index of player 1 increases.

Case 6: Finally, if we delete an edge of type e_6 (either between the players of weights b_1 and y_1 or those of weights b_2 and y_2), for instance:



the Shapley–Shubik index of player 1 will change to

$$
\begin{aligned}
\varphi((\mathcal{G},G_{\setminus\{e_6\}}),1) &= n\binom{n}{\lfloor\frac{n}{2}\rfloor}\frac{(\lfloor\frac{n}{2}\rfloor+3)!(\lceil\frac{n}{2}\rceil+n+1)!}{(2n+5)!}+\xi\frac{(\frac{2}{3}n+3)!(2n+1)!}{(\frac{8}{3}n+5)!}\\
&< \varphi((\mathcal{G},G),1).
\end{aligned}
$$

In each of these cases, no matter which edge we deleted, the Shapley–Shubik power index of the distinguished player has changed, which completes the proof. □

Finally, we cover the case of control by only deleting edges incident to the distinguished player when our goal is to decrease his or her power index.[12] For this case, we will show NP-hardness for both power indices we consider. By essentially the same reduction, we additionally show coNP-hardness for control by deleting edges with the goal of maintaining

12. Recall from Theorem 8 that this case is trivial for the goal of increasing the distinguished player's power, i.e., it is never possible to increase this power by deleting edges incident to the distinguished player.

the distinguished player's power with respect to both these indices, no matter whether these deleted edges each are incident to this player, each are not incident to him or her, or whether they can be deleted anywhere in the graph.

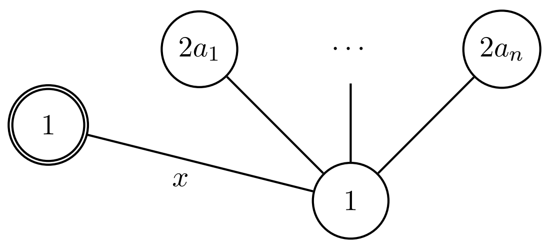
Theorem 12. *For* PI $\in$ {PBI, SSI}, Control-by-Deleting-Edges-ODP-to-Decrease-PI *is* NP-*hard and* Control-by-Deleting-Edges-ODP-to-Maintain-PI *is* coNP-*hard.*

Proof. We provide a reduction from either Partition or its complement to our control problems to show either their NP-hardness or coNP-hardness. Let $(a_1, \ldots, a_n)$ be a given instance of Partition (respectively, its complement), let $\alpha = \sum_{i=1}^{n} a_i$, and let $\xi = \#\text{Partition}(a_1, \ldots, a_n)$ be the number of its solutions.

Construct the graph-restricted weighted voting game $(\mathcal{G}, G)$:

$$\mathcal{G} = (1, 2a_1, \ldots, 2a_n, 1; \alpha + 2)$$

with $n + 2$ players, distinguished player 1, and the communication structure $G = (N, E)$ where player $n + 2$ is connected with all other players and there are no further edges:



Therefore, the only edge that can be removed is edge x.

Set the deletion limit to $k = 1$. We will prove that

$$\begin{aligned}
(\exists e \in E_1)\ [\beta((\mathcal{G}, G_{\setminus\{e\}}), 1) - \beta((\mathcal{G}, G), 1) < 0] &\iff \xi > 0,\\
(\exists e \in E_1)\ [\varphi((\mathcal{G}, G_{\setminus\{e\}}), 1) - \varphi((\mathcal{G}, G), 1) < 0] &\iff \xi > 0,\\
(\exists e \in E_1)\ [\beta((\mathcal{G}, G_{\setminus\{e\}}), 1) - \beta((\mathcal{G}, G), 1) = 0] &\iff \xi = 0, \text{ and}\\
(\exists e \in E_1)\ [\varphi((\mathcal{G}, G_{\setminus\{e\}}), 1) - \varphi((\mathcal{G}, G), 1) = 0] &\iff \xi = 0.
\end{aligned}$$

Player 1 can be pivotal only for the coalitions containing player $n+2$ (otherwise, it will not be connected with the rest of the coalition), and for player 1 to be pivotal for a coalition $C \cup \{n + 2\}$, C has to have a total weight of α.

Suppose that $\xi = 0$. Then

$$\beta((\mathcal{G}, G), 1) = \varphi((\mathcal{G}, G), 1) = 0,$$

and removing x will not change the indices.

Suppose now that $\xi > 0$. We have

$$\beta((\mathcal{G}, G), 1) > 0 \quad \text{and} \quad \varphi((\mathcal{G}, G), 1) > 0,$$

and if we delete x, the indices of our distinguished player will decrease to 0 because we will isolate it from the rest of the players, i.e., there will be no connected coalition (except player 1 alone) containing 1. □

5. Discussion of the Upper Bounds of the Control Complexity

Finally, let us discuss the best known upper bounds for our problems. Recall that it is #P-complete to compute these power indices in graph-restricted weighted voting games (Skibski et al., 2015). Therefore, we can nondeterministically guess a subset of edges that can be added to the communication graph (or a subset of edges that can be removed from the graph), and then checking whether such an addition (or such a deletion) has increased, decreased, or maintained a distinguished player's power index can be done in polynomial time by means of a #P oracle (equivalently, by a PP oracle, due to the well-known result that $\mathrm{P}^{\mathrm{PP}} = \mathrm{P}^{\#\mathrm{P}}$) that computes the distinguished player's power index in the game before the control action and the new power index after adding (or deleting) the guessed edges. Therefore, an obvious upper bound of all our problems is $\mathrm{NP}^{\mathrm{PP}}$, which is a huge complexity class; for example, it contains the entire polynomial hierarchy by the celebrated result of Toda (1991).

Can we provide a better upper bound for some of our problems? Indeed, we will show that some of the restricted problems where only edges incident to the distinguished player can either be added or deleted can in fact be solved in $\mathrm{NP}^{\mathrm{NP}}$, the second level of the polynomial hierarchy. To this end, let us start by introducing yet another decision problem:

GRWVG-Pivot-Diff	
Given:	A weighted voting game $\mathcal{G} = (w_1, \ldots, w_n; q)$ with player set $N = \{1, \ldots, n\}$, a distinguished player $p \in N$, two communication graphs $G = (N, E)$ and $G' = (N, E')$ for this game such that $E' \subseteq E$ and $E \setminus E' \subseteq E_p$.
Question:	Does there exist a coalition in $N \setminus \{p\}$ for which p is pivotal in the game $(\mathcal{G}, G)$ but is not pivotal in the game $(\mathcal{G}, G')$?

This problem will serve as the NP oracle when we show the $\mathrm{NP}^{\mathrm{NP}}$ upper bound of some of our control problems. First, let us prove its NP-completeness.

Theorem 13. GRWVG-Pivot-Diff *is* NP-*complete.*

Proof. Let p be our distinguished player with weight w_p. Note that for p to be pivotal for coalitions $C \subseteq N \setminus \{p\}$, they have to have a subset $C' \subseteq C$ such that $C' \cup \{p\}$ is connected and winning and there exists no $C'' \subseteq C'$ such that C'' would also be connected and winning, i.e., no $C'' \subseteq C'$ is connected and satisfies that $w_{C''} \geq q$ for the given quota q.

To see that GRWVG-Pivot-Diff is in NP, given an instance $(\mathcal{G}, p, G, G')$ of GRWVG-Pivot-Diff with player set N, nondeterministically guess a coalition $C \subseteq N \setminus \{p\}$ and check deterministically whether the distinguished player p is pivotal for C in $(\mathcal{G}, G)$ and is not pivotal for C in $(\mathcal{G}, G')$. Let C_1 and C_2 be subsets of this coalition C such that $C_1 \cup \{p\}$ and $C_2 \cup \{p\}$ are connected components of $G[C \cup \{p\}]$ and $G'[C \cup \{p\}]$, respectively, which can be winning coalitions—we can find all connected components of a given graph in polynomial time (see, e.g., Hopcroft & Tarjan, 1973). First, we check if $C_1 \cup \{p\}$ is a winning coalition in $(\mathcal{G}, G)$, i.e., if the total weight of the players in it is at least the quota (if $C_1 \cup \{p\}$ were a losing coalition, p cannot be pivotal for it and therefore not for the whole coalition C, either), and if $C_2 \cup \{p\}$ is a winning coalition in $(\mathcal{G}, G')$. Next, for each connected component of C_1, we check if any of them is also a winning coalition—if there is none, p is pivotal for

C in $(\mathcal{G}, G)$. In this case, if $C_2 \cup \{p\}$ is a losing coalition in $(\mathcal{G}, G')$, the guessed coalition C is our correct answer, i.e., we accept on this computation path and have a yes-instance of our problem. Otherwise, if $C_2 \cup \{p\}$ is a winning coalition in $(\mathcal{G}, G')$, we check among all connected components of C_2 if there exists a component with weight at least the quota. If there exists such a component, the guessed coalition C is the coalition we were looking for; we accept on this computation path and have a yes-instance of our problem. Otherwise, it is not true that p is pivotal for the guessed coalition C in $(\mathcal{G}, G)$ and is not pivotal for C in $(\mathcal{G}, G')$, so we reject on this computation path. If all guessed coalitions C lead to rejection, we have a no-instance of our problem. Therefore, it is in NP.

We prove NP-hardness of GRWVG-Pivot-Diff via a reduction from the NP-complete SubsetSum problem. Let $((a_1, \ldots, a_n), q)$ be an instance of SubsetSum. Define the graph-restricted weighted voting game $(\mathcal{H}, H')$ with

$$\mathcal{H} = (1, a_1, \ldots, a_n, 2, 1, 10^u; 10^u + q + 2),$$

where $u \in \mathbb{N}$ is chosen such that $10^u > 4 + \sum_{j=1}^{n} a_j$, and the following communication graph H': All players but player $n+3$ (with weight 1) form a complete connected subgraph, and this player $n+3$ is connected only with weight-2 player $n+2$ (see Figure 7).

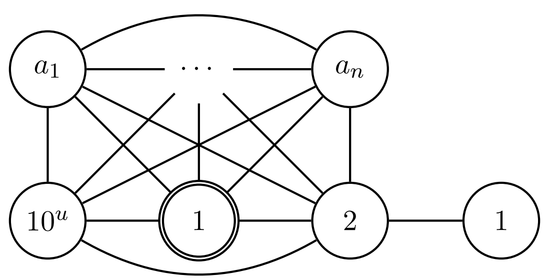


Figure 7: Communication structure of the game $(\mathcal{H}, H')$ from the proof of Theorem 13

Let H be the communication graph created from H' by adding an edge between players 1 and $n+3$ (both with weight 1) with which player 1 is not connected in H' (see Figure 8).

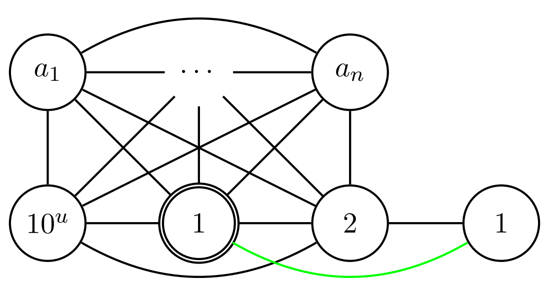


Figure 8: Communication structure of the game $(\mathcal{H}, H)$ from the proof of Theorem 13

Player 1 is our distinguished player. Let us analyze which coalitions player 1 is pivotal for in $(\mathcal{H}, H')$: Since 10^u is larger than the total weight of the rest of the players, player $n+4$ is contained in all winning coalitions, so it is enough to consider only these coalitions. Next,

- if a coalition also contains player $n+2$, then it is connected with and without the distinguished player 1, who thus is pivotal for it if its weight is $10^u + q + 1$;
- if a coalition does not contain player $n+2$, player 1 is pivotal for it if the coalition contains players from $\{2, \dots, n+1\}$ with total weight $q+1$. Note that the weight has to be greater than q. Player 1 is also pivotal for the coalitions created from the coalitions we just mentioned by adding to them the remaining player $n+3$ (who will not be connected with the former part of the coalitions).

In $(\mathcal{H}, H)$, player 1 is pivotal for all the coalitions listed above (in the second case, note that although player $n+3$ is connected with the other part of the coalition, there are still the same connected components after removing the distinguished player). But player 1 can be additionally pivotal for coalitions containing player $n+4$, not containing player $n+2$, containing players from $\{2, \dots, n+1\}$ with total weight q, and player $n+3$. Let us consider such a coalition C consisting of players $n+3$, $n+4$, and the players from $\{2, \dots, n+1\}$ whose weights sum up to q: The coalition has weight 10^u+q+1 and $C \cup \{1\}$ has weight 10^u+q+2, i.e., exactly the quota. That means that C always loses, and $C \cup \{1\}$ wins only if it is connected—$C \cup \{1\}$ is connected in H but has 2 connected components in H'. Therefore, our problem is equivalent to the problem of whether there exists a sublist of $(a_1, \dots, a_n)$ whose elements sum up to q. It follows from this reduction that GRWVG-Pivot-Diff is NP-hard. □

We now improve the trivial upper bound of $\mathrm{NP}^{\mathrm{PP}}$ to $\mathrm{NP}^{\mathrm{NP}}$ for control by adding edges incident to the distinguished player with the goal of increasing or maintaining his or her power with regard to either of our two power indices (and, by Theorem 3, the upper bound for control by adding edges to increase these indices in the unrestricted scenario is also improved). Recalling the NP- and coNP-hardness results from Theorem 7, this still leaves a complexity gap for these problems between the first and the second level of the polynomial hierarchy.

Theorem 14. *For* $\mathrm{PI} \in \{\mathrm{PBI}, \mathrm{SSI}\}$*, the problems* Control-by-Adding-Edges-ODP-to-Increase-PI *(which, by Theorem 3, is the same as* Control-by-Adding-Edges-to-Increase-PI*) and* Control-by-Adding-Edges-ODP-to-Maintain-PI *are in* $\mathrm{NP}^{\mathrm{NP}}$.

Proof. Let $(\mathcal{G}, G)$ be a graph-restricted weighted voting game with player set $N = \{1, \dots, n\}$, $\mathcal{G} = (w_1, \dots, w_n; q)$, and communication graph $G = (N, E)$, and let $i \in N$ be our distinguished player. If we connect some set of players $T \subseteq N \setminus (\mathcal{N}(i) \cup \{i\})$ with i using edges E', i may become pivotal for some coalitions for which i was not pivotal before. But if there does not exist any coalition for which i becomes pivotal after this control action, the power index remains unchanged.[13] That means that player i's indices increase if

13. Recall that—by the property of *"global monotonicity with respect to added communication possibilities"* (Napel et al., 2012) mentioned in the proof of Theorem 2—it is impossible to make i stop being pivotal for any coalition for which i was pivotal before adding edges.

and only if $(\mathcal{G}, i, G_{\cup E'}, G) \in$ GRWVG-Pivot-Diff, and they do not change if and only if $(\mathcal{G}, i, G_{\cup E'}, G) \notin$ GRWVG-Pivot-Diff. Therefore, in $(\mathcal{G}, G)$, we can guess which up to k players we connect with i, and for each subset of added edges E' guessed, we use the NP oracle GRWVG-Pivot-Diff to check whether there exists a coalition for which i becomes pivotal, and if there is any, the index will increase; otherwise, it does not change. This shows that all four problems are in $\mathrm{NP}^{\mathrm{NP}}$. □

Finally, we show the same result for control by deleting edges incident to the distinguished player with the goal of decreasing or maintaining his or her power, again leaving a complexity gap for these problems between the first and the second level of the polynomial hierarchy (recall the NP- and coNP-hardness results from Theorem 12).

Theorem 15. *For* $\mathrm{PI} \in \{\mathrm{PBI}, \mathrm{SSI}\}$*, the problems* Control-by-Deleting-Edges-ODP-to-Decrease-PI *and* Control-by-Deleting-Edges-ODP-to-Maintain-PI *are in* $\mathrm{NP}^{\mathrm{NP}}$*.*

Proof. Let $(\mathcal{G}, G)$ be a graph-restricted weighted voting game with player set $N = \{1, \ldots, n\}$, $\mathcal{G} = (w_1, \ldots, w_n; q)$, and communication graph $G = (N, E)$, and let $i \in N$ be our distinguished player. Let $E' \subseteq E_i$ be a subset of edges that are removed from $(\mathcal{G}, G)$ so as to decrease the power indices of i. The indices decrease if there exists a coalition for which i is pivotal in $(\mathcal{G}, G)$ but is no longer pivotal in $(\mathcal{G}, G_{\setminus E'})$. However, if there does not exist any coalition for which i stops being pivotal after the deletion, the indices remain unchanged.[14] That means that

- player i's indices decrease if and only if $(\mathcal{G}, i, G, G_{\setminus E'}) \in$ GRWVG-Pivot-Diff, and
- they do not change if and only if $(\mathcal{G}, i, G, G_{\setminus E'}) \notin$ GRWVG-Pivot-Diff.

Therefore, in $(\mathcal{G}, G)$, we can guess which up to k players we disconnect from i, and for each subset of deleted edges $E' \subseteq E_i$ guessed, we use the NP oracle GRWVG-Pivot-Diff to check whether there exists a coalition for which i stops being pivotal, and if there is any, the index will decrease; otherwise, it does not change. This shows that all four problems are in $\mathrm{NP}^{\mathrm{NP}}$. □

We leave it as an interesting open problem whether the complexity gap between the $\mathrm{NP}^{\mathrm{NP}}$ upper bounds from Theorems 14 and 15 and the lower bounds (NP- and coNP-hardness) shown in Theorems 7 and 12 can be closed.

6. Power Index Comparison in Graph-Restricted Weighted Voting Games

Faliszewski and Hemaspaandra (2009) proved that the problem of comparing two values of a parsimoniously #P-complete[15] function is PP-complete. They used this problem to show PP-completeness of the power index comparison problem for both the probabilistic

14. As in the previous proof, recall that by the property of the indices shown by Napel et al. (2012) and mentioned in the proof of Theorem 8, the distinguished player i cannot become pivotal for any coalition for which i was not pivotal before our control action, so i's indices cannot increase.
15. Recall this notion from Footnote 6.

Penrose–Banzhaf and the Shapley–Shubik index: Given two weighted voting games, each containing the same player p, in which of these games is p's power index value higher?

Extending their definition, we can use the same motivation to study the power index comparison problem for both the probabilistic Penrose–Banzhaf and the Shapley–Shubik index in *graph-restricted* weighted voting games as well. For a power index PI, define the following problem:

GRWVG-Compare-PI	
Given:	Two graph-restricted weighted voting games, $(\mathcal{G}, G)$ and $(\mathcal{H}, H)$, each containing the distinguished player p.
Question:	Is it true that $\mathrm{PI}((\mathcal{G}, G), p) > \mathrm{PI}((\mathcal{H}, H), p)$?

It is easy to observe that the control problems we have studied earlier are closely related to this problem. Indeed, our control problems boil down to comparing the power of a distinguished player occurring in two graph-restricted weighted voting games, one before adding (or deleting) edges and the other one after this control action. For example, we can modify the reduction from Compare-#SubsetSum-RR to Control-by-Deleting-Edges-ONDP-to-Decrease-PI, with $\mathrm{PI} \in \{\mathrm{PBI}, \mathrm{SSI}\}$, given in the proof of Theorem 9 to create an instance of GRWVG-Compare-PI instead, by mapping a given instance A of Compare-#SubsetSum-RR to the following GRWVG-Compare-PI instance: Two graph-restricted weighted voting games, $(\mathcal{G}, G)$ and $(\mathcal{G}, G_{\setminus\{x\}})$ as constructed in the proof of Theorem 9 (with graph G shown in Figure 3 and graph $G_{\setminus\{x\}}$ shown in Figure 4), each containing the distinguished player 1. As we have shown in that proof, $A \in$ Compare-#SubsetSum-RR if and only if $\mathrm{PI}((\mathcal{G}, G), 1) > \mathrm{PI}((\mathcal{G}, G_{\setminus\{x\}}), 1)$, i.e., $((\mathcal{G}, G), (\mathcal{G}, G_{\setminus\{x\}}), 1) \in$ GRWVG-Compare-PI. Therefore, GRWVG-Compare-PI is PP-hard.

On the other hand, Skibski et al. (2015) have shown that both the probabilistic Penrose–Banzhaf power index and the Shapley–Shubik power index can be computed in #P. Therefore, the arguments of Faliszewski and Hemaspaandra (2009, Lemma 2.2) can be used to show that GRWVG-Compare-PI is in PP, and thus we have:

Theorem 16. *For* $\mathrm{PI} \in \{\mathrm{PBI}, \mathrm{SSI}\}$*,* GRWVG-Compare-PI *is* PP*-complete.*

7. Conclusions

In this work, we have analyzed the problem of controlling the communication structures in graph-restricted weighted voting games by adding or deleting edges in the graphs. In particular, we have determined upper and lower bounds on how much such control actions can change a distinguished player's power. Further, we have studied the computational complexity of the problems of whether such control actions can increase, decrease, or maintain the probabilistic Penrose–Banzhaf or the Shapley–Shubik power index of a given player.

We have also considered restrictive scenarios of these problems where we are allowed to control only a subset of the edges: either only the edges incident to the distinguished player or only the edges not incident to him or her. Specifically, we have substantially improved and extended the results obtained in our preliminary conference papers (Kaczmarek & Rothe, 2021; Kaczmarek, Rothe, & Talmon, 2023). Our main results are summarized in Tables 1 and 2. An obvious conclusion to draw from these results is that controlling the communication structure of a graph-restricted weighted voting game is, at least in terms of classical worst-case complexity, a very hard task, potentially harder than NP-hard (assuming NP $\neq$ PP, as is widely believed). Another conclusion is that the complexity of these problems does depend on what kind of control is allowed: The problems seem to be easier when only edges incident to the distinguished player can be added or removed. When adding edges, decreasing the probabilistic Penrose–Banzhaf or the Shapley–Shubik power index is even in P; and so is increasing these power indices when deleting edges.[16] This insight is also supported by the improved $\mathrm{NP}^{\mathrm{NP}}$ upper bounds in Section 5. In addition, we have considered the related problem of comparing the power index of a given player who is contained in two given graph-restricted weighted voting games, analogously to the problem defined and studied by Faliszewski and Hemaspaandra (2009), and like them we have obtained PP-completeness results for our problem regarding both the probabilistic Penrose–Banzhaf and the Shapley–Shubik power index.

Of course, our hardness results may be disappointing at first glance for someone interested in controlling a graph-restricted weighted voting game. On second thought, however, he or she might consider ways to circumvent this by standard techniques including integer linear programming for small instances, or heuristic approaches or approximation techniques for larger instances. We propose to study these topics in the future.

Further interesting tasks for future research include the question of whether our complexity lower bounds in Tables 1 and 2 can be raised even further and, eventually, whether we can precisely pinpoint the complexity of these problems by providing lower bounds matching the $\mathrm{NP}^{\mathrm{PP}}$ and $\mathrm{NP}^{\mathrm{NP}}$ upper bounds shown in Section 5. Kaczmarek and Rothe (2024c, 2024a) have obtained $\mathrm{NP}^{\mathrm{PP}}$-completeness results for the related problems of control by adding players to weighted voting games via a novel proof technique, and we do hope that their technique can be suitably tailored to also apply to our problems of control by adding edges to graph-restricted weighted voting games or by deleting edges from them.

Another interesting task for future research is to explore fixed-parameter tractability and parameterized complexity (Downey & Fellows, 2013; Niedermeier, 2006) for our problems. This approach of a multivariate complexity analysis may have a high potential for applications by providing deeper practical insights. Natural parameters to study are, evidently, the number of edges that can be added or deleted. Perhaps even less obvious parameters—such as the maximum degree or the size of a largest connected component in the communication graphs—might give some additional insights into the parameterized complexity of our problems. Furthermore, there may be some limitations of our model from a practical point of view. For instance, power indices may have some limitation when it comes to making political predictions. Exploring changes to our model that take practical considerations better into account is another interesting task for future research.

Moreover, what can be said about the complexity of control in graph-restricted weighted voting games when the underlying communication structures is a *directed* graph? A directed edge from one player to another can be interpreted as an asymmetric relation between them:

16. A possible reason for this apparent difference in the hardness of the ONDP and ODP variants of our problems may be that in the latter case we are more strictly limited in our control action by adding or deleting only edges incident to the distinguished player, whereas in the former case we have much more freedom where in the graph we can exert control.

In the blockchain setting, for example, one player might trust another player but not vice versa. Finally, we propose to study variants of our problems by considering further power indices such as the normalized Penrose–Banzhaf index (Banzhaf III, 1965; Penrose, 1946).

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