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# Recursion in Abstract Argumentation is Hard — On the Complexity of Semantics Based on Weak Admissibility

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## Abstract

We study the computational complexity of abstract argumentation semantics based on weak admissibility, a recently introduced concept to deal with arguments of self-defeating nature. Our results reveal that semantics based on weak admissibility are of much higher complexity (under typical assumptions) compared to all argumentation semantics which have been analysed in terms of complexity so far. In fact, we show PSPACE-completeness of all non-trivial standard decision problems for weak-admissible based semantics. We then investigate potential tractable fragments and show that restricting the frameworks under consideration to certain graph-classes significantly reduces the complexity. We also show that weak-admissibility based extensions can be computed by dividing the given graph into its strongly connected components (SCCs). This technique ensures that the bottleneck when computing extensions is the size of the largest SCC instead of the size of the graph itself and therefore contributes to the search for fixed-parameter tractable implementations for reasoning with weak admissibility.

## 1. Introduction

Abstract argumentation frameworks (AFs) as introduced by Dung (1995) are nowadays identified as key concept to understand the fundamental mechanisms behind formal argumentation and non-monotonic reasoning. In these frameworks, it is solely the attack relation between (abstract) arguments that is used to determine the semantics of a given AF, i.e. jointly acceptable sets of arguments called extensions.

Most of the existing argumentation semantics were either based on the concept of naivety or admissibility (van der Torre & Vesic, 2017). The former is satisfied if the selected sets are maximal conflict-free. For the latter, it is required that the sets defend themselves (each attacker of an argument in the set is counter-attacked by the set).

There is a wide consensus that the absence of defense in naive extensions potentially leads to undesired results. However, already Dung noticed that also the concept of defense can be seen problematic; in particular, when self-defeating arguments are involved, that is, arguments which attack themselves directly or indirectly through an odd loop of arguments. Such "dummy" arguments may block the acceptance state of other reasonable ones, while never standing a chance of being accepted themselves. This issue has been known for a long time, and inspired several approaches to mitigate the effect of self-defeating arguments, see e.g. (Amendola & Ricca, 2019; Bodanza & Tohmé, 2009; Dondio, 2019; Dondio & Longo, 2019; Fazzinga, Flesca, & Furfaro, 2020a). However, no semantics for abstract argumentation among the numerous invented so far (Baroni, Caminada, & Giacomin, 2011) has addressed this problem in a commonly agreed way.

In a recent paper, Baumann, Brewka, and Ulbricht (2020b) propose a mediating position between naivity and admissibility and introduced the concept of weak admissibility. This new concept aims at limiting the effect of self-defeating arguments by verifying the credibility of arguments in a recursive fashion: any conflict-free set of arguments is considered acceptable unless attacked by some serious rival. On top of handling self-defeating arguments in a more reasonable way, the introduced semantics possess several promising theoretical properties which were already pointed out in (Baumann et al., 2020b) by showing that weak admissibility inherits many of the desirable properties of its classical Dung-style counterpart. These observations triggered further investigations of these semantics w.r.t. well-known postulates discussed in the literature (Baroni, Caminada, & Giacomin, 2018; van der Torre & Vesic, 2017): in particular, Dauphin, Rienstra, and van der Torre (2020) have studied the aforementioned postulates in a comprehensive fashion, while in (Baumann, Brewka, & Ulbricht, 2020a) concepts like strong equivalence for semantics based on weak admissibility are adressed.

In light of these solid theoretical results an investigation from a computational point of view stands to reason as well. In this paper, we take several steps towards this direction by thoroughly analyzing the computational complexity of weak admissibility. The complexity analysis we provide is of particular interest, since all known complexity results of standard tasks for argumentation semantics are located within the first two layers of the polynomial hierarchy (Dvořák & Dunne, 2018). This holds even for semantics which have a certain recursive nature like cf2 or stage2 semantics; see (Gaggl & Woltran, 2013; Dvořák & Gaggl, 2016) for the respective complexity analyses. In contrast, reasoning with weak admissibility based semantics will turn out to be $\mathsf{PSPACE}$-complete in general. We recall that under the assumption that the polynomial hierarchy does not collapse, problems complete for $\mathsf{PSPACE}$ are rated as significantly harder than problems located at lower levels of the polynomial hierarchy. Our results are mirrored in the complexity landscape of (propositional) non-monotonic reasoning in the broad sense, where decision problems for many prominent formalisms (like default logic or circumscription) are located on the second level of the polynomial hierarchy (see, e.g. (Cadoli & Schaerf, 1993; Thomas & Vollmer, 2010) for survey articles), and only a few formalisms reach $\mathsf{PSPACE}$-hardness. Examples for the latter are nested circumscription (Cadoli, Eiter, & Gottlob, 2005), nested counterfactuals (Eiter & Gottlob, 1996), model-preference default logic (Papadimitriou, 1991), and theory curbing (Eiter & Gottlob, 2006).

In the literature, several techniques have been developed in order to circumvent the worst case complexity of reasoning tasks as good as possible. A quite straightforward approach is restricting the structural properties of the given AF. Commonly investigated sub-classes are odd-cycle free, noeven, bipartite, symmetric, and acyclic AFs (Dvořák & Dunne, 2018). We will amplify our investigation by examining these cases as well. Thereby, we will obtain a significant drop in the computational complexity in many cases. Other approaches aim at dividing the given AF into certain smaller parts and evaluating them separately. This, however, not only relies on the graph structure but also on the semantics under consideration. Noteworthy examples for such techniques are SCC-recursiveness and splitting (Baumann, 2011). Regarding the former, the idea is that many AF semantics allow for the computation of extensions SCC-wise (Baroni, Giacomin, & Guida, 2005) which, very roughly speaking, implicates that the most relevant input value is no longer the size of the given AF, but the size of its largest SCC. Computing AF semantics this way is a common strategy in abstract argumentation and contributed not only to a theoretical understanding of existing semantics, but also inspired the proposal of novel ones (Dvořák & Gaggl, 2016). Splitting (Baumann, 2011) can be seen as a restricted version of SCC-recursiveness. Here, the AF is divided into two parts and the computation can be performed step-wise. Similar techniques have been investigated for several non-monotonic formalisms like logic programs (Lifschitz & Turner, 1994), auto-epistemic logic (Gelfond & Przymusinska, 1992) and default logic (Turner, 1996). The overall picture we obtain in this paper is that techniques of this kind are hard to adapt to weak admissibility-based semantics. However, we will make several contributions into this direction. Notably, we show that the verification of weakly preferred extensions (i.e. $\subseteq$-maximal weakly admissible sets) is fixed-parameter tractable in the size of the largest SCC of the graph, i.e. we present an algorithm for verifying preferred extensions with a running time that scales exponentially with the size of the largest SCC but polynomially with the size of the argumentation framework.

Our main contributions can be summarized as follows:

- Starting with the most general case we show that all standard decision problems for weak-admissible based semantics (with the exception of the trivial skeptical weakly admissible acceptance) are $\mathsf{PSPACE}$-complete.
- We analyze the effect of restricting the AFs under consideration to certain graph-classes. In most cases this renders the "weak" semantics to be computationally comparable to their Dung-style counterparts, which is a significant drop in their complexity. In particular, we investigate symmetric, acyclic, bipartite, odd-cycle free, and noeven AFs.
- We study to which extent SCC-recursiveness or splitting techniques can be applied to weak admissibility based semantics. In particular, we show SCC-recursiveness of weakly preferred semantics and develop approaches to compute weakly admissible extensions SCC-wise as well.

This submission combines and extends several previously published works regarding computational aspects of weak admissibility based semantics. The reduction which is used to prove PSPACE-completeness of weakly admissible semantics was reported in (Dvořák, Ulbricht, & Woltran, 2021); see also (Dvořák, Ulbricht, & Woltran, 2020). The paper (Dvořák et al., 2021) also provides the basis for our investigation of symmetric, acyclic, odd-cycle free, and noeven AFs. Crucial theoretical insights in the behavior of these subclasses stem from (Baumann et al., 2020a, Section 6). SCC-recursiveness of weakly preferred semantics was reported in (Friese & Ulbricht, 2021). The present paper combines these results, provides full proofs of all claims, significantly extends the overall presentation and contains a more comprehensive selection of examples. Novel results in this article concern fixed-parameter tractability (Corollary 5.9), SCC-wise computation of weakly admissible semantics (Propositions 5.11 and 5.13) as well as splitting results (Section 5.2).

## 2. Background

Let us start by giving the necessary preliminaries.

### 2.1 Standard Concepts and Classical Semantics

We fix a non-finite background set $\mathcal{U}$. An *argumentation framework (AF)* (Dung, 1995) is a directed graph $F = (A, R)$ where $A \subseteq \mathcal{U}$ represents a set of arguments and $R \subseteq A \times A$ models *attacks* between them. We denote with $\mathcal{F}$ the set of all finite AFs over $\mathcal{U}$; we shall consider finite AFs only. The union $F \cup G$ of two AFs $F = (A, R)$ and $G = (B, S)$ is given as $(A \cup B, R \cup S)$; the intersection $F \cap G$ is the AF $(A \cap B, R \cap S \cap (A \cap B \times A \cap B))$.

Now assume $F = (A, R)$. For $S \subseteq A$ we let $F{\downarrow_S} = (A \cap S, R \cap (S \times S))$. For $a, b \in A$, if $(a, b) \in R$, then we say that *$a$ attacks $b$*. An argument *$a$ attacks* a set $E$ if there is some $b \in E$ s.t. $a$ attacks $b$. Analogously, $E$ attacks $a$ if there is some $b \in E$ attacking $a$. A set $U \subseteq A$ is called *unattacked* if there is no $a \in A \setminus U$ attacking $U$. Moreover, $E$ is *conflict-free* in $F$ (for short, $E \in cf(F)$) iff for no $a, b \in E$, $(a, b) \in R$. We say a set $E$ *classically defends* (c-defends) an argument $a$ (in $F$) if any attacker of $a$ is attacked by some argument of $E$, i.e. for each $(b, a) \in R$, there is $c \in E$ such that $(c, b) \in R$.

Finally, for a given $F = (B, S)$ we let $A(F) = B$ and $R(F) = S$.

A *semantics* $\sigma$ is a mapping $\sigma\colon \mathcal{F} \to 2^{2^{\mathcal{U}}}$ where $F \mapsto \sigma(F) \subseteq 2^A$, i.e. given an AF $F = (A, R)$ a semantics returns a subset of $2^A$; $E \in \sigma(F)$ is called a *$\sigma$-extension* of $F$. We consider here *admissible*, *complete*, *grounded*, *preferred*, and *stable* semantics (abbr. *adm*, *co*, *gr*, *pr*, *stb*).

**Definition 2.1.** Let $F = (A, R)$ be an AF and $E \in cf(F)$.

1. $E \in adm(F)$ iff $E$ c-defends all its elements,
2. $E \in co(F)$ iff $E \in adm(F)$ and, for any $x$ c-defended by $E$, we have $x \in E$,
3. $E \in gr(F)$ iff $E$ is $\subseteq$-minimal in $co(F)$,
4. $E \in pr(F)$ iff $E$ is $\subseteq$-maximal in $adm(F)$.
5. $E \in stb(F)$ iff $E \in cf(A)$ and any $a \in A \setminus E$ is attacked by $E$.

In the following, we abuse notation and let $\bigcup \mathcal{S} = \bigcup_{M \in \mathcal{S}} M$ for a set $\mathcal{S}$ of sets; in particular, $\bigcup \sigma(F)$ is the union of all $\sigma$-extensions of $F$.

### 2.2 Weak Admissible-Based Semantics

The reduct is the central notion in the definition of weak admissible semantics (Baumann et al., 2020b). For a compact definition, we use, given an AF $F = (A, R)$, $E^+_F = \{a \in A \mid$

$E$ attacks $a$ in $F\}$ as well as $E^{\oplus}_F = E \cup E^+_F$. The latter set is known as the *range* of $E$ in $F$. When clear from the context, we omit the subscript $F$.

**Definition 2.2.** Let $F = (A, R)$ be an AF and let $E \subseteq A$. The $E$-reduct of $F$ is the AF $F^E = (E^*, R \cap (E^* \times E^*))$ where $E^* = A \setminus E^{\oplus}_F$.

By definition, $F^E$ is the subframework of $F$ obtained by removing the range of $E$ as well as corresponding attacks, i.e. $F^E = F{\downarrow}_{A \setminus E^{\oplus}}$. The intuition of the $E$-reduct is as follows. Suppose the set $E$ is considered accepted, the arguments in $E^+$ are rejected and we are not (yet) certain about the status of the remaining arguments. Then, the $E$-reduct contains those arguments which belong to the latter case, i.e. the arguments whose status is not (yet) decided. Weak admissibility formalizes the intuition that an argument $y$ which attacks the candidate set $E$ needs to be a "serious" threat, for otherwise it can be disregarded. Whether or not $y$ is serious enough is decided by checking the reduct $F^E$ in a recursive fashion.

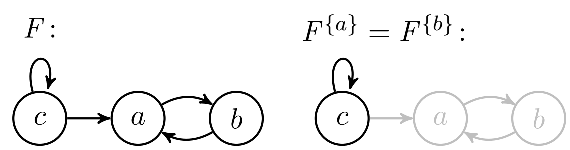
**Definition 2.3.** For an AF $F = (A, R)$, $E \subseteq A$ is called *weakly admissible* (or *w-admissible*) in $F$ ($E \in adm^w(F)$) if

1. $E \in cf(F)$ and
2. for any attacker $y$ of $E$ we have $y \notin \bigcup adm^w\left(F^E\right)$.

The major difference between the standard definition of admissibility and the "weak" one is that extensions do not have to defend themselves against *all* attackers: attackers which do not appear in any w-admissible set of the reduct can be neglected.

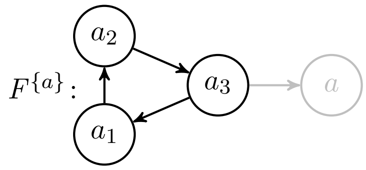
Observe that the definition of $adm^w(F)$ is recursive since the second item makes use of $adm^w(F^E)$. However, given $E \neq \emptyset$, the reduct $F^E$ contains strictly less arguments than $F$. Hence the recursion will always end with an empty AF (which does not contain any non-empty set of arguments) or a self-attacker (which is not conflict-free); thus the definition is well-defined. In the following, let us familiarize ourselves with weak admissibility using some illustrative examples.

**Example 2.4.** Consider the following simple example:

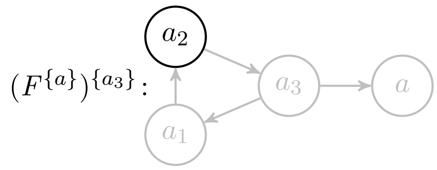



While we observe $\{a\} \notin adm(F)$, we can verify weak admissibility of $\{a\}$ in $F$. Obviously, $\{a\}$ is conflict-free in $F$ (condition 1). Since $c$ is the only attacker of $\{a\}$ in $F^{\{a\}}$ we have to check $c \notin \bigcup adm^w\left(F^{\{a\}}\right)$ (condition 2). Since $\{c\}$ is not conflict-free in the reduct $F^{\{a\}} = (\{c\}, \{(c, c)\})$ we find $\{c\} \notin adm^w\left(F^{\{a\}}\right)$ yielding $\bigcup adm^w\left(F^{\{a\}}\right) = \emptyset$. Hence, $c \notin \bigcup adm^w\left(F^{\{a\}}\right)$, and thus $\{a\} \in adm^w(F)$. ◇

**Example 2.5.** Now assume $a$ is attacked by an odd cycle $a_1, a_2, a_3$. Let us check whether $\{a\} \in adm^w(F)$:



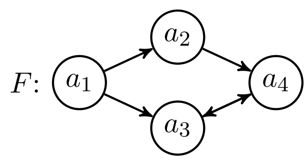

In fact, the only conflict-free set attacking $a$ is $\{a_3\}$. However, in the reduct $(F^{\{a\}})^{\{a_3\}}$ the set $\{a_2\}$ is weakly admissible. Since $a_2$ attacks $a_3$, $\{a_3\} \notin adm^w\left((F^{\{a\}})^{\{a_3\}}\right)$:



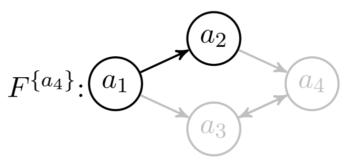

We conclude that $\{a\} \in adm^w(F)$. ◇

Let us consider another example illustrating the mechanisms of weak admissibility beyond self-defeating arguments.

**Example 2.6.** Consider the following AF $F$.




Let us verify—although this seems a bit surprising at first glance—that $\{a_4\} \in adm^w(F)$. To see this, we note that $a_2$ is the only attacker in the corresponding reduct:

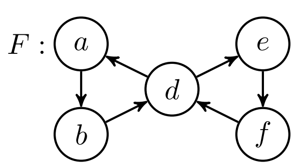



Now since $a_2$ is attacked by $a_1$, it stands no chance of being w-admissible in $F^{\{a_4\}}$, although it is not a self-defeating argument. Thus $\{a_4\} \in adm^w(F)$. ◇

Although Example 2.6 may appear somewhat counter-intuitive, it is similar in spirit to Example 2.5: in both cases, weak admissibility verifies whether a certain attacker can be neglected. In Example 2.5, $a_3$ does no harm since it is contained in a self-defeating odd loop; in Example 2.6, $a_2$ does no harm since it is defeated by the undisputed $a_1$.

The following AF shows that weak admissibilty can model choices without using even cycles. This observation will be useful later.

**Example 2.7.** Consider now an AF with two odd cycles sticking together.



Let $E = \{a\}$. Then the reduct $F^E$ consists of an odd cycle only from which we already know that it does not possess any non-empty weakly admissible extension. The same is true for $E' = \{e\}$. However, $E \cup' E$ is not weakly admissible since the reduct contains the unattacked argument $d$. In summary, we obtain $adm^w(F) = \{\emptyset, \{a\}, \{e\}\}$. $\diamond$

Following the classical Dung-style semantics, *weakly preferred* extensions are defined as $\subseteq$-maximal w-admissible extensions.

**Definition 2.8.** For an AF $F = (A, R)$, $E \subseteq A$ is called *weakly preferred* (or *w-preferred*) in $F$ ($E \in pr^w(F)$) iff $E$ is $\subseteq$-maximal in $adm^w(F)$.

In order to define the "weak" counterparts to Dung's grounded and complete extensions, the following notion of "weak defense" has been proposed in (Baumann et al., 2020b):

**Definition 2.9.** Let $F = (A, R)$ be an AF. Given two sets $E, X \subseteq A$. We say $E$ *weakly defends* (or *w-defends*) $X$ iff for any attacker $y$ of $X$ we have,

1. $E$ attacks $y$, or (c-defense)
2. $y \notin \bigcup adm^w\left(F^E\right)$, $y \notin E$ and $X \subseteq X' \in adm^w(F)$.

Now weakly complete and weakly grounded extensions can be defined analogously to complete and grounded ones:

**Definition 2.10.** For an AF $F = (A, R)$, $E \subseteq A$ is called *weakly complete* (or just *w-complete*) in $F$ ($E \in co^w(F)$) iff $E \in adm^w(F)$ and for any $X$, s.t. $E \subseteq X$ and $X$ w-defended by $E$, we have $X \subseteq E$.

A set $E \subseteq A$ is called *weakly grounded* (or *w-grounded*) in $F$ ($E \in gr^w(F)$) iff $E$ is $\subseteq$-minimal in $co^w(F)$.

The following relations between weak admissibility semantics and their Dung-style counterparts (Baumann et al., 2020b, Proposition 5.6) will be useful throughout the present paper:

$$\begin{aligned} adm(F) &\subseteq adm^w(F) \subseteq cf(F) \\ pr^w(F) &\subseteq co^w(F) \subseteq adm^w(F) \\ gr^w(F) &\subseteq co^w(F) \\ stb(F) &\subseteq pr^w(F) \end{aligned}$$

Moreover, as it is the case for the classical semantics, a set $E \subseteq A$ is w-preferred in $F$ iff it is $\subseteq$-maximal in $co^w(F)$ (Baumann et al., 2020b, Theorem 5.3).

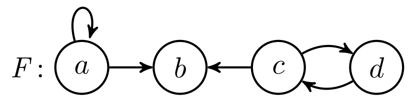
Towards a more convenient notion of weak defense, the following characterization has been developed in (Baumann et al., 2020a); it is suitable in all cases that "matter", i.e. cases where w-completeness of a given set is to be verified:

**Proposition 2.11.** *Let $F$ be an AF and let $E \in adm^w(F)$. Then, for any $X = E \,\dot\cup\, D$ we have that $E$ w-defends $X$ if*

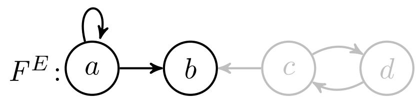
1. *for any attacker $y$ of $D$, $y \notin \bigcup adm^w(F^E)$, and*

*2. there is a set $D \subseteq D'$ with $D' \in adm^w\left(F^E\right)$.*

**Example 2.12.** Consider the AF $F$:




Let us verify that $E = \{d\}$ w-defends $X = \{b, d\}$. Since $\{d\}$ itself is w-admissible, the conditions of the above proposition are met. We thus consider the reduct $F^E$:




Now $D = \{b\}$ is not attacked by a w-admissible argument (since $a$ is a self-attacker) and is itself w-admissible in $F^E$. Hence $X = E \cup D$ is w-defended by $E$. Thus $\{b\}$ is not w-complete (but of course $\{b, d\}$ is). It is thus easy to verify that $co^w(F) = \{\emptyset, \{c\}, \{b, d\}\}$. ◇

**Example 2.13.** Recall Example 2.5. The empty set $\emptyset$ weakly defends $\{a\}$ since both conditions in Proposition 2.11 are clearly satisfied (by considering $F^{\emptyset} = F$). We find therefore that $\{a\}$ is the unique w-complete extension of this AF. ◇

**Example 2.14.** Regarding Example 2.7, $\emptyset$ weakly defends both $\{a\}$ and $\{e\}$, but neither of them defends any other non-empty set. Hence these are the two w-complete extensions of the AF depicted in Example 2.7. ◇

Throughout the paper, the following results will be frequently applied. The first one is taken from (Baumann et al., 2020a, Corollaries 4.2 and 4.6, Theorem 4.13, Proposition 4.14); the second is due to (Baumann et al., 2020a, Theorem 4.5).

**Theorem 2.15.** *Let $F = (A, R)$ be an AF and $\sigma \in \{adm^w, co^w, gr^w, pr^w\}$. If $E \in \sigma(F)$ and $E' \in \sigma(F^E)$, then $E \cup E' \in \sigma(F)$.*

**Theorem 2.16.** *Let $F = (A, R)$ be an AF. Then $E \in pr^w(F)$ if and only if $E \in cf(F)$ and $\bigcup adm^w(F^E) = \emptyset$.*

For more details regarding the definition and basic properties of weak admissibility we refer the reader to (Baumann et al., 2020b).

### 2.3 Complexity Classes and Decision Problems

We assume the reader to be familiar with the basic concepts of computational complexity theory (Arora & Barak, 2009; Papadimitriou, 1994) as well as the standard classes P (polynomial time), NP (non-deterministic polynomial time) and coNP (complementary class to NP). Moreover, we consider the class DP that contains those problems that can be solved via an NP algorithm together with an coNP algorithm such that an instance is accepted iff it is accepted by both the NP and the coNP algorithm. In the following, we briefly recapitulate the concept of oracle machines and related complexity classes relevant for this work. To this end, let $\mathsf{C}$ denote some complexity class. By a $\mathsf{C}$-oracle machine we mean a (polynomial time) Turing machine which can access an oracle that decides a given (sub)-problem in $\mathsf{C}$ within one computation step. We denote such machines as $\mathsf{P}^{\mathsf{C}}$ if the underlying Turing machine is deterministic; and $\mathsf{NP}^{\mathsf{C}}$ if the underlying Turing machine is non-deterministic. In this work we consider complexity classes using $\mathsf{NP}$-oracles. First, the class $\Sigma_2^{\mathsf{P}} = \mathsf{NP}^{\mathsf{NP}}$ denotes the set of problems which can be decided by a non-deterministic polynomial time algorithm that has (unrestricted) access to an $\mathsf{NP}$-oracle. The class $\Pi_2^{\mathsf{P}} = \mathsf{coNP}^{\mathsf{NP}}$ is defined as the complementary class of $\Sigma_2^{\mathsf{P}}$, i.e. $\Pi_2^{\mathsf{P}} = \mathsf{co}\Sigma_2^{\mathsf{P}}$. Finally, $\mathsf{L}$ (logarithmic space) and $\mathsf{PSPACE}$ (polynomial space) contain the problems that can be solved using only logarithmic or polynomial space of memory, respectively. Notice, that space complexity classes have implicit bounds on the running time, i.e. problems in $\mathsf{L}$ can be solved in polynomial time while problems in $\mathsf{PSPACE}$ can be solved in exponential time. We have:

$$\mathsf{L} \subseteq \mathsf{P} \subseteq \begin{matrix} \mathsf{NP} \\ \mathsf{coNP} \end{matrix} \subseteq \mathsf{DP} \subseteq \begin{matrix} \Sigma_2^{\mathsf{P}} \\ \Pi_2^{\mathsf{P}} \end{matrix} \subseteq \mathsf{PSPACE}$$

Moreover, we sometimes consider problem parameters for a more fine-grained analysis, making use of concepts from parameterized complexity (Cygan, Fomin, Kowalik, Lokshtanov, Marx, Pilipczuk, Pilipczuk, & Saurabh, 2015). In particular we consider the class of fixed parameter tractable problems ($\mathsf{FPT}$) which contains the problems that can be solved in $O(f(k) \cdot p(|I|)$, where $f$ is an arbitrary computable function, $k$ is a problem parameter, $p$ is a polynomial and $|I|$ is the size of the instance. That is, the running time may scale exponentially (or even worse) w.r.t. the parameter but scales polynomially with the instance size.

For an AF $F = (A, R)$ and a semantics $\sigma$, we say an argument $a \in A$ is *credulously accepted* (*skeptically accepted*) in $F$ w.r.t. $\sigma$ if $a \in \bigcup \sigma(F)$ ($a \in \bigcap \sigma(F)$). The corresponding decision problems for a semantics $\sigma$, given an AF $F$ and argument $a$, are as follows:

- *Credulous Acceptance* ($\mathit{Cred}_\sigma$): Given an AF $F = (A, R)$ and argument $a \in A$, is $a$ contained in some $S \in \sigma(F)$?
- *Skeptical Acceptance* ($\mathit{Skept}_\sigma$): Given an AF $F = (A, R)$ and argument $a \in A$, is $a$ contained in each $S \in \sigma(F)$?

Furthermore we consider the following decision problems:

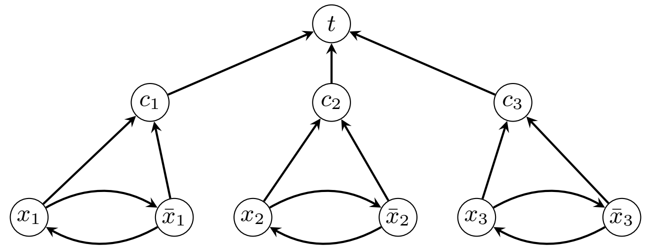
- *Verification* ($\mathit{Ver}_\sigma$): Given an AF $F = (A, R)$ and a set $S \subseteq A$, is $S \in \sigma(F)$?
- *Non-emptiness* ($\mathit{NEmpty}_\sigma$): Given an AF $F = (A, R)$, is there a non-empty set $S \subseteq A$ such that $S \in \sigma(F)$?

## 3. Complexity Analysis

In this section, we investigate the complexity of the standard decision problems in argumentation for the four semantics based on weak admissibility.

Let us start by building up some intuition about the complexity of weak admissibility semantics. As for most semantics the verification problem is a suitable groundwork.

**Example 3.1.** Consider the following AF $F$, adapted from the well-known standard translation from propositional formulas to AFs:




Let us check whether $E = \{t\} \in adm^w(F)$. This is the case if none of the attackers $c_1$, $c_2$, or $c_3$ occur in a w-admissible extension of the reduct $F^E$, which is obtained by removing the argument $t$ from $F$.

Now take $c_1$. We see that $c_1$ does not occur in a w-admissible extension in $F^E$: It is attacked by both $x_1$ and $\bar{x}_1$ which are in turn both w-admissible in any relevant sub-AF of $F$. Similarly, neither $c_2$ nor $c_3$ occur in a w-admissible extension of $F^E$. Thus $E \in adm^w(F)$. $\diamond$

Although this example was quite straightforward, several observations can be made:

- weak admissibility does not appear to be a local property: the reason why $E = \{t\}$ is w-admissible in the previous example are the arguments $x_1, \ldots, \bar{x}_3$ which are not contained in $E$; we also see that this example is quite small and can be extended to chains of arbitrary length,
- unless there is some shortcut, several sub-AFs need to be computed, inducing a recursion with depth in $\mathcal{O}(|A|)$ in the worst case,
- it is not clear at first glance whether deciding credulous acceptance is actually much easier, because guessing a suitable set (here $\{t, x_1, x_2, x_3\}$) might skip computationally expensive recursive steps.

The main contribution of this paper is to formally prove that there are no shortcuts and no suitable guessing in any case: All considered non-trivial problems are PSPACE-complete. Our results are summarized in Table 1 together with the results for admissibility-based semantics summarized by Dvořák and Dunne (2018).

### 3.1 Membership Results

We provide an algorithm that runs in PSPACE and closely follows the definition of w-admissibility.

**Lemma 3.2.** *$Ver_{adm^w}$ is in* PSPACE*.*

*Proof.* An algorithm for verifying that $E \in adm^w(F)$ proceeds as follows: (1) test whether $E \in cf(F)$; if not return false, (2) compute the reduct $F^E$, (3) iterate over all subsets $S$ of $F^E$ that contain at least one attacker of $E$ and test whether $S$ is w-admissible; if so return false; else return true. Notice that the last step involves recursive calls. However,

Table 1: Complexity of classical and weak-admissible based semantics ("c" is used as shorthand for "complete").

| $\sigma$ | $Cred_\sigma$ | $Skept_\sigma$ | $Ver_\sigma$ | $NEmpty_\sigma$ |
|---|---|---|---|---|
| $adm$ | NP-c | trivial | in P | NP-c |
| $co$ | NP-c | P-c | in P | NP-c |
| $gr$ | P-c | P-c | P-c | in $P$ |
| $pr$ | NP-c | $\Pi_2^{\mathsf{P}}$-c | coNP-c | NP-c |
| $adm^w$ | PSPACE-c | trivial | PSPACE-c | PSPACE-c |
| $co^w$ | PSPACE-c | PSPACE-c | PSPACE-c | PSPACE-c |
| $gr^w$ | PSPACE-c | PSPACE-c | PSPACE-c | PSPACE-c |
| $pr^w$ | PSPACE-c | PSPACE-c | PSPACE-c | PSPACE-c |

the size of the considered AF is decreasing in each step and thus the recursion depth is in $O(|A|)$. Moreover, we only need to store the current AF as well as the set $S$ to verify. Finally, iterating over all subsets of an AF can be done in PSPACE as well. Hence, the above algorithm is in PSPACE. $\square$

Given that verification is in PSPACE we can adapt standard algorithms to obtain the PSPACE membership of the other problems. Notice that $Skept_{adm^w}$ is always false as the empty-set is always w-admissible.

**Proposition 3.3.** *For* $\sigma \in \{gr^w, adm^w, co^w, pr^w\}$, $Cred_\sigma$, $Skept_\sigma$, $Ver_\sigma$, *and* $NEmpty_\sigma$ *can be solved in* PSPACE.

*Proof.* $Ver_{adm^w} \in$ PSPACE is by Lemma 3.2. The other memberships are by the following algorithms that can be easily implemented in PSPACE with calls to other PSPACE problems, e.g. $Ver_{adm^w}$, and thus are themselves in PSPACE.

$Ver_{pr^w}$ can be solved by first verifying that the set is w-admissible and then iterating over all super-sets and verifying that they are not w-admissible.

$Ver_{co^w} \in$ PSPACE: To test whether a set $E$ is w-complete, first test whether it is w-admissible, then compute $Cred = \bigcup adm^w(F^E)$ (which is in PSPACE as we show below), and finally test for each set $D \subseteq A \setminus E$ whether it is w-defended by $E$. The latter can be done by first testing whether $Cred$ attacks $D$ and then iterating over all $D' \supsetneq D$ and test $D' \in adm^w\left(F^E\right)$ (which by the above is in PSPACE). If none of the sets $D$ is w-defended by $E$ then $E$ is w-complete and thus we obtain $Ver_{co^w} \in$ PSPACE. $Ver_{gr^w} \in$ PSPACE: To test whether a set $E$ is w-grounded, we first test whether it is w-complete, and then that each $E'$ with $E' \subsetneq E$ is not w-complete.

For $Cred_\sigma$ with $Ver_\sigma \in$ PSPACE: we iterate over all sets of the arguments that contain the query argument and test whether the set is a $\sigma$-extension. As soon as we find a subset that is a $\sigma$-extension we can stop and return that the argument is credulously accepted. Otherwise if none of the sets is a $\sigma$-extension, then the argument is not credulously accepted.

For $Skept_\sigma$ we iterate over all subsets of the arguments that do not contain the query argument and test whether the set is a $\sigma$-extension. As soon as we find a subset that is a $\sigma$-extension we can stop and return that the argument is not skeptically accepted. Otherwise if none of the sets is a $\sigma$-extension, then the argument is skeptically accepted.

For $NEmpty_\sigma$ we iterate over all non-empty subsets of the arguments and test whether the set is a $\sigma$-extension. If one of them is a $\sigma$-extension we terminate and return true otherwise we return false. □

### 3.2 Hardness Results

We show hardness by a reduction from the PSPACE-complete problem of deciding whether a QBF is valid. The problem remains PSPACE-complete if we consider QBFs of the following form

$$\Phi = \forall x_n \exists x_{n-1} \ldots \forall x_2 \exists x_1 \colon \phi(x_1, x_2, \ldots, x_{n-1}, x_n).$$

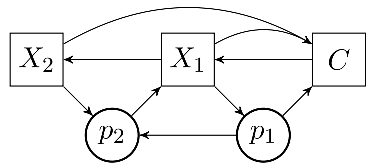
Notice that $\Phi$ might start with a universal or existential quantifier and then alternates between universal and existential quantifiers after each variable and ends with an existential quantifier. Moreover, we can assume that $\phi$ is a propositional formula in CNF given by a set of clauses $C$, i.e. $\phi = \bigwedge_{c \in C} \bigvee_{l \in c} l$ where each $l$ is a literal over the atoms $\{x_1, x_2, \ldots, x_n\}$. We call a QBF starting with a universal quantifier a $\forall$-QBF and a QBF starting with an existential quantifier an $\exists$-QBF. Finally, observe that we named variables in reverse order to avoid renaming variables in our proofs by induction. We further assume that each clause contains one of the literals $x_1, \neg x_1$, in order to avoid empty clauses when eliminating variables in the following proofs.

We start with a reduction that maps QBFs to AFs such that the validity of the QBF can be read off by inspecting the w-admissible sets of the AF. We will later extend this reduction to encode the specific decision problems under our considerations.

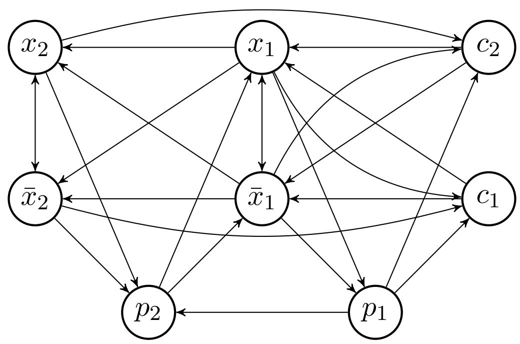
**Reduction 3.4.** Given a QBF $\Phi$ with propositional formula $\phi(x_1, \ldots, x_n)$ over clauses $C$, we define the AF $G_\Phi = (A, R)$ with

$$\begin{aligned}
A = & \{x_i, \bar{x}_i, p_i \mid 1 \leq i \leq n\} \cup \{c \mid c \in C\} \text{ and} \\
R = & \{(x_i, \bar{x}_i), (\bar{x}_i, x_i) \mid 1 \leq i \leq n\} \cup \\
& \{(x_i, x_{i+1}), (x_i, \bar{x}_{i+1}) \mid 1 \leq i < n\} \cup \\
& \{(\bar{x}_i, x_{i+1}), (\bar{x}_i, \bar{x}_{i+1}) \mid 1 \leq i < n\} \cup \\
& \{(x_i, c) \mid x_i \in c \in C\} \cup \{(\bar{x}_i, c) \mid \neg x_i \in c \in C\} \cup \\
& \{(c, x_1), (c, \bar{x}_1) \mid c \in C\} \cup \{(p_i, p_{i+1}) \mid 1 \leq i < n\} \cup \\
& \{(x_i, p_i), (\bar{x}_i, p_i) \mid 1 \leq i \leq n\} \cup \\
& \{(p_i, x_{i-1}), (p_i, \bar{x}_{i-1}) \mid 2 \leq i \leq n\} \cup \{(p_1, c) \mid c \in C\}.
\end{aligned}$$

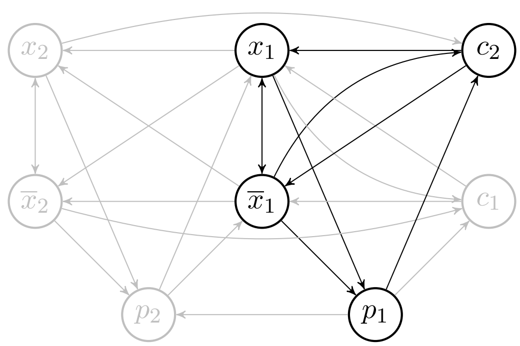
**Example 3.5.** Let us consider the valid QBF $\forall x_2 \exists x_1 : \phi$ with $\phi = c_1 \wedge c_2 = (\neg x_2 \vee x_1) \wedge (x_2 \vee \neg x_1)$ and apply Reduction 3.4 to obtain an AF $F$. It will be convenient to think of several layers, each one induced by a variable occurring in the QBF at hand. We thus have two layers here, with $x_i$ and $\bar{x}_i$ attacking each other in the expected way and each layer attacked by its predecessor. The $x$-arguments attack the $c$-arguments in the natural way. The $c$-arguments only attack $x_1$ and $\bar{x}_1$. The arguments $p_1$ and $p_2$ induce odd cycles to forbid certain possible extensions which will become clear later. Schematically, this looks as follows. Thereby, we cluster the $X$-variables, i.e. $X_i$ encapsulates $x_i$ and $\bar{x}_i$.




In detail, Reduction 3.4 applied to our QBF yields the following AF:



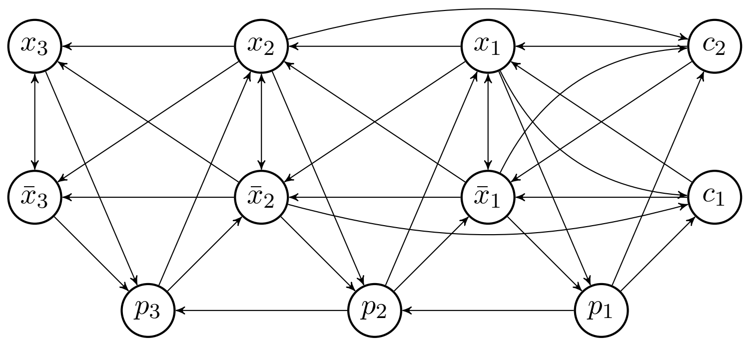

Now regarding our QBF note that setting $x_2$ to true requires $x_1$ is to be true as well and setting $x_2$ to false requires $x_1$ to be false. This translates to $F$ as follows: Take $E = \{\bar{x}_2\}$, corresponding to setting $x_2$ to false. The set $E$ is not w-admissible in $F$. To see this, consider the reduct $F^E$:




Here $\{\bar{x}_1\}$ (corresponding to $\neg x_1$ in the QBF) is w-admissible in $F^E$ (even admissible) and attacks $\bar{x}_2$ witnessing that $E \notin adm^w(F)$. Similarly, $\{x_2\}$ is not w-admissible since it is attacked by $x_1$ in the corresponding reduct.

Let us now consider a QBF which evaluates to false. For this, we move from $\phi$ to $\phi' = C_1 \wedge C_2 = (\neg x_2 \vee x_1) \wedge (x_2)$. Note that $\phi'$ is obtained from $\phi$ by removing $\neg x_1$ from $C_2$. Consider the induced QBF $\forall x_2 \exists x_1 : \phi'$. Let $F'$ be the AF obtained by applying Reduction 3.4. This time, $E = \{\bar{x}_2\}$ is w-admissible: The reduct is the same as the one depicted above, with the attack from $\bar{x}_1$ to $c_2$ removed. Thus neither $\{x_1\}$ nor $\{\bar{x}_1\}$ are w-admssible in $(F')^E$: The argument $c_2$ occurs in both $((F')^E)^{\{x_1\}}$ and $((F')^E)^{\{\bar{x}_1\}}$ and hence witnesses that both arguments are not w-admissible in $(F')^E$. This means in turn that $E = \{\bar{x}_2\}$ is w-admissible in $F'$. $\diamond$

**Example 3.6.** For the sake of demonstrating our construction, let us assume our QBF consists of three variables, i.e. consider $\exists x_3 \forall x_2 \exists x_1 : \phi$ with $\phi = (\neg x_2 \vee x_1) \wedge (x_2 \vee \neg x_1)$ as above. The AF $F$ induced by Reduction 3.4 is the following:




Now $E = \{x_3\}$ is w-admissible: The reduct $F^{\{x_3\}}$ is the AF $F$ from the previous example, where both $\{x_2\}$ and $\{\bar{x}_2\}$ are not w-admissible (recall that we consider the formula $\phi$ from above). Thus $E$ is w-admissible. $\diamond$

The previous examples hint at the following behavior of Reduction 3.4:

- if a QBF of the form $\forall x_2 \exists x_1 : \phi$, evaluates to true then neither $\{x_2\}$ nor $\{\bar{x}_2\}$ is w-admissible,
- if a QBF of the form $\exists x_3 \forall x_2 \exists x_1 : \phi$, evaluates to true, then at least one of $\{x_3\}$ and $\{\bar{x}_3\}$ is w-admissible, and
- analogous reasoning applies to QBFs which evaluate to false.

We also want to mention that, e.g. in Example 3.6, the arguments $x_3$ and $\bar{x}_3$ are the only possible candidates for w-admissible extensions:

- $p_2$, for example, is attacked by $x_2$ and $\bar{x}_2$ in the corresponding reduct; this can only be prevented by also including $x_1$ or $\bar{x}_1$, which in turn are in conflict with $p_2$,
- $c_1$, for example, is attacked by $p_1$ which can only be removed from the reduct by including $x_1$ or $\bar{x}_1$, but both are attacked by $c_1$,
- $x_2$, for example, is attacked by $p_3$, but attacks $x_3$, $\bar{x}_3$ and $p_2$ which are the only attackers of $p_3$.

The following lemma formalizes that these observations are true in general.

**Lemma 3.7.** *For a QBF $\Phi$ and each integer $n$ we have the following:*

1. *Suppose $\Phi$ is of the form $\exists x_n \forall x_{n-1} \dots \exists x_1 : \phi(x_1, x_2, \dots, x_n)$. Then we have that $adm^w(G_\Phi) \cap \{\{x_n\}, \{\bar{x}_n\}\} \neq \emptyset$ if $\Phi$ is valid and $adm^w(G_\Phi) = \{\emptyset\}$ otherwise.*
2. *Suppose $\Phi$ is of the form $\forall x_n \exists x_{n-1} \dots \exists x_1 : \phi(x_1, x_2, \dots, x_n)$. Then we have that $adm^w(G_\Phi) = \{\emptyset\}$ if $\Phi$ is valid and $adm^w(G_\Phi) \cap \{\{x_n\}, \{\bar{x}_n\}\} \neq \emptyset$ otherwise.*

*Moreover, in both cases $adm^w(G_\Phi) \subseteq \{\{x_n\}, \{\bar{x}_n\}, \emptyset\}$.*

We briefly sketch the main ideas of the proof. First, we have that all conflict-free sets $E$ that contain an argument $a$ different from $x_n$ and $\bar{x}_n$ yield a reduct $G_\Phi^E$ with unattacked argument $b$ that attacks $a$ in $G_\Phi$ and thus $E$ is not w-admissible. That is, $\{x_n\}$, $\{\bar{x}_n\}$, and $\emptyset$ are the only candidates for being w-admissible. The remainder of the proof is then by induction on the number of variables $n$, starting with $n = 1$. In the induction step we exploit that, when considering one of the sets $E = \{x_n\}$ or $E = \{\bar{x}_n\}$ respectively, we have that the reduct $G_\Phi^E$ corresponds to the AF $G'_\Phi$ where $\Phi'$ is the QBF we obtain from $\Phi$ when eliminating the variable $x_n$ by setting it to true, false respectively, and simplifying the CNF formula. That is, we remove the quantifier for $x_n$ and delete all clauses that contain $x_n$, $\neg x_n$ respectively, and delete $\neg x_n$, $x_n$ respectively, from the remaining clauses.

That is, we have that $\{x_n\}$ is weakly admissible in $G_\Phi$ if and only if neither $\{x_{n-1}\}$ nor $\{\bar{x}_{n-1}\}$ is weakly admissible in $G'_\Phi$ and as $\Phi'$ has only $n - 1$ variables one can exploit the induction hypothesis.

The full proof of the above proposition is by the following lemmas.

**Lemma 3.8.** *For a QBF $\Phi$ with $n$ quantifier alternations, $adm^w(G_\Phi) \subseteq \{\{x_n\}, \{\bar{x}_n\}, \emptyset\}$.*

*Proof.* Let $E \in adm^w(G_\Phi)$.

Assume $p_i \in E$ for some $i \geq 2$. Since $E$ must be conflict-free, we have $x_{i-1} \notin E$, $\bar{x}_{i-1} \notin E$, and $p_{i+1} \notin E$ as well as $x_i \notin E$ and $\bar{x}_i \notin E$. Thus both $x_i$ and $\bar{x}_i$ occur in the reduct $F^E$ and do not have any attacker in $F^E$. In this case $E$ cannot be w-admissible since $p_i \in E$ is attacked by $\{x_i\}$ and $\{\bar{x}_i\}$ which are w-admissible in $F^E$. So the assumption $p_i \in E$ for some $i \geq 2$ must be false.

Consider now $p_1 \in E$. Similarly, if $E \in cf(F)$, then $c_j \notin E$ for each $j$ as well as $x_1, \bar{x}_1 \notin E$ and hence, $p_1$ is attacked by $x_1$ and $\bar{x}_1$ which are unattacked in $F^E$; contradiction.

Now assume $c_j \in E$ for some $j$. Since $c_j$ attacks both $x_1$ and $\bar{x}_1$, $\{p_1\}$ is w-admissible in $F^E$ which in turn attacks $c_j$. Thus $c_j \in E$ is impossible.

Finally, if $x_i \in E$ or $\bar{x}_i \in E$ for $i \leq n - 1$, then either $p_{i+1}$ is unattacked in $F^E$ (which attacks both arguments) or $p_i \in E$, $x_{i+1} \in E$, or $\bar{x}_{i+1} \in E$ (which contradicts $E \in cf(F)$). Hence $x_i \notin E$; a contradiction.

Hence, none of these cases can occur, i.e. $E \in \{\{x_n\}, \{\bar{x}_n\}, \emptyset\}$. □

The remainder of the proof proceeds by induction on the number of variables: Lemma 3.9 is the base case and the consecutive lemmas constitute the induction step.

**Lemma 3.9.** *For $\Phi = \exists x_1 : \phi(x_1)$ we have that $adm^w(G_\Phi) \cap \{\{x_1\}, \{\bar{x}_1\}\} \neq \emptyset$ iff $\Phi$ is valid.*

*Proof.* By the above lemma it suffices to consider the sets $\{x_1\}, \{\bar{x}_1\}$. The formula $\Phi$ is valid iff $x_1$ or $\neg x_1$ appears in all clauses.

$\Rightarrow$: Assume $\{x_1\}$ is a w-admissible set but $\Phi$ is not valid, i.e. there is a $c \in C$ such that $x_1 \notin c$. By construction $x_1$ attacks $p_1$ and is attacked by $c$ and thus $c$ is unattacked in the reduct and thus $\{c\}$ is w-admissible in the reduct which is in contradiction to $\{x_1\}$ being w-admissible. A similar reasoning applies to the case where $\{\bar{x}_1\}$ is w-admissible but $\Phi$ is not valid.

$\Leftarrow$: Assume that the formula is valid and w.l.o.g. assume that $x_1$ appears in all clauses. Then by construction $x_1$ attacks all the other arguments in $G_\Phi$ and thus $\{x_1\}$ is a w-admissible set. □

In the following lemmas we will modify a CNF formula $\phi(x_1, x_2, \dots, x_n)$ by setting a variable to true or false. Here we use $\phi(x_1, x_2, \dots, \top)$ to denote the formula that we obtain by deleting all clauses containing $x_n$ and removing $\neg x_n$ from the remaining clauses and $\phi(x_1, x_2, \dots, \bot)$ to denote the formula that we obtain by deleting all clauses containing $\neg x_n$ and removing $x_n$ from the remaining clauses

**Lemma 3.10.** *If Lemma 3.7 holds for $\exists$-QBFs with $n-1$ variables then it also holds for $\forall$-QBFs with $n$ variables.*

*Proof.* Consider a $\forall$-QBF $\Phi = \forall x_n \exists x_{n-1} \dots \exists x_1 : \phi(x_1, x_2, \dots, x_n)$. We have that $\Phi$ is valid iff both $\Phi_1 = \exists x_{n-1} \forall x_{n-2} \dots \exists x_1 : \phi(x_1, x_2, \dots, \top)$ and $\Phi_2 = \exists x_{n-1} \forall x_{n-2} \dots \exists x_1 : \phi(x_1, x_2, \dots, \bot)$ are valid. Moreover, $G_\Phi^{\{x_n\}} = G_{\Phi_1}$ and $G_\Phi^{\{\bar{x}_n\}} = G_{\Phi_2}$.

First assume that $\Phi$ is valid and consider $\{x_n\}$ (the argument for $\{\bar{x}_n\}$ is analogous). We show that $\{x_n\}$ is not w-admissible. Consider $G_\Phi^{\{x_n\}} = G_{\Phi_1}$. By the induction hypothesis we have that $\{x_{n-1}\}$ or $\{\bar{x}_{n-1}\}$ is weakly admissible in $G_\Phi^{\{x_n\}}$ and as both $x_{n-1}$ and $\bar{x}_{n-1}$ attack $x_n$ we have that $\{x_n\}$ is not w-admissible.

Now assume that $\Phi$ is not valid and w.l.o.g. assume that $\Phi_1$ is not valid. By the induction hypothesis we have that neither $\{x_{n-1}\}$ nor $\{\bar{x}_{n-1}\}$ is weakly admissible in $G_\Phi^{\{x_n\}} = G_{\Phi_1}$ and thus $\{x_n\}$ is w-admissible. □

**Lemma 3.11.** *If Lemma 3.7 holds for $\forall$-QBFs with $n-1$ variables then it also holds for $\exists$-QBFs with $n$ variables.*

*Proof.* Consider an $\exists$-QBF $\Phi = \exists x_n \forall x_{n-1} \dots \exists x_1 : \phi(x_1, x_2, \dots, x_n)$. We have that $\Phi$ is valid iff one of $\Phi_1 = \forall x_{n-1} \exists x_{n-2} \dots \exists x_1 : \phi(x_1, x_2, \dots, \top)$ and $\Phi_2 = \forall x_{n-1} \exists x_{n-2} \dots \exists x_1 : \phi(x_1, x_2, \dots, \bot)$ is valid. Moreover, $G_\Phi^{\{x_n\}} = G_{\Phi_1}$ and $G_\Phi^{\{\bar{x}_n\}} = G_{\Phi_2}$.

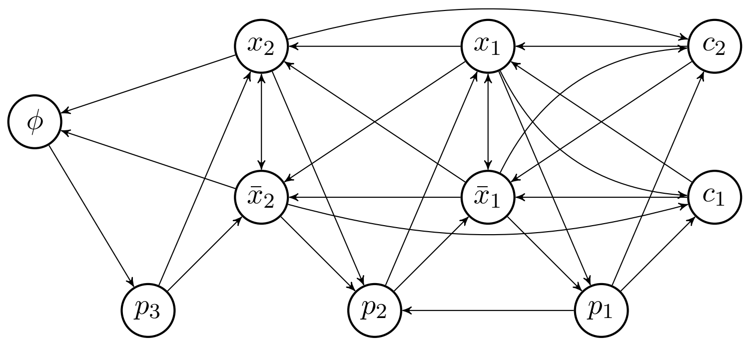
First assume that $\Phi$ is valid and w.l.o.g. assume that $\Phi_1$ is valid. We show that $\{x_n\}$ is w-admissible. Consider $G_\Phi^{\{x_n\}} = G_{\Phi_1}$. By the induction hypothesis we have that neither $\{x_{n-1}\}$ nor $\{\bar{x}_{n-1}\}$ is weakly admissible in $G_\Phi^{\{x_n\}}$ and thus $\{x_n\}$ is w-admissible.

Now assume that $\Phi$ is not valid and and consider $\{x_n\}$ (the argument for $\{\bar{x}_n\}$ is analogous). By the induction hypothesis we have that $\{x_{n-1}\}$ or $\{\bar{x}_{n-1}\}$ is weakly admissible in $G_\Phi^{\{x_n\}}$ and as both $x_{n-1}$ and $\bar{x}_{n-1}$ attack $x_n$ we have that $\{x_n\}$ is not w-admissible. □

We next extend our reduction by two further arguments $\phi$ and $p_{n+1}$ in order to show our hardness results.

**Reduction 3.12.** Given a $\forall$-QBF $\Phi = \forall x_n \exists x_{n-1} \dots \exists x_1 : \phi(x_1, \dots, x_n)$ we define $F_\Phi = G_\Phi \cup (\{\phi, p_{n+1}\}, \{(\phi, p_{n+1}), (p_n, p_{n+1}), (p_{n+1}, x_n), (p_{n+1}, \bar{x}_n), (x_n, \phi), (\bar{x}_n, \phi)\})$.

**Example 3.13.** Recall the valid QBF from our first example: $\forall x_2 \exists x_1 : \phi$ with $\phi = C_1 \wedge C_2 = (\neg x_2 \vee x_1) \wedge (x_2 \vee \neg x_1)$. Augmenting Reduction 3.4 with Reduction 3.12 yields the following AF $F$:




Note the similarity to Example 3.6: Basically, $\phi$ replaces the pair $x_3, \bar{x}_3$ of arguments. Hence it is easy to see that $\{\phi\}$ is w-admissible since the reduct $F^{\{\phi\}}$ is again the first AF from Example 3.5 possessing no w-admissible argument. $\diamond$

We now formally characterize the potential w-admissible sets in Reduction 3.12.

**Lemma 3.14.** *For a QBF $\Phi$, $adm^w(F_\Phi) \subseteq \{\emptyset, \{\phi\}\}$.*

*Proof.* In comparison to Lemma 3.7, we are only left to consider $p_{n+1}$. The assumption $p_{n+1} \in E \in adm^w(F)$ yields an analogous contradiction: $E \in cf(F)$ implies $\phi, x_n, \bar{x}_n \notin E$ and hence $p_{n+1}$ is attacked by $\{\phi\} \in adm^w(F^E)$. $\square$

**Proposition 3.15.** *Given a $\forall$-QBF $\Phi = \forall x_n \exists x_{n-1} \ldots \exists x_1 : \phi(x_1, x_2, \ldots, x_n)$ we have that $\Phi$ is valid if and only if $adm^w(F_\Phi) = \{\emptyset, \{\phi\}\}$, and $adm^w(F_\Phi) = \{\emptyset\}$ otherwise.*

*Proof.* We have that the empty-set is always w-admissible and by Lemma 3.14 that $\{\phi\}$ is the only candidate for being a w-admissible set. Now consider $\{\phi\}$ and the reduct $F_\Phi^{\{\phi\}}$. We have that $F_\Phi^{\{\phi\}} = G_\Phi$ and $x_n$ and $\bar{x}_n$ being the attackers of $\phi$. By Lemma 3.7 we have that $\{x_n\}$ or $\{\bar{x}_n\}$ is w-admissible in the reduct iff $\Phi$ is not valid. Thus $\{\phi\}$ is w-admissible iff $\Phi$ is valid. $\square$

**Theorem 3.16.** *All of the following problems are* PSPACE*-complete: $Cred_{adm^w}$, $Ver_{adm^w}$, $NEmpty_{adm^w}$, as well as $Cred_\sigma$, $Skept_\sigma$, $Ver_\sigma$, $NEmpty_\sigma$ for any $\sigma \in \{co^w, gr^w, pr^w\}$.*

*Proof.* The membership results are by Proposition 3.3. The hardness results are all by Reduction 3.12 and Proposition 3.15. It only remains to state the precise problem instances that are equivalent to testing the validity of the $\forall$-QBF $\Phi$. First, consider $Cred_{adm^w} = Cred_{co^w} = Cred_{pr^w}$. In the AF $F_\Phi$ we have that $\phi$ is credulously accepted w.r.t. w-admissible semantics iff $\{\phi\} \in adm^w(F_\Phi)$ iff $\Phi$ is valid. Now, consider $Ver_{adm^w}$ and $Ver_{pr^w}$. We have that $\{\phi\} \in adm^w(F_\Phi)$ iff $\{\phi\} \in pr^w(F_\Phi)$ iff $\Phi$ is valid. Next, consider $Skept_{pr^w}$. We have that $\phi$ is skeptically accepted iff $pr^w(F_\Phi) = \{\{\phi\}\}$ iff $\Phi$ is valid. Moreover, for $NEmpty_{adm^w} = NEmpty_{pr^w}$, the only w-preferred/w-admissible extension is the empty-set iff $\Phi$ is not valid.

For the remaining problems it suffices to show $gr^w(F_\Phi) = co^w(F_\Phi) = \{\{\phi\}\}$ whenever $\Phi$ is valid and otherwise, $gr^w(F_\Phi) = co^w(F_\Phi) = \{\emptyset\}$. Regarding the former, if $\Phi$ is valid, then $adm^w(F_\Phi) = \{\emptyset, \{\phi\}\}$. We show that in this case, $\emptyset \notin co^w(F_\Phi)$. To this end we show that $\emptyset$ w-defends $\{\phi\}$. We may apply Proposition 2.11 to the w-admissible set $E = \emptyset$ and

$X = E \dot{\cup} D = \emptyset \cup \{\phi\} = \{\phi\}$, and see that $\emptyset$ w-defends $\{\phi\}$ since (1) no attacker of $\phi$ can be w-admissible in $F_\Phi^E = F_\Phi^\emptyset = F_\Phi$, and (2) $\{\phi\}$ itself is w-admissible in $F_\Phi^E = F_\Phi^\emptyset = F_\Phi$. If, on the other hand, $\Phi$ is not valid, then $adm^w(F_\Phi) = \{\emptyset\}$, so the only candidate for a w-complete or w-grounded extension is $\emptyset$. Since there is no other w-admissible set in the reduct $F_\Phi^\emptyset = F_\Phi$, $\emptyset$ does not w-defend any set and is thus itself w-complete and hence w-grounded. Hence, we obtain for $\sigma \in \{gr^w, co^w\}$ that $\Phi$ is valid iff $\phi$ is credulously, skeptically respectively, accepted in $F_\Phi$, iff $\{\phi\} \in \sigma(F_\Phi)$ iff $F_\Phi$ has a non-empty $\sigma$-extensions. Thus, Reduction 3.12 provides a reduction from $\forall$-QBF to all of the considered problems, and as it can be clearly performed in polynomial time, the PSPACE-hardness follows. $\square$

The results of this section are summarized in Table 2.

Table 2: Complexity of weak-admissible based semantics.

| $\sigma$ | $Cred_\sigma$ | $Skept_\sigma$ | $Ver_\sigma$ | $NEmpty_\sigma$ |
|---|---|---|---|---|
| $adm^w$ | PSPACE-c | trivial | PSPACE-c | PSPACE-c |
| $co^w$ | PSPACE-c | PSPACE-c | PSPACE-c | PSPACE-c |
| $gr^w$ | PSPACE-c | PSPACE-c | PSPACE-c | PSPACE-c |
| $pr^w$ | PSPACE-c | PSPACE-c | PSPACE-c | PSPACE-c |

## 4. Complexity for Specific Graph-Classes

In the previous section we have shown the standard reasoning problems to be computationally hard. A common approach towards tractability is to consider AFs that have a special graph structure (Dunne, 2007). To this end, we consider graph classes that have been shown to be tractable fragments for the traditional argumentation semantics. As we will see, some results follow from the fact that weak-admissible semantics coincide with the standard semantics on certain classes of AFs. However, some cases require a dedicated analysis.

### 4.1 Symmetric AFs

First, we consider the class of symmetric AFs $(A, R)$ (Coste-Marquis, Devred, & Marquis, 2005) which require that if $(a, b) \in R$ then also $(b, a) \in R$. In symmetric AFs we have that conflict-free and admissible sets coincide and thus also w-admissible and conflict-free sets coincide (recall that we always have $cf(F) \supseteq adm^w(F) \supseteq adm(F)$). If we additionally assume that there is no self-attack then all arguments are credulously accepted and an argument is only w-defended if it is not attacked at all. That is, c-defence and w-defence coincide and thus also complete and w-complete as well as grounded and w-grounded semantics coincide.

**Lemma 4.1.** *For symmetric AFs $F$ we have $adm^w(F) = adm(F)$ and $pr^w(F) = pr(F)$. Moreover, if $F$ has no self-attacks then also $co^w(F) = co(F)$ and $gr^w(F) = gr(F)$.*

By (Baumann et al., 2020b, Theorems 3.10, 5.14), we can remove self-attacking arguments without changing the extensions of weakly-admissible based semantics and thus by the known complexity results for the standard semantics (Dvořák & Dunne, 2018) we obtain that reasoning on symmetric AFs is in logarithmic space.

**Proposition 4.2.** *For symmetric AFs and $\sigma \in \{gr^w, adm^w, co^w, pr^w\}$, the problems $Cred_\sigma$, $Skept_\sigma$, $Ver_\sigma$ and $NEmpty_\sigma$ can be solved in* $\mathsf{L}$.

Table 3: Complexity of w-admissible semantics in symmetric AFs

| $\sigma$ | $Cred_\sigma$ | $Skept_\sigma$ | $Ver_\sigma$ | $NEmpty_\sigma$ |
|---|---|---|---|---|
| $gr^w$ | in $\mathsf{L}$ | in $\mathsf{L}$ | in $\mathsf{L}$ | in $\mathsf{L}$ |
| $adm^w$ | in $\mathsf{L}$ | trivial | in $\mathsf{L}$ | in $\mathsf{L}$ |
| $co^w$ | in $\mathsf{L}$ | in $\mathsf{L}$ | in $\mathsf{L}$ | in $\mathsf{L}$ |
| $pr^w$ | in $\mathsf{L}$ | in $\mathsf{L}$ | in $\mathsf{L}$ | in $\mathsf{L}$ |

### 4.2 Acyclic AFs

Next we consider graph classes that are based on the absence of (certain types) of cycles. First, if the AF under consideration is acyclic, then the unique preferred extension coincides with the grounded one. Given the relationships between the classical semantics and our "weak" ones, we may infer a similar result.

**Definition 4.3.** Given an AF $F = (A, R)$. A sequence $a_1, a_2, \dots, a_n, a_{n+1}$ of arguments with $a_i \in A$, $a_1 = a_{n+1}$ and $(a_i, a_{i+1}) \in R$ for all $i$ is called a *cycle* in $F$. If $n$ is odd, then it is an *odd cycle*. We call $F$ *acyclic* if there is no cycle and *odd-cycle free* if there is no odd cycle in $F$.

We will use the following known result from the literature regarding acyclic AFs.

**Proposition 4.4** ((Dung, 1995), Theorem 30)**.** *If $F$ is acyclic, then there is exactly one complete extension $E$ which is also grounded, preferred, and stable.*

We next extend this result to semantics based on weak admissibility.

**Proposition 4.5.** *If $F$ is acyclic, then there is exactly one w-complete extension $E$ which is also w-grounded, w-preferred, complete, grounded, preferred, and stable.*

*Proof.* Notice that the grounded extension $G$ of an acyclic AF is the unique stable extension (Dung, 1995). Moreover, the grounded extension is contained in each w-complete and thus in each w-preferred extension. As each stable extension is also w-preferred, we obtain that $G$ is the unique w-complete extension $E$ which is also w-grounded, w-preferred, complete, grounded, preferred, and stable. $\square$

Since the unique grounded extension is stable for acyclic AFs (Dung, 1995), this in particular implies that the semantics coincide with their "weak" versions.

**Corollary 4.6.** *For any acyclic AF $F$ and any semantics $\sigma \in \{gr, pr, co\}$ we have: $\sigma(F) = \sigma^w(F)$.*

We can now exploit the fact that the grounded extension is the only w-preferred extension and can be computed in polynomial time.

**Proposition 4.7.** *For acyclic AFs and $\sigma \in \{gr^w, adm^w, co^w, pr^w\}$, $Cred_\sigma$, $Skept_\sigma$, $Ver_\sigma$, and $NEmpty_\sigma$ are in $\mathsf{P}$. Moreover, the complexity results in Table 4 hold.*

*Proof.* For $\sigma \in \{gr^w, co^w, pr^w\}$ the results are by Corollary 4.6 and the corresponding results for the classical semantics (Dvořák & Dunne, 2018). Clearly, $Cred_{adm^w} = Cred_{pr^w}$, $NEmpty_{adm^w} = NEmpty_{pr^w}$ and $Skept_{adm^w}$ is trivially false as the empty set is always weakly admissible. To decide $Ver_{adm^w}$ for a set $E$ we simply test whether it is a subset of the unique grounded extension. □

Table 4: Complexity of w-admissible semantics in acyclic AFs

| $\sigma$ | $Cred_\sigma$ | $Skept_\sigma$ | $Ver_\sigma$ | $NEmpty_\sigma$ |
|---|---|---|---|---|
| $gr^w$ | P-c | P-c | in L | in L |
| $adm^w$ | P-c | trivial | P-c | in L |
| $co^w$ | P-c | P-c | in L | in L |
| $pr^w$ | P-c | P-c | in L | in L |

An interesting observation is that even in acyclic AFs w-admissible semantics differs from admissible sets which can already be seen by a simple chain like, e.g. $F = (\{a, b, c\}, \{(a, b), (b, c)\})$ with $\{c\} \in adm^w(F)$.

### 4.3 Odd-Cycle Free AFs

Next, let us consider odd-cycle free AFs. For the standard semantics, odd-cycle free AFs are not a tractable fragment but $Ver_{pr}$ becomes tractable and the complexity of $Skept_{pr}$ drops to coNP-complete (Dvořák & Dunne, 2018). For w-admissible based semantics we have a similar effect with a more drastic drop in complexity.

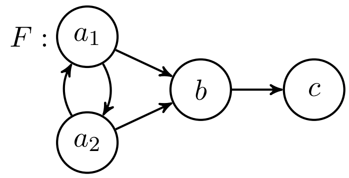
To this end, we first provide characterizations that are crucial for the following complexity investigations: (a) for odd-cycle free AFs the w-preferred and preferred extensions coincide; and (b) for odd-cycle free AFs there is a unique w-grounded extension that consists of the skeptically preferred accepted arguments. We then combine these characterization with the complexity results for preferred semantics (Dvořák & Dunne, 2018) on odd cycle free AFs and obtain that reasoning with w-admissible based semantics is NP-complete/coNP-complete.

Let us first recall the result for standard semantics and odd-cycle free AFs (Dung, 1995, Theorem 33 and Corollary 36).

**Proposition 4.8.** *If $F$ is odd-cycle free, then $stb(F) = pr(F)$. In particular, $stb(F) \neq \emptyset$.*

The main motivation for weak admissibility as well as weak defense was to reduce the effect of self-defeating arguments on the acceptability of other arguments. We have already seen that the deletion of such arguments does not influence the newly introduced semantics (Baumann et al., 2020b). Furthermore, one might expect that classical semantics and their associated weak counterparts even coincide in the absence of self-defeating arguments. Consider therefore the following example.

**Example 4.9.** Let $F$ be an odd-cycle free AF as depicted below.




Although surprising at first glance, $\{c\}$ is a w-admissible extension of $F$. The intuitive reason is that the definition of w-admissibility identifies $b$ as a negligible argument since it is not w-admissible in $F^{\{c\}}$. Moreover, $\{c\}$ defends neither $\{a_1, c\}$ nor $\{a_2, c\}$, so it is even w-complete. In summary, $adm(F) \neq adm^w(F)$ and $co(F) \neq co^w(F)$. ⋄

The prediction of some arguments being negligible as we have seen in the previous example renders some sets w-admissible/w-complete which are not classically admissible/complete, even in the absence of odd-cycles. In consideration of the next assertion we can however guarantee that no further arguments are credulously accepted.

**Proposition 4.10.** *For any odd-cycle free AF $F$, $pr(F) = pr^w(F)$.*

*Proof.* ($\subseteq$) Let $E \in pr(F)$. Since $F$ is odd-cycle free, $E$ is a stable extension, i.e. no proper superset of $E$ can be conflict-free and, as $F^E$ is the empty AF, $E$ is a w-admissible extensions. Thus $E$ is w-preferred.

($\supseteq$) Let $E^w \in pr^w(F)$. We show that $F^{(E^w)}$ is the empty AF, i.e. $E^w$ is stable and hence preferred. Assume the contrary, i.e. $F^{(E^w)}$ contains some arguments. Since this AF is odd-cycle free as well, there is a non-empty complete extension (Dung, 1995, Lemma 34), say $E'$. There is hence a w-complete extension of $F^{(E^w)}$, say $(E')^w$, with $E' \subseteq (E')^w$. By Theorem 2.15 we infer $E^w \cup (E')^w \in co^w(F)$, a contradiction. □

Since credulous reasoning coincides for admissible, complete, and preferred semantics (for both the classical as well as the weak versions), we can now infer the following result.

**Corollary 4.11.** *For any odd-cycle free AF $F$ and any semantics $\sigma \in \{adm, pr, co\}$ we have that $\bigcup \sigma(F) = \bigcup \sigma^w(F)$.*

However, the most interesting (and presumably most surprising) observation we are going to make about odd-cycle free AFs is related to w-grounded semantics. More precisely, it turned out that in absence of odd-cycles first, w-grounded semantics returns exactly one unique extension and secondly, it coincides with the skeptical reasoning regarding classical preferred semantics. This is similar in spirit to relatively grounded AFs (Dung, 1995, Definition 31) where $\bigcap pr(F)$ is required to coincide with the grounded extension. Before giving our result, we prove the following auxiliary lemma.

**Lemma 4.12.** *For any odd-cycle free AF $F$ and $E \in adm^w(F)$ there exists an extension $E^* \in stb\left(F^E\right)$, such that $E \cup E^* \in stb(F)$.*

*Proof.* The AF $F^E$ is odd-cycle free as well with at least one stable extension $E^*$. By Theorem 2.15, $E \cup E^* \in adm^w(F) \subseteq cf(F)$ and by $E^* \in stb\left(F^E\right)$, the AF $F^{E \cup E^*} = \left(F^E\right)^{E^*}$ is empty; this together with conflict-freeness characterizes $E \cup E^* \in stb(F)$. □

**Theorem 4.13.** *For any odd-cycle free AF $F$ we have that $\bigcap pr(F)$ is the unique w-grounded extension, i.e. $gr^w(F) = \{\bigcap pr(F)\}$*

*Proof.* We show that $G^w = \bigcap stb(F)$ is the least w-complete extension. It then immediately follows that

1. there is only one w-grounded extension,
2. $gr^w(F) \ni \bigcap stb(F) = \bigcap pr(F)$.

(w-admissibility) The set $G^w$ is conflict-free because the stable extensions are conflict-free. Now consider the reduct $F^{G^w}$. We observe that each stable extension $E$ of $F$ has the form $E = G^w \cup E'$, where $E'$ is a stable extension of $F^{G^w}$ and vice versa, each stable extension $E'$ of $F^{G^w}$ extends to a stable extension of $F$. Let $y$ be an attacker of $G^w$ which occurs in $F^{G^w}$. Attacking $\bigcap stb(F)$, $y$ is attacked by each stable extension of $F$ and hence, it is attacked by each stable extension of $F^{G^w}$ (Lemma 4.12). It is therefore attacked by each preferred extension of $F^{G^w}$ and can thus not occur in a w-admissible extension of $F^{G^w}$. Since $y$ was an arbitrary attacker of $G^w$ occurring in $F^{G^w}$, $G^w$ is w-admissible.

(w-complete) Assume $X$ satisfies $G^w \subseteq X$ and is w-defended by $G^w$. We set $X = G^w \dot\cup D$ and apply Proposition 2.11, i.e.

1. for any attacker $y$ of $D$, $y$ is not w-admissible in $F^{G^w}$, and
2. there is a set $D \subseteq D'$ with $D' \in adm^w\left(F^{G^w}\right)$.

For the sake of contradiction assume $G^w \subsetneq X$ and take $x \in D$. Since $x \notin \bigcap stb(F)$, there is a stable extension $E$ attacking $x$. As before, $E = G^w \cup E'$ where $E'$ is a stable extension of $F^{G^w}$. So $x$ is attacked by a $y \in E'$. As this contradicts the first item, we conclude $X = G^w$.

(least) We show: If $E^w$ is w-complete, then $E^w$ w-defends $G^w \cup E^w$, implying $G^w \cup E^w \subseteq E^w$ by definition of w-completeness and hence $G^w \subseteq E^w$, i.e. $G^w$ is the least w-complete extension.

Let $E^w \in co^w(F)$ and $X = E^w \cup G^w = E^w \dot\cup D$. Note that we have $D \subseteq G^w$. We have to show that

1. for any attacker $y$ of $D$, $y$ is not w-admissible in $F^{E^w}$, and
2. there is a set $D \subseteq D'$ with $D' \in adm^w\left(F^{E^w}\right)$.

This can be seen as follows:

1. Let $y$ be an attacker of $D$ occurring in $F^{E^w}$. Then $y$ attacks $G^w = \bigcap stb(F)$ and is thus attacked by each stable extension of $F$. However, by Lemma 4.12 each stable extension of $F^{E^w}$ extends to a stable extension of $F$. Hence $\{y\}$ cannot be extended to a stable extension of $F^{E^w}$. It can thus not be extended to a w-preferred extension of $F^{E^w}$ and hence does not occur in $\bigcup adm^w(F^{E^w})$.

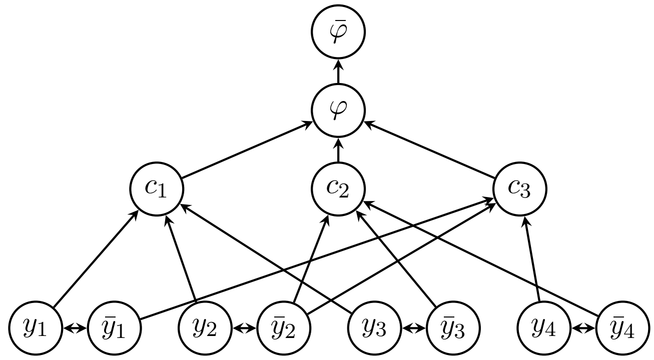



Figure 1: Illustration of the standard translation $G_\varphi$, for the propositional formula $\varphi$ with clauses $\{\{y_1, y_2, y_3\}, \{\bar{y}_2, \bar{y}_3, \bar{y}_4)\}, \{\bar{y}_1, \bar{y}_2, y_4\}\}$.

2. We first observe that $E^w$ cannot attack $G^w$; otherwise, $E^w$ would be attacked by each stable extension of $F$. However, there is at least one stable extension $E = E^w \cup E^*$ with $E^* \in stb(F^{E^w})$ containing $E^w$ (Lemma 4.12). Since $G^w \subseteq E$, this would contradict conflict-freenes of $E$. So $E^w$ does not attack $G^w$. In particular, $D = D \cap A(F^{E^w})$. Since $F^{E^w}$ is odd-cycle free, there is a stable extension $D'$ of $F^{E^w}$. We are left to show $D \subseteq D'$. Since $D = D \cap A(F^{E^w})$, there are only to cases: either $D \subseteq D'$ or $D'$ attacks $D$. Assume the latter is the case. Then $D'$ attacks $\bigcap stb(F)$. Since $D'$ can be extended to a stable extension of $F$ (again by Lemma 4.12), this yields a contradiction. □

By the known complexity results for the standard semantics (Dvořák & Dunne, 2018) we obtain the following results for credulous and skeptical acceptance w.r.t. w-admissible semantics in odd-cycle free AFs.

**Proposition 4.14.** *For odd-cycle free AFs, $Cred_\sigma$ is* NP*-complete for $\sigma \in \{adm^w, co^w, pr^w\}$, and $Skept_{co^w} = Skept_{gr^w} = Cred_{gr^w}$ and $Skept_{pr}$ are* coNP*-complete.*

We next consider the verification problems. To this end we first consider a variant of the standard translation from propositional logic to argumentation as used by Dvořák and Dunne (2018).

**Reduction 4.15.** For a propositional formula $\varphi$ in CNF given by a set of clauses $C$ over the atoms $Y$, we define the *standard translation* from $\varphi$ as $G_\varphi = (A, R)$, where

$$\begin{aligned} A =& \{\varphi, \bar{\varphi}\} \cup C \cup Y \cup \bar{Y} \\ R =& \{(\varphi, \bar{\varphi})\} \cup \{(c, \varphi) \mid c \in C\} \cup \\ & \{(x, c) \mid x \in c, c \in C\} \cup \{(\bar{x}, c) \mid \bar{x} \in c, c \in C\} \cup \\ & \{(x, \bar{x}), (\bar{x}, x) \mid x \in Y\} \end{aligned}$$

By (Dvořák & Dunne, 2018) we have that $\bar{\varphi}$ is skeptically accepted in $G_\varphi$ w.r.t. preferred semantics iff $\varphi$ is unsatisfiable. By Proposition 4.10 and Theorem 4.13 we obtain the following lemma.

**Lemma 4.16.** *For a propositional formula $\varphi$ in CNF and the associated AF $G_\varphi$ we have the following:*

- *$\bar{\varphi}$ is skeptically accepted in $G_\varphi$ w.r.t. w-preferred semantics iff $\varphi$ is unsatisfiable.*
- *If $\varphi$ is not a tautology, then no other argument than $\bar{\varphi}$ in $G_\varphi$ is skeptically accepted w.r.t. w-preferred semantics.*
- *$\{\bar{\varphi}\} \in gr^w(G_\varphi)$ iff $\varphi$ is unsatisfiable.*

**Proposition 4.17.** *For odd-cycle free AFs we have that $Ver_{gr^w}$ is $\mathsf{DP}$-complete, $Ver_{adm^w}$ is $\mathsf{coNP}$-complete, $Ver_{co^w}$ is $\mathsf{DP}$-complete, and $Ver_{pr^w}$ is in $\mathsf{L}$.*

*Proof.* First consider $Ver_{pr^w} \in \mathsf{L}$. As discussed above we have that w-preferred, preferred and stable extensions coincide. We can thus exploit that $Ver_{stb} \in \mathsf{L}$ (Dvořák & Dunne, 2018, Table 1) to also verify w-preferred in $\mathsf{L}$.

Next consider that $Ver_{adm^w}$ is $\mathsf{coNP}$-complete. In order to verify that $S \in adm^w(F)$ one can proceed as follows: first test whether $S$ is conflict-free, then compute the reduct $F^E$, and finally verify that none of the attackers of $S$ is credulously accepted. The first two steps are in polynomial time and the final step is in $\mathsf{coNP}$. For hardness we use that $Cred_{adm^w}$ is $\mathsf{NP}$-hard for odd-cycle free AFs. So take some odd-cycle free AF $F = (A, R)$ and any $a \in A$. We construct the odd-cycle free AF $F' = (A \cup \{t\}, R \cup \{(a, t)\})$ and obtain that $\{t\} \in adm^w(F')$ iff $a$ is not credulously accepted in $F'$ iff $a$ is not credulously accepted in $F$. We thus obtain that $Ver_{adm^w}$ is $\mathsf{coNP}$-hard.

For $Ver_{gr^w} \in \mathsf{DP}$ consider the following algorithm of a given set $S$: For each $s \in S$ we test whether it is skeptically accepted w.r.t. preferred semantics ($\in \mathsf{coNP}$) and for each $s \in A \setminus S$ we test whether it is not skeptically accepted w.r.t. preferred semantics ($\in \mathsf{NP}$).

For $Ver_{co^w} \in \mathsf{DP}$ consider the following algorithm for a given set $S$: First test whether $S$ is w-admissible (this can be done in $\mathsf{coNP}$) and then for each $s \in A \setminus S$ test that is not w-defended by $S$. To this end consider the odd-cycle free AF $F^S$. We have that stable and preferred extension coincide and thus an argument is skeptically accepted w.r.t. preferred semantics iff it is not attacked by any admissible set. We now guess sets $A_s$ for each $s \in A \setminus S$ and then check whether they are admissible and attack $s$. If so, then we have that none of the $s \in A \setminus S$ is defended as they are attacked by an accepted argument. Otherwise, if there is an $s$ such that there is no admissible set attacking it, we have $s$ is skeptically accepted in $F^S$ w.r.t. (w-)preferred semantics and thus defended by $S$ in $F$. Notice that we can guess and check all the sets $A_s$ independently of each other and thus have an $\mathsf{NP}$-procedure for the second part. Combining the two parts we obtain a $\mathsf{DP}$-procedure.

The $\mathsf{DP}$-hardness for $Ver_{gr^w}$ and $Ver_{co^w}$ is obtained by the following reduction from the $\mathsf{DP}$-complete SAT-UNSAT problem. Given an instance $(\varphi, \psi)$ of SAT-UNSAT, i.e. $\varphi$ and $\psi$ are propositional formulas in CNF, we apply the standard translation to both in $\varphi$ and $\psi$ and rename the arguments of the second AF such that the argument sets are disjoint. Let $G_\varphi$, $G_\psi$ be the resulting AFs and consider the union $F_{\varphi,\psi} = G_\varphi \cup G_\psi$. By Lemma 4.16 we have that $\{\bar{t}\}$ is a w-grounded, w-complete respectively, extension of $F_{\varphi,\psi}$ iff $\varphi$ is satisfiable and $\psi$ is unsatisfiable iff $(\varphi, \psi)$ is a valid SAT-UNSAT instance. That is, we have a reduction from SAT-UNSAT to $Ver_{gr^w}$ and $Ver_{co^w}$ and thus those problems are $\mathsf{DP}$-hard. $\square$

We now turn to the *NEmpty* problems.

Table 5: Complexity of w-admissible semantics in odd-cycle free AFs

| $\sigma$ | $Cred_\sigma$ | $Skept_\sigma$ | $Ver_\sigma$ | $NEmpty_\sigma$ |
|---|---|---|---|---|
| $gr^w$ | coNP-c | coNP-c | DP-c | coNP-c |
| $adm^w$ | NP-c | trivial | coNP-c | in L |
| $co^w$ | NP-c | coNP-c | DP-c | in L |
| $pr^w$ | NP-c | coNP-c | in L | in L |

**Proposition 4.18.** *For odd-cycle free AFs we have that $NEmpty_{gr^w}$ is* coNP*-complete, $NEmpty_{adm^w} \in$ L, $NEmpty_{co^w} \in$ L, and $NEmpty_{pr^w} \in$ L.*

*Proof.* The $NEmpty_{\sigma^w} \in$ L results are by the facts that odd-cycle free AFs have at least one stable extension and this stable extension is w-admissible, w-complete and w-preferred and that if the AF is non-empty, then the stable extension must contain at least one argument.

For $NEmpty_{gr^w}$ we consider the complementary problem of testing whether the unique w-grounded extension is empty. That is, for each $s \in A$ we have to show that it is not skeptically accepted w.r.t. preferred semantics. We have that stable and preferred extension coincide and thus an argument is skeptically accepted w.r.t. preferred semantics iff it is not attacked by any admissible set. We now guess sets $A_s$ for each $s \in A$ and then check whether they are admissible and attack $s$. If so, then we have that none of the $s \in A$ is skeptically accepted. Otherwise, if there is an $s$ such that there is no admissible set attacking it we have $s$ is skeptically accepted w.r.t. preferred semantics and thus in the w-grounded extension. The above is an NP-procedure for the complementary problem and thus yields a coNP-procedure for $NEmpty_{gr^w}$.

The coNP-hardness for $NEmpty_{gr^w}$ is by the following reduction from the coNP-complete UNSAT problem. Given an instance $\varphi$ of UNSAT, i.e. a propositional formula in CNF, we apply the standard translation to $\varphi$ and consider the resulting AF $G_\varphi$. By Lemma 4.16 we have that the w-grounded extension is non-empty iff iff $\varphi$ is unsatisfiable. Hence, $NEmpty_{gr^w}$ is also coNP-hard. □

Our results for odd-cycle free AFs are summarized in Table 5.

### 4.4 Even-Cycle Free AFs

Next we investigate the class of even-cycle free AFs (noeven AFs) (Dvořák & Dunne, 2018) which allow to decide admissible based semantics in polynomial-time. We have that noeven AFs have a unique preferred extension, which is however not true for w-preferred extensions. Consider the AF $F$ in Figure 2 which has only odd cycles. We have that $adm^w = \{\emptyset, \{a\}, \{e\}\}$ and thus $pr^w = \{\{a\}, \{e\}\}$ (moreover $co^w = gr^w = \{\{a\}, \{e\}\}$). We next show that noeven AFs are not a tractable fragment for weak admissibility-based semantics. To this end we use the above AF $F$ to adapt the standard translation from propositional logic in order to obtain a noeven AF. That is we replace the mutual attacks of the $y$ arguments which model setting a variable to either true or false, by sub-AFs in the style of Figure 2. Recall that we discussed this AF in Example 2.7. This yields the AF depicted in Figure 3.

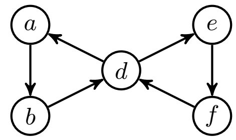



Figure 2: Noeven AF with two $pr^w$ extensions $\{a\}$ and $\{e\}$.

**Reduction 4.19.** Given a propositional formula $\phi(x_1, \ldots, x_n) = \bigwedge_{c \in C} \bigvee_{l \in c} l$ we define the AF $H_\phi = (A, R)$ with $A$ and $R$ as follows.

$$\begin{aligned} A =& \{x_i, \bar{x}_i, b_i, d_i, f_i \mid 1 \leq i \leq n\} \cup \{c \mid c \in C\} \cup \{t, \bar{t}\} \\ R =& \{(x_i, b_i), (b_i, d_i), (d_i, x_i), (d_i, \bar{x}_i), (\bar{x}_i, f_i), (f_i, d_i) \\ & \mid 1 \leq i \leq n\} \cup \{(c, t) \mid c \in C\} \cup \{(t, \bar{t})\} \cup \\ & \{(x_i, c) \mid x_i \in c \in C\} \cup \{(\bar{x}_i, c) \mid \neg x_i \in c \in C\} \end{aligned}$$

**Lemma 4.20.** *For every propositional formula $\phi$ we have that*

1. *$\phi$ is satisfiable iff $t$ is credulously accepted in $H_\phi$ w.r.t. $\sigma$, for $\sigma \in \{gr^w, adm^w, co^w, pr^w\}$; and*
2. *$\phi$ is unsatisfiable iff $\bar{t}$ is skeptically accepted in $H_\phi$ w.r.t. $\sigma$, for $\sigma \in \{gr^w, co^w, pr^w\}$.*

By the above, the NP-hard SAT problem can be reduced to credulous acceptance and the coNP-hard UNSAT problem can be reduced to skeptical acceptance.

**Proposition 4.21.** *For noeven AFs, $Cred_\sigma$ is NP-hard for $\sigma \in \{gr^w, adm^w, co^w, pr^w\}$ and $Skept_\tau$ is coNP-hard for $\tau \in \{gr^w, co^w, pr^w\}$.*

Note that here we do not have yet results for matching upper bounds and leave this for future work; we also do so for the problems $Ver_\sigma$ and $NEmpty_\sigma$.

### 4.5 Bipartite AFs

Finally, we consider the class of bipartite AFs. We have that bipartite AFs are a subclass of odd-cycle free AFs and thus we can again use the correspondence of w-preferred and preferred semantics. Moreover, preferred semantics has been shown to be tractable on bipartite AFs (Dunne, 2007), i.e., credulous and skeptical acceptance of an argument w.r.t. preferred semantics can be decided in polynomial time (Dunne, 2007, Theorem 6).

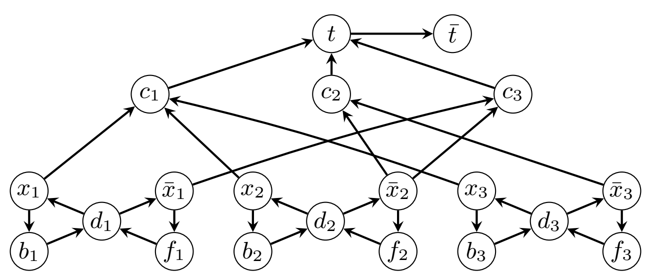



Figure 3: Illustration of the AF $H_\phi$, for $\phi$ with clauses $\{\{x_1, x_2, x_3\}, \{\bar{x}_2, \bar{x}_3\}, \{\bar{x}_1, \bar{x}_2\}\}$.

**Proposition 4.22.** *For bipartite AFs and $\sigma \in \{gr^w, adm^w, co^w, pr^w\}$, the problems $Cred_\sigma$, $Skept_\sigma$, $Ver_\sigma$, and $NEmpty_\sigma$ are in* $\mathsf{P}$ *and* $Skept_{pr^w} \in \mathsf{L}$

*Proof.* Let us first consider $Cred_\sigma$, $Skept_\sigma$: The results for $pr^w$ are directly from the corresponding results for $pr$ in (Dunne, 2007, Theorem 6). Furthermore, we have that $Cred_{gr^w} = Skept_{gr^w} = Skept_{pr^w}$ and thus reasoning with $gr^w$ is in $\mathsf{P}$. The results for $adm^w, co^w$ semantics are by its correspondences with the problems for $gr^w$ or $pr^w$. For $Ver_\sigma$, and $NEmpty_\sigma$ we first exploit that bipartite AFs are in particular odd-cycle free to obtain the $\mathsf{L}$ membership results (cf. Table 5). Moreover, the unique $gr^w$ can be computed in polynomial time by computing the skeptically accepted arguments and thus also $Ver_{gr^w}$, and $NEmpty_{gr^w}$ are in $\mathsf{P}$. Finally, verifying is $adm^w$ and $co^w$ is in $\mathsf{P}$ as the reduct of an argument set in a bipartite AF is a bipartite AF and thus we can test in polynomial time whether an extension is attacked by a weakly admissible set of its reduct and whether the unique $gr^w$ extension of the reduct is empty. □

Table 6: Complexity of w-admissible semantics in bipartite AFs

| $\sigma$ | $Cred_\sigma$ | $Skept_\sigma$ | $Ver_\sigma$ | $NEmpty_\sigma$ |
|---|---|---|---|---|
| $gr^w$ | $\mathsf{P}$-c | $\mathsf{P}$-c | in $\mathsf{P}$ | in $\mathsf{P}$ |
| $adm^w$ | $\mathsf{P}$-c | trivial | in $\mathsf{P}$ | in $\mathsf{L}$ |
| $co^w$ | $\mathsf{P}$-c | $\mathsf{P}$-c | in $\mathsf{P}$ | in $\mathsf{L}$ |
| $pr^w$ | $\mathsf{P}$-c | $\mathsf{P}$-c | in $\mathsf{L}$ | in $\mathsf{L}$ |

## 5. Utilizing Graph-Specific Properties

Since reasoning with weakly admissible semantics is $\mathsf{PSPACE}$-complete in general, potential implementations will heavily benefit from techniques reducing the size of the input AF that has to be processed. The removal of self-attackers (Baumann et al., 2020b, Theorem 3.10) is a handy pre-processing step, but clearly not sufficient to handle large problem instances. A possibly more promising approach is to partition a given AF into its strongly connected components (SCCs).

However, as pointed out in previous work (Dauphin et al., 2020, Proposition 11), weakly admissible, weakly complete, and weakly grounded semantics cannot be computed in the way SCC-recursiveness is usually defined. On the other hand, we answer an open problem (Dauphin et al., 2020, Open Question 2) affirmatively by showing that the technique from (Baroni et al., 2005) can indeed be applied to $pr^w$.

In this section, we recall the basic definitions for SCC-recursiveness (Baroni et al., 2005), discuss why $adm^w$ extensions cannot be computed this way and point out the differences to $pr^w$. Moreover, we discuss a workaround for $adm^w$ which also has the property that, very roughly speaking, the most relevant input value for the computation of extensions is no longer the size of the given AF, but the size of its largest SCC. We conclude the section by showing that $pr^w(F)$ can also be computed by applying the splitting technique (Baumann, 2011, Theorem 2), which can be seen as a simplification of SCC-recursiveness.

### 5.1 SCC Recursiveness

Although the idea behind SCC-recursiveness is natural and intuitive, formalizing the required concepts is a quite technical endeavor at first glance. Therefore, let us recall the basic intuition before introducing the technicalities. The first step is to cluster an AF according the strongly connected components (SCCs). This yields a directed acyclic graph (DAG). Now we consider some "initial" SCC $S$, i.e. an SCC which has no in-going edge from another SCC according to our DAG. We then apply a *base function* $\bar{\sigma}$ to $S$ which returns a set $E$ of arguments which is a $\bar{\sigma}$-extension of $F{\downarrow_S}$ (in a certain sense compatible with previously chosen sets of arguments). This choice influences the other SCCs in our DAG, so we need to re-calculate them which intuitively corresponds to calculating the reduct $F^E$ and then removing arguments in $S$ since we are finished with this SCC. The technical part is that we need to keep in mind which arguments are attacked by $S$ in a way that $E$ cannot provide any defense; therefore we make use of three different sets $D_F(S,E)$, $P_F(S,E)$, and $U_F(S,E)$ (see below) in order to keep track of all required information. When this procedure is done, we take the union over all extensions we have obtained by successive applications of the base function $\bar{\sigma}$; this union yields a $\sigma$-extension of the whole AF which is why $\bar{\sigma}$ is called base-function for $\sigma$.

Formally, given an AF $F = (A,R)$ a *strongly connected component* (or SCC) is a maximal set of arguments $S$ s.t. in $F{\downarrow_S}$ the following condition holds: For any two $a_1, a_n \in A\,(F{\downarrow_S})$ there is a sequence $a_1, \ldots, a_n$ of arguments with $a_i \in S$ and $(a_i, a_j) \in R$. We denote by $SCCS_F$ the set of all SCCs of $F$. For SCCs $S$ and $P$, if there is some $a \in P$ and $b \in S$ s.t. $(a,b) \in R$, then we call $P$ a *parent* of $S$. By $S^{\prec}$ we denote all ancestors of $S$ which are induced by this parent relation (not including $S$ itself). If $S^{\prec} = \emptyset$, then $S$ is called *initial*. For an SCC $S$ and $E \subseteq A$ we consider the following sets:

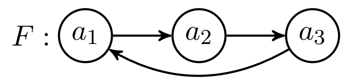
- $D_F(S,E) = \{a \in S \mid \exists P \in S^{\prec}, b \in E \cap P:\ (b,a) \in R\}$,
- $P_F(S,E) = \{a \in S \mid \exists b \in P \in S^{\prec} : (b,a) \in R,\ E \text{ does not attack } b\} \setminus D_F(S,E)$,
- $U_F(S,E) = S \setminus (D_F(S,E) \cup P_F(S,E))$.

Intuitively, $D_F(S,E)$ contains these arguments in $S$ which are defeated by arguments in $E$ due to previously considered SCCs. The set $P_F(S,E)$ contains arguments that are attacked by an argument in a previously considered SCC that is not counter attacked. The set $U_F(S,E)$ contains the remaining arguments. That is, for admissibility-based semantics our choice is restricted to arguments occurring in $U_F(S,E)$.

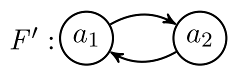
We let $UP_F(S,E) = U_F(S,E) \cup P_F(S,E)$. We say a semantics $\sigma$ is SCC-recursive if for any AF $F = (A,R)$, we have $\sigma(F) = \bar{\sigma}(F,A)$, where for any AF $F = (A,R)$ and any $C \subseteq A$, $\bar{\sigma}(F,C) \subseteq 2^A$ is given as follows: $E \subseteq A$ satisfies $E \in \bar{\sigma}(F,C)$ iff

- if $|SCCS(F)| = 1$, then $E \in \bar{\sigma}_b(F,C)$ for a "base function" $\bar{\sigma}_b(F,C)$,
- otherwise, for all $S \in SCCS(F)$ it holds that $E \cap S \in \bar{\sigma}(F{\downarrow_{UP_F(S,E)}}, U_F(S,E) \cap C)$.

Let us first consider an example illustrating why $adm^w$ is not SCC-recursive and to gain some intuition why this problematic mechanism does not apply to $pr^w$. The problem with SCC-recursiveness is that it is sometimes impossible to tell which attacks are meaningful (that is, coming from a weakly admissible extension of the reduct $F^E$) and which not. To illustrate this, let us quickly compare an even and an odd cycle and then move to a simple example consisting of just two SCCs. For $F$ forming an odd cycle

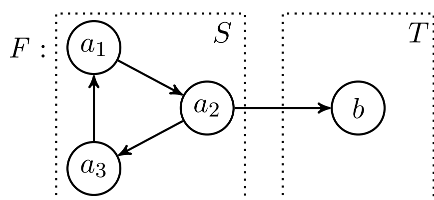



there is no non-empty weakly admissible extension: For example, set $E = \{a_1\}$. Then the reduct $F^E$ consists of the unattacked argument $a_3$ attacking $a_1$; thus $E \notin adm^w(F)$. In contrast, the even cycle

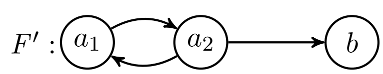



has two non-empty extensions $\{a_1\}$ and $\{a_2\}$; both are even stable and hence clearly weakly admissible as well. The problem with SCC-recursiveness can now be seen as in the following example.

**Example 5.1.** Consider the following AF $F$, consisting of two SCCs:



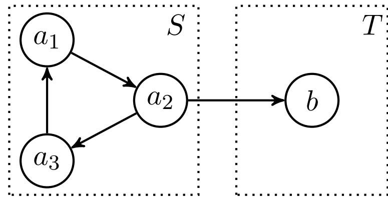

The only weakly admissible extension of the initial SCC $S$ (consisting of the odd cycle induced by $a_1$, $a_2$, and $a_3$) is $\emptyset$, as we just saw. Regarding the second SCC $T$, $\{b\}$ is an extension; however, $b$ receives an attack from an undefeated argument $a_2$. In this case, $b$ is acceptable since $a_2$ does not occur in any weakly admissible extension of $F^{\{b\}}$. However, if we turn the odd cycle into an even one



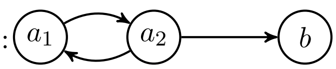

then suddenly $\{b\} \notin adm^w(F')$. However from the perspective of the second SCC $T$ the situation did not change: There is one argument with an input attack from an undecided argument $a_2$. ◇

This is why $adm^w$ is not SCC-recursive; but what is the catch for $pr^w$? To see the difference we characterize $pr^w$ with (Baumann et al., 2020a, Theorem 4.5): $E \in pr^w(F)$ if and only if $E \in cf(F)$ and $\bigcup adm^w\left(F^E\right) = \emptyset$; that is, if $E$ is weakly preferred, then *no* argument in the reduct $F^E$ is weakly admissible. This means, the unpredictable behavior as illustrated in the previous example does not occur for $pr^w$. To see this, let us recall the example from the point of view of weakly preferred semantics.

**Example 5.2.** Suppose $F$ is given as before:

$F$ :



We start by considering the initial SCC $S$: The only weakly preferred extension is $\emptyset$. So, moving to the second SCC $T$ we know that no argument in $S$ is acceptable and hence we can be sure that $\{b\}$ is acceptable in the whole AF; thus $pr^w(F) = \{\{b\}\}$ can be inferred. In case of an even cycle,

$F'$ :



the two weakly preferred extensions of the initial SCC are $\{a_1\}$ and $\{a_2\}$; hence it is correctly inferred that $pr^w(F') = \{\{a_1, b\}, \{a_2\}\}$. $\diamond$

Therefore, from an intuitive point of view, weakly preferred semantics are SCC-recursive for the same reason as stable semantics are, which is explained in the following. Consider some argument which is not in a given extension. For stable semantics, such arguments are defeated and therefore do not play any role anymore. In case of weakly preferred semantics, we know from Theorem 2.16 that these arguments do not occur in any weakly admissible extension of the reduct. Due to the definition of weak admissibility this means that these arguments can therefore be disregarded as well. The goal of the following considerations is to formalize this intuition.

**Lemma 5.3.** *Let $F = (A, R)$ be an AF and $U \subseteq A$ unattacked* [1].

- *If $E \in adm^w(F)$, then $E_u = E \cap U \in adm^w(F)$.*
- *Assume $E \subseteq U$. Then for all sets $U', U''$, s.t. $U \subseteq U' \subseteq A$ and $U \subseteq U'' \subseteq A$ we have that $E \in adm^w(F{\downarrow_{U'}})$ iff $E \in adm^w(F{\downarrow_{U''}})$.*

*Proof.* We show both claims simultaneously by induction over $n = |A|$. The base case $n = 0$ is trivial. We thus assume for a fixed integer $n$ and $|A| \leq n$ the following:
If $E \in adm^w(F)$, then $E_u = E \cap U \in adm^w(F)$. (IH-1)
If $E \subseteq U$, then $\forall U \subseteq U', U'' \subseteq A : E \in adm^w(F{\downarrow_{U'}})$ iff $E \in adm^w(F{\downarrow_{U''}})$. (IH-2)
Now assume that $F = (A, R)$ is given with $|A| = n + 1$. Let $U \subseteq A$ be unattacked.

- Let $E \in adm^w(F)$. By $E \in cf(F)$, $E_u \in cf(F)$ as well. Thus if $E_u \notin adm^w(F)$, then there is some $E' \in adm^w(F^{E_u})$ attacking $E_u \neq \emptyset$. In particular, $E' \cap U \neq \emptyset$ must attack $E_u$ since $U$ is unattacked. By (IH-1), $E'_u := E' \cap U \in adm^w(F^{E_u})$, too. Now consider $F^{E_u} := (A_u, R_u)$, the unattacked set $U_u = U \setminus E_u^{\oplus}$, $U' = A_u$, and $U'' = A_u \setminus E^{\oplus}$. By (IH-2) we find $E'_u \in adm^w\left(F^E\right)$ (note that $F^E = F^{E_u}{\downarrow_{U''}}$). This contradicts $E \in adm^w(F)$.

1. Recall that a set $U \subseteq A$ is called *unattacked* if there is no $a \in A \setminus U$ attacking $U$.

- Let $E \subseteq U$. As $E = \emptyset$ is trivial, we may assume $E \neq \emptyset$ yielding access to the induction hypothesis in the reduct. Let $U \subseteq U', U'' \subseteq A$. ($\Rightarrow$) Assume $E \in adm^w(F\downarrow_{U'})$. If $E \notin adm^w(F\downarrow_{U''})$ then there is some $E' \in adm^w((F\downarrow_{U''})^E)$ attacking $E$. By (IH-1), $E'_u = E' \cap U \in adm^w((F\downarrow_{U''})^E)$ as well. Since $U$ is unattacked, $E'_u$ must attack $E$. By (IH-2), $E'_u \in adm^w((F\downarrow_{U'})^E)$ which is in contradiction with $E \in adm^w(F\downarrow_{U'})$. ($\Leftarrow$) The reverse direction is by the same argument but exchanging the roles of $U'$ and $U''$. $\square$

We continue by giving two auxiliary lemmas, the first is to establish the required connection between consideration of the reduct of an SCC and the set $F\downarrow_{UP_F(S,E)}$. Recall that $UP_F(S,E) = \{a \in S \mid \nexists b \in E \setminus S : (b,a) \in R\}$.

**Lemma 5.4.** *Let $F = (A, R)$ be an AF, $E \subseteq A$ with $E \in cf(F)$ and $S \in SCCS_F$. Then*

$$\left(F\downarrow_{UP_F(S,E)}\right)^{E\cap S} = \left(F^E\right)\downarrow_{UP_F(S,E)}$$

*Proof.* Let $a \in S$.

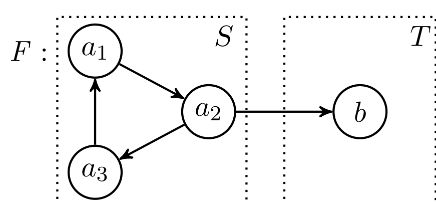
($\subseteq$) If $a \notin A\left(\left(F^E\right)\downarrow_{UP_F(S,E)}\right)$, then either (a) there is some $b \in E$ s.t. $a \in \{b\}^\oplus$, or (b) there is some $b \in E \setminus S$ s.t. $(b,a) \in R$. However, if (a) is not true, then neither is (b) so it suffices to discuss (a).

Now consider two cases: (1) If $b \in E \setminus S$, then $a$ does not occur in $F\downarrow_{UP_F(S,E)}$. (2) If $b \notin E \setminus S$, then $b \in E \cap S$. Since $E \in cf(F)$, $b$ occurs in $F\downarrow_{UP_F(S,E)}$. Since $a \in \{b\}^\oplus$, we infer $a \notin A\left(\left(F\downarrow_{UP_F(S,E)}\right)^{E\cap S}\right)$.

($\supseteq$) If $a \notin A\left(\left(F\downarrow_{UP_F(S,E)}\right)^{E\cap S}\right)$, then either (a) there is some $b \in E \setminus S$ s.t. $(b,a) \in R$, or (b) there is some $e \in E \cap S$ occurring in $F\downarrow_{UP_F(S,E)}$ with $a \in \{e\}^\oplus$.

If (a) is true, then $a$ does not occur in $F^E$ and hence $a \notin A\left(\left(F^E\right)\downarrow_{UP_F(S,E)}\right)$. If (b) is true, take $e \in E \cap S$ as described. From $e \in E$ and $a \in e^\oplus$ we again infer that $a$ does not occur in $F^E$. $\square$

Moreover, we need to be able to turn non-empty extensions of SCCs into non-empty extensions of the whole AF and vice versa. To illustrate this, recall the AF




from before: Here we see that $F$ possesses some non-empty weakly admissible extension since the second SCC does ($\{b\}$). In this case, the initial SCC does not possess one, so $\{b\} \in adm^w(F)$. If there was an SCC possessing some weakly admissible argument attacking $b$, then we would move to this SCC and continue the argument inductively. For technical reasons, we need to formalize this for the reduct $F^E$ for some extension $E$ instead of $F$ itself. To this end we will use the *directionality property* of w-admissible and w-preferred semantics, which we recall in the next lemma.

**Lemma 5.5** ((Baumann et al., 2020a; Dauphin et al., 2020))**.** *For any AF $F$ and $U \subseteq A$ unattacked we have (a) $\{E \cap U \mid E \in adm^w(F)\} = adm^w(F{\downarrow_U})$ and (b) $\{E \cap U \mid E \in pr^w(F)\} = pr(F{\downarrow_U})$.*

**Lemma 5.6.** *For any AF $F = (A, R)$, $U \subseteq A$ unattacked, and $adm^w(F{\downarrow_U}) = \{\emptyset\}$ we have that $adm^w(F) = adm^w(F{\downarrow_{A\setminus U}})$.*

*Proof.* The proof is by induction on the size of $B = A\setminus U$. If $|B| = 0$ the statement is trivially true. For the induction step we assume that the claim holds for $|B| \leq n$ and show that then it also holds for $|B| = n+1$. By definition $\emptyset \in adm^w(F)$ as well as $\emptyset \in adm^w(F{\downarrow_{A\setminus U}})$ and thus it is sufficient to consider non-empty extensions.

($\Rightarrow$) Consider non-empty $E \in adm^w(F)$. By $adm^w(F{\downarrow_U}) = \{\emptyset\}$ and the directionality property we have $E \subset A \setminus U$. Towards a contradiction assume that $E \notin adm^w(F{\downarrow_{A\setminus U}})$. As $E$ is conflict-free, there is a $D \in adm^w(F^E{\downarrow_{A\setminus U}})$ (note that $F^E{\downarrow_{A\setminus U}} = (F{\downarrow_{A\setminus U}})^E$) that attacks $E$. On $F^E$ we can apply the induction and obtain $D \in adm^w(F^E)$ which is in contradiction to $E \in adm^w(F)$. Hence $E \in adm^w(F{\downarrow_{A\setminus U}})$.

($\Leftarrow$) Consider non-empty $E \in adm^w(F{\downarrow_{A\setminus U}})$. Towards a contradiction assume that $E \notin adm^w(F{\downarrow_{A\setminus U}})$. As $E$ is conflict-free, there is an $D \in adm^w(F^E)$ that attacks $E$. In $F^E$ we can apply the induction hypothesis and obtain $D \in adm^w(F^E{\downarrow_{A\setminus U}})$ which is in contradiction to $E \in adm^w(F{\downarrow_{A\setminus U}})$. Hence $E \in adm^w(F)$. $\square$

This yields the following:

**Lemma 5.7.** *Let $F = (A, R)$ and let $E \subseteq A$. There is an SCC $S \in SCCS_F$ with $adm^w\left((F^E){\downarrow_{UP_F(S,E)}}\right) \neq \{\emptyset\}$ if and only if $adm^w\left(F^E\right) \neq \{\emptyset\}$.*

*Proof.* The proof is by induction on the number of SCCs. The statement clearly holds when $F$ is strongly connected, i.e. there is just one SCC. For the induction step we assume that the claim holds if the number of SCCs is at most $n$ and show that then it also holds if the number of SCCs is $n+1$.

($\Rightarrow$) Assume that there is such an SCC $S$ with $adm^w\left((F^E){\downarrow_{UP_F(S,E)}}\right) \neq \{\emptyset\}$. First consider the AF $G = F{\downarrow_{\bigcup_{P\in S^{\prec}} P}}$ which has at most $n$ SCCs. If $adm^w\left((G^E){\downarrow_{UP_G(P,E)}}\right) = adm^w\left((F^E){\downarrow_{UP_F(P,E)}}\right) \neq \{\emptyset\}$ for some $P \in S^{\prec}$, then by induction hypothesis we have that $adm^w\left(G^E\right) \neq \{\emptyset\}$. As $\bigcup_{P\in S^{\prec}} P$ is unattacked in $F$, we can apply the directionality property and obtain that $adm^w\left(F^E\right) \neq \{\emptyset\}$.

Now consider the case where $adm^w\left(G^E\right) = \{\emptyset\}$ and consider the AF $H = F{\downarrow_{(S\cup\bigcup_{P\in S^{\prec}} P)}}$. We have a non-empty $E_0 \in adm^w\left(H^E{\downarrow_{UP_H(S,E)}}\right)$ and by Lemma 5.6 we obtain $E_0 \in adm^w\left(H^E\right)$. As $S \cup \bigcup_{P\in S^{\prec}} P$ is unattacked in $F$ we can apply the directionality property to obtain $adm^w\left(F^E\right) \neq \{\emptyset\}$.

($\Leftarrow$) If $adm^w\left(F^E\right) \neq \{\emptyset\}$, then there is some w-preferred extension $\emptyset \neq E_0 \in adm^w\left(F^E\right)$. Let $S \in SCCS_F$ be an SCC with $E_0 \cap S \neq \emptyset$ s.t. $E_0 \cap P = \emptyset$ for any $P \in S^{\prec}$. For the AF $H = F{\downarrow_{(S\cup\bigcup_{P\in S^{\prec}} P)}}$, by directionality we have that $E_0 \cap S \in adm^w(H^E)$. Further by Lemma 5.6 we obtain $E_0 \cap S \in adm^w\left(H^E{\downarrow_{UP_H(S,E)}}\right)$ which is equivalent to $E_0 \cap S \in adm^w\left(F^E{\downarrow_{UP_F(S,E)}}\right)$. $\square$

Now we follow (Baroni et al., 2005), Section 5.2, where $\sigma = stb$ is considered, with adjustments to make it work for $pr^w$:

**Theorem 5.8.** *Let $F = (A, R)$ be an AF. Then $E \in pr^w(F)$ iff for any SCC $S \in SCCS_F$, $E \cap S \in pr^w(F{\downarrow}_{UP_F(S,E)})$.*

*Proof.* ($\Rightarrow$) Let $E \in pr^w(F)$. Let $S \in SCCS_F$.

(well-defined) It is clear that $E \cap S \subseteq UP_F(S, E)$ since otherwise, $E \notin cf(F)$. Thus $E \cap S$ in an extension in $F{\downarrow}_{UP_F(S,E)}$.

(cf) We have that $E \cap S \in cf(F{\downarrow}_{UP_F(S,E)})$ due to $E \in cf(F)$.

(w-pref) Since $E$ is w-preferred, $F^E$ does not contain any w-admissible argument. Now assume $E \cap S$ is not w-preferred in $F{\downarrow}_{UP_F(S,E)}$. Since $E \cap S$ is conflict-free, this means there must be a non-empty w-admissible extension in $\left(F{\downarrow}_{UP_F(S,E)}\right)^{E\cap S}$. Hence by Lemma 5.4 there is a non-empty w-admissible extension in

$$\left(F{\downarrow}_{UP_F(S,E)}\right)^{E\cap S} = \left(F^E\right){\downarrow}_{UP_F(S,E)} .$$

Thus by Lemma 5.7 there is a non-empty w-admissible extension in $F^E$, i.e. $E \notin pr^w(F)$; a contradiction.

($\Leftarrow$) We have to show that $E \in cf(F)$ and $adm^w\left(F^E\right) = \{\emptyset\}$. The former is clear since each argument is chosen among $UP_F(S, E)$. Furthermore, in each SCC $F$ we have that

$$\{\emptyset\} = adm^w\left(\left(F{\downarrow}_{UP_F(S,E)}\right)^{E\cap S}\right) = adm^w\left(\left(F^E\right){\downarrow}_{UP_F(S,E)}\right)$$

yielding $adm^w\left(F^E\right) = \{\emptyset\}$ by Lemma 5.7. □

We can use the above characterization to give a fixed-parameter tractable algorithm (w.r.t. size of the SCCs) for the verification problem. This is a significant improvement over the PSPACE-hardness in the general case.

**Corollary 5.9.** *Let $F = (A, R)$ be an AF where each SCC of $F$ contains at most $k$ arguments and $E \subseteq A$. Verifying whether $E \in pr^w(F)$ is in time $O(k^k \cdot poly(|A|))$.*

*Proof.* By Theorem 5.8 we can consider each SCC $S \in SCCS_F$ separately and have to check whether $E \cap S \in pr^w(F{\downarrow}_{UP_F(S,E)})$. Notice that the number of SCCs is in $O(|A|)$ and we can compute the SCCs $S \in SCCS_F$ as well as all the sub-AFs $F{\downarrow}_{UP_F(S,E)}$ in polynomial time ($poly(|A|)$).

Now consider an SCC $S$, $E' = E \cap S$ and $G = (A_G, R_G) = F{\downarrow}_{UP_F(S,E)}$. We know that either (i) $G$ is of size at most $k - 1$ or (ii) $G$ is strongly connected and of size at most $k$. If $E'$ has a conflict, then we are done, otherwise we have to check that $\bigcup adm^w(G^{E'}) = \emptyset$ (cf. Theorem 2.16). We do that by iterating over all non-empty sets $D \subseteq A_G$ and testing whether $D \in adm^w(G^{E'})$. For the last step recall the straightforward algorithm for verifying a weakly admissibility set:

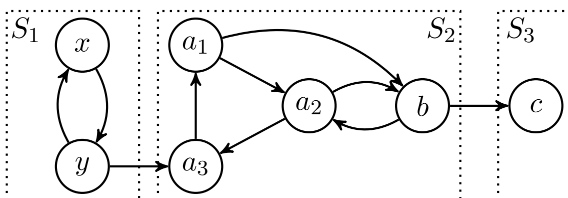
1. test whether $D \in cf(G)$; if not return false,
2. compute the reduct $G^D$,
3. iterate over all subsets $S$ of $G^D$ that contain at least one attacker of $E$ and test whether $S$ is w-admissible; if so return false; else return true.

Notice that the recursion depth of the algorithm is bounded by $k-1$ as the AF shrinks by at least one argument in each call. In case (ii) we additionally have, as $G$ is strongly connected, that in the first call at least two arguments are removed. Now consider a branch of the recursion tree. Such a branch is identified by the arguments that appear in the sets that are tested at each of the recursion levels. That is, a branch is a partition of the arguments $A_G$ into $k$ sets, the first $k-1$ corresponding to the sets that are tested on the corresponding recursion levels and the last set corresponds to the arguments that do not appear in any of these sets. That is we have at most $O(k^{|A_G|}) = O(k^k)$ branches in the recursion tree and thus the verification algorithm runs in time $O(k^k \cdot poly(|k|))$. In total, we need to run the verification algorithm $O(|A|)$ many times which give us a total time of $O(k^k \cdot poly(|A|))$. $\square$

As already pointed out, $adm^w$ is not SCC-recursive. However, in the following we want to prove that there is a workaround for this problem, i.e. $adm^w$ can be computed by iteratively considering all SCCs of the given AF. The difference is that we require additional information and therefore do not adhere by the properties proposed in the definition of SCC-recursiveness.

Let us consider an example in order to understand the underlying intuition.

**Example 5.10.** Consider the following AF $F$:




With the usual SCC-recursive scheme we cannot correctly decide whether $\{b\}$ is weakly admissible, although it is a w-admissible extension of the second SCC $S_2$. However, $\{y, a_1\} \in adm^w(F^{\{b\}})$ shows that $\{b\} \notin adm^w(F)$. This information can be extracted by inspecting only w-admissible arguments of SCCs that have already been considered. $\diamond$

Indeed, the following result formalizes that one can verify weak admissibility of some $E \subseteq A$ by computing $pr^w(F^E)$ with the aforementioned SCC-recursive scheme and then checking certain conditions for each SCC.

**Proposition 5.11.** *Let $F = (A, R)$ be an AF. Then $E \in adm^w(F)$ iff for any SCC $S \in SCCS_F$, letting $G = \bigcup_{P \in S^{\prec}} (F^E)\downarrow_P$, we have that*

- *no extension $E' \in pr^w(G)$ attacks $E \cap S$,*
- *the following two conditions must hold within $S$:*
  - *$E \cap S \in cf(F\downarrow_{UP_F(S,E)})$,*
  - *there is no $E' \in pr^w(G)$ s.t. there is $E'' \in pr^w\left(\left((F^E)^{E'}\right)\downarrow_S\right)$ attacking $E \cap S$.*

*Proof.* ($\Rightarrow$) Let $E \in adm^w(F)$. Let $S \in SCCS_F$.

- Towards a contradiction assume that an extension $D \in adm^w(G)$ attacks $E \cap S$, then by directionality (see Lemma 5.5), there is a set $D' \in adm^w\left(F^E\right)$ with $D \subseteq D'$ that attacks $E$.

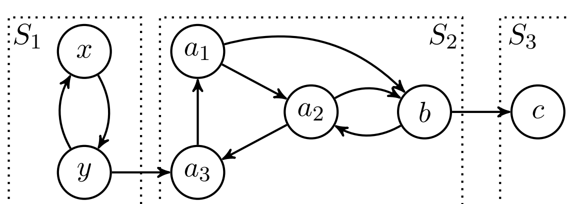
- Now let us turn to the arguments within $S$.
  - $E \cap S \in cf(F{\downarrow_{UP_F(S,E)}})$ is clear as $E \in cf(F)$.
  - Now suppose there is $E' \in pr^w(G)$ s.t. there is a weakly admissible extension $E'' \in adm^w\left(\left((F^E)^{E'}\right){\downarrow_S}\right)$ attacking $E \cap S$. We can now infer that $E' \cup E'' \in adm^w(F^E)$, thus $E \cap S$ (and hence $E$) possesses some attacker in the corresponding reduct; contradiction.

($\Leftarrow$) Now suppose the described properties are satisfied. We have to show $E \in adm^w(F)$. First observe that $E$ is clearly conflict-free. Now consider the reduct $F^E$ and assume there is some $E' \in pr^w(F^E)$ attacking $E$. There must be some SCC $S$ s.t. $E'$ attacks $E \cap S$. In case $S$ is initial, the second or third condition must be false; therefore assume $S$ is not initial.

Let $G = \bigcup_{P \in S^{\prec}} \left(F^E\right){\downarrow_P}$. Without loss of generality, we assume that for each $P \in S^{\prec}$ it is not true that $E'$ attacks $E \cap P$. In case some extension in $adm^w(G)$ attacks $E \cap S$, we found a counterexample using the same arguments as in Lemma 5.7. Analogously, if there is some $E' \in pr^w(G)$, then $E' \in adm^w(F^E)$ as well. Now suppose $E'' \in adm^w\left(\left((F^E)^{E'}\right){\downarrow_S}\right)$. Since $E' \in pr^w(G)$, there is no weakly admissible argument in $G^{E'}$. We therefore deduce that $E' \cup E'' \in adm^w(F^E)$; contradiction. □

To be able to proceed as described in Proposition 5.11, we require all w-preferred extensions of the current reduct $F^E$. This can be done iteratively due to Theorem 5.8: Given $S \in SCCS_F$ and the w-preferred extensions of $\bigcup_{P \in S^{\prec}} \left(F^E\right){\downarrow_P}$, we obtain the w-preferred extensions of $\bigcup_{P \in S^{\prec}} \left(F^E\right){\downarrow_P} \cup (F^E){\downarrow_S}$ by augmenting each $E' \in pr^w\left(\bigcup_{P \in S^{\prec}} \left(F^E\right){\downarrow_P}\right)$ with some $E'' \in pr^w\left(((F^E)^{E'}){\downarrow_S}\right)$.

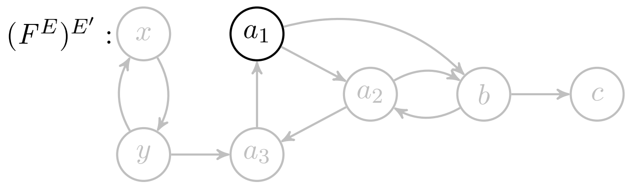
**Example 5.12.** Recall the previous AF $F$ with three SCCs $S_1 = \{x, y\}$, $S_2 = \{a_1, a_2, a_3, b\}$, and $S_3 = \{c\}$.




Let us check whether $E = \{b\}$ is weakly admissible. So regarding the initial SCC $S_1$ we consider $E_1 = \emptyset$ since $E \cap S_1 = \emptyset$. The reduct $(F^{E_1}){\downarrow_{S_1}} = F{\downarrow_{S_1}}$ has the w-preferred extensions $\{x\}$ and $\{y\}$. Now consider $S_2$ and the extension $E_2 = \{b\}$. We have $F{\downarrow_{UP_F(S,E)}} = F{\downarrow_{S_2}}$. Let us check the requirements:

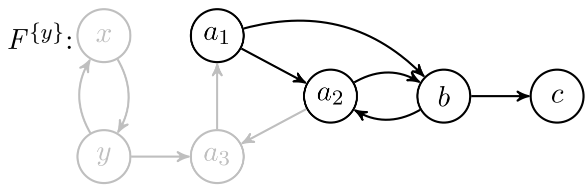
- Neither $\{x\}$ nor $\{y\}$ attacks $\{b\}$.

- Now consider $S_2$:
  - $\{b\} \in cf(F{\downarrow}_{UP_F(S,E)})$ is also true, but:
  - letting $E' = \{y\}$, the AF $(F^E)^{E'}$ has $\{a_1\}$ as unattacked set which attacks $b$.




Therefore, our procedure rightfully detects that $\{b\} \notin adm^w(F)$; the counterexample $\{y, a_1\} \in adm^w(F^E)$ is found.

Now we try $E = \{y, c\}$. For the initial SCC $S_1$ we have $E_1 = \{y\}$. Since $E_1$ is w-preferred in the initial SCC $S_1$, we proceed. The reduct $(F^{E_1}){\downarrow}_{S_1}$ is empty and has no non-empty w-preferred extension.




Now consider $S_2$ and the extension $E_2 = \emptyset$. There is nothing to check so we only update the w-preferred extensions of the reduct. Since $\{a_1\}$ is the only candidate, we now have $\{a_1\}$ as the only w-preferred extension in the reduct $(F^E){\downarrow}_{S_1} \cup (F^E){\downarrow}_{S_2}$.

Regarding $S_3$, one can now check that $\{c\}$ satisfies all requirements since $\{a_1\}$ defeats the only attacker $\{b\}$. Therefore, we rightfully obtain $\{y, c\} \in adm^w(F)$. $\diamond$

The drawback of the technique we just mentioned is that computing $pr^w(F^E)$ is required. In the following, we give an alternative characterization where the reduct only needs to be examined for the current SCC. The difference is that $pr^w(F)$ is required instead of $pr^w(F^E)$. In order to establish the following result, let us mention some technical background we are going to utilize. First, if $E = E' \dot\cup E''$ is conflict-free, then

$$(F^E)^{E'} = F^{E \cup E'} = F^E = F^{E' \cup E} = (F^{E'})^E$$

and moreover, the modularization property for $adm^w$ works in both directions, i.e. if $E = E' \dot\cup E''$ with $E' \in adm^w(F)$, then $E \in adm^w(F)$ if and only if $E'' \in adm^w(F^{E'})$. We also recall that $E \in pr^w(F)$ iff $E \in cf(F)$ and $\bigcup adm^w(F^E) = \emptyset$. From these properties it follows that if $E = E' \dot\cup E''$ with $E' \in adm^w(F)$, then $E \in pr^w(F)$ if and only if $E'' \in pr^w(F^{E'})$. Throughout the following proof, these properties are frequently used.

**Proposition 5.13.** *Let $F = (A, R)$ be an AF. Then $E \in adm^w(F)$ iff for any SCC $S \in SCCS_F$, letting $H = F{\downarrow}_{\bigcup_{P \in S^{\prec}} P}$, we have that*

- *$E \in cf(F)$,*

- *the following condition must hold within $S$:*
  - *there is no $E' \in pr^w(H)$ with $E \cap (\bigcup_{P \in S^{\prec}} P) \subseteq E'$ s.t. $E \cap S \notin adm^w((F^{E'})\downarrow_S)$.*

*Proof.* ($\Rightarrow$) Suppose the claim is not true for some SCC $S$. We show $E \notin adm^w(F)$. Without loss of generality assume $E \cap A(H) \in adm^w(H)$. The case $E \notin cf(F)$ is clear. So suppose there is some $E' \in pr^w(H)$ with $E \cap (\bigcup_{P \in S^{\prec}} P) \subseteq E'$ and $E \cap S \notin adm^w((F^{E'})\downarrow_S)$.

First let us make the following observation: Letting $E^* = E' \setminus E$ we have that $E^*$ is weakly admissible in $H^E$ since $E' = (E \cap A(H)) \cup E^*$ and the fact that $E \cap A(H) \in adm^w(H)$. Moreover, $(H^E)^{E^*} = H^{E'}$. Therefore, the existence of our w-preferred $E'$ yields some $E^* \in adm^w(H^E)$ with $(H^E)^{E^*} = H^{E'}$. Since $A(H)$ is unattacked in $F$ and hence also $A(H^E)$ in $F^E$, our previous considerations yield $E^* \in adm^w(F^E)$. Let us fix this $E^* = E' \setminus E$. We now consider the cases why $E \cap S \notin adm^w((F^{E'})\downarrow_S)$ might hold.

1. Assume $E \cap S \notin cf((F^{E'})\downarrow_S)$. Since $E \in cf(F)$ this must be due to the fact that some $a \in E$ does not occur in $(F^{E'})\downarrow_S$. Since $E \in cf(F)$, $E^*$ must attack $E \cap S$; from $E^* \in adm^w(F^E)$ we infer $E \notin adm^w(F^E)$.

2. Now suppose $E \cap S \in cf((F^{E'})\downarrow_S)$, but there is some weakly admissible attacker in $E'' \in adm^w(((F^{E'})^{E \cap S})\downarrow_S)$. Since $E' \in pr^w(H)$, $H^{E'}$ does not possess any non-empty w-admissible extension. Since $A(H)$ is unattacked, the same is true in $F^{E'}\downarrow_{\bigcup_{P \in S^{\prec}} P}$. From Lemma 5.6 we therefore deduce $E'' \in adm^w((F^{E'})^{E \cap S} \downarrow_{(S \cup \bigcup_{P \in S^{\prec}} P)})$ and thus by directionality and Lemma 5.3, $E'' \in adm^w((F^{E'})^{E \cap S})$. Since
$$(F^{E'})^{E \cap S} = (F^{E \cap A(H) \cup E^*})^{E \cap S}$$
holds, modularization yields $E'' \cup E^* \in adm^w(F^{E \cap (A(H) \cup S)})$ and finally by directionality and Lemma 5.3, $E'' \cup E^* \in adm^w(F^E)$. This yields $E \notin adm^w(F)$.

We obtain $E \notin adm^w(F)$, a contradiction to the initial assumption.

($\Leftarrow$) Suppose $E \notin adm^w(F)$. Let $S$ be an SCC s.t. $E \cap S$ possesses w-admissible attackers in $F^E$ and suppose the same is not true in $H$. Let $E_0 \in pr^w(F^E)$ attack $E \cap S$. We distinguish two cases:

1. Suppose $E_0 \cap A(H)$ attacks $E \cap S$. By directionality $E_H := E_0 \cap A(H) \in pr^w(H^E)$ as well. By modularization, $E' := (E_H \cup E) \cap A(H) \in pr^w(H)$. Since $E'$ attacks $E \cap S$, we obtain $E \cap S \notin cf^w((F^{E'})\downarrow_S)$ and thus $E \cap S \notin adm^w((F^{E'})\downarrow_S)$ .

2. Otherwise, $E_0 \cap S$ attacks $E \cap S$. By SCC-recursiveness of $pr^w$, we have that $E_0 \in pr^w(F^E)$ implies that $E_0 \cap S \in pr^w(((F^E)^{E_0 \cap A(H)})\downarrow_S)$. By choice of $E'$ as in case 1, this is yields $E_0 \cap S \in pr^w((F^{E'})^{E \cap S}\downarrow_S)$ showing $E \cap S \notin adm^w((F^{E'})\downarrow_S)$.

We obtain $E \cap S \notin adm^w((F^{E'})\downarrow_S)$, a contradiction to the initial assumption. $\square$

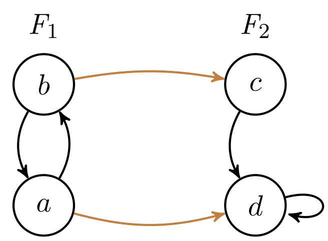
### 5.2 Splitting

In this section, we will briefly discuss splitting (Baumann, 2011). In accordance with the previous section, we will see that the usual notion of splitting can be applied to weakly preferred semantics.

**Definition 5.14.** Let $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$ be two AFs with $A_1 \cap A_2 = \emptyset$ and let $R_3 \subseteq A_1 \times A_2$. We call $(F_1, F_2, R_3)$ a *splitting* of the AF $F = (A_1 \cup A_2, R_1 \cup R_2 \cup R_3)$.

Please note that the subframework $F_2$ does not attack arguments in $F_1$. Consider the following example.

**Example 5.15.** Let $F_1 = (A_1, R_1)$ with $A_1 = \{a, b\}$ and $R_1 = \{(a, b), (b, a)\}$ as well as $F_2 = (A_2, R_2)$ with $A_2 = \{c, d\}$ and $R_2 = \{(c, d), (d, d)\}$, and let $R_3 = \{(a, d), (b, c)\}$. Then, $(F_1, F_2, R_3)$ is a splitting of the following AF.

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The underlying idea of a splitting is as follows: Once we have computed an extension $E_1$ of the AF $F_1$, we want to construct a reduced version of $F_2$ which reflects the acceptance of $E_1$. Then we compute an extension $E_2$ of this reduced AF to obtain an extension $E_1 \cup E_2$ of $F$. In the following we define how to reduce $F_2$ for stable semantics.

The following theorem taken from (Baumann, 2011, Theorem 2) states that this observation is no coincidence. Indeed, we can indeed find stable extensions $E$ of the whole AF $F$ by computing a stable extension $E_1$ of $F_1$ and merge it (in case of existence) with a stable extension of the reduced version of $F_2$.

**Theorem 5.16.** *Let $F = (A, R)$ be an AF and let $(F_1, F_2, R_3)$ be a splitting of $F$ with $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$.*

- *If $E_1 \in stb(F_1)$ and $E_2 \in stb(F^{E_1} \cap F_2)$, then $E_1 \cup E_2 \in stb(F)$.*
- *Vice versa, if $E \in stb(F)$, then $E \cap A_1 \in stb(F_1)$ and $E \cap A_2 \in stb(F^{E_1} \cap F_2)$.*

In case of stable semantics we have that an extension $E_1$ of the bottom part $F_1$ attacks any remaining argument in $F_1$ by definition. This is definitely not true for all other semantics considered in this article. This is a problem in case of admissibility-based semantics since potential attackers of an extension $E_2$ of the top part (namely the $(E_1, R_3)$-reduct of $F_2$) might be unattacked. Indeed, this is the main reason why the proposed splitting method does not work for this family of semantics. We refer the reader to (Baumann, 2014, Example 4.5.) for an illustration. What about semantics based on weak admissibility? In case of weakly preferred semantics we have that the reduct $F_1^{E_1}$ does not contain any non-trivial weakly admissible set. Thus, possibly existing attackers of weakly preferred extensions in $F^{E_1} \cap F_2$ are not *serious* ones. Indeed, it turns out that this property is sufficient to disregarded them providing us with a similar splitting result for weakly preferred semantics.

**Theorem 5.17.** *Let $F = (A, R)$ be an AF and let $(F_1, F_2, R_3)$ be a splitting of $F$ with $F_1 = (A_1, R_1)$ and $F_2 = (A_2, R_2)$.*

- *If $E_1 \in pr^w(F_1)$ and $E_2 \in pr^w(F^{E_1} \cap F_2)$, then $E_1 \cup E_2 \in pr^w(F)$.*
- *Vice versa, if $E \in pr^w(F)$, then $E \cap A_1 \in pr^w(F_1)$ and $E \cap A_2 \in pr^w(F^{E_1} \cap F_2)$.*

*Proof.* The proof relies on the modularization property of w-admissible extensions and the following characterization of $pr^w$: $E \in pr^w(F)$ iff $E \subseteq cf(F)$ and $adm^w\left(F^E\right) = \{\emptyset\}$.

- If $E_1$ is a w-preferred extension of $F_1$, then $E_1$ is w-admissible in the whole AF $F$ (Lemma 5.3) and $F_1^{E_1}$ does not contain w-admissible arguments. By the latter, a w-preferred extension $E_2$ of $F^{E_1} \cap F_2$ is w-admissible in $F^{E_1}$ and by modularization, $E_1 \cup E_2 \in adm^w(F)$. Being a w-preferred extension of $F^{E_1} \cap F_2$, $E_2$ does not tolerate w-admissible arguments in the reduct, i.e. $\left(F^{E_1} \cap F_2\right)^{E_2}$ possesses no w-admissible argument. We now show

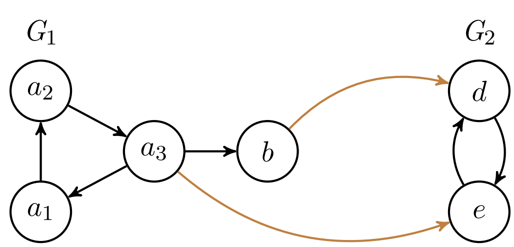
$$F^{E_1 \cup E_2} \cap F_2 = \left(F^{E_1} \cap F_2\right)^{E_2}.$$

  - ($\subseteq$) Assume $a \in A\left(F^{E_1 \cup E_2} \cap F_2\right)$. Then $a \in A(F_2)$ such that it is neither attacked by or contained in $E_1$ nor $E_2$. Hence $a$ occurs in $F_2$ and is not attacked by $E_1$, so $a \in A(F^{E_1} \cap F_2)$. Since $a$ is not attacked by $E_2$, $a \in A\left(\left(F^{E_1} \cap F_2\right)^{E_2}\right)$ follows.
  - ($\supseteq$) Now let $a \in A\left(\left(F^{E_1} \cap F_2\right)^{E_2}\right)$. Hence $a \in A(F_2)$ such that $a$ is not attacked by or contained in $E_1$. Since $E_1 \dot\cup E_2$ is conflict-free, $E_2 \subseteq A(F^{E_1})$ and by $E_2 \subseteq A(F_2)$ we get $E_2 \subseteq A(F^{E_1} \cap F_2)$. So from $a \in A\left(\left(F^{E_1} \cap F_2\right)^{E_2}\right)$ we conclude that $a$ is not attacked by or contained in $E_2$, either. So $a \in A(F_2)$ such that neither $E_1$ nor $E_2$ attack or contain $a$, i.e. $a \in A\left(F^{E_1 \cup E_2} \cap F_2\right)$.

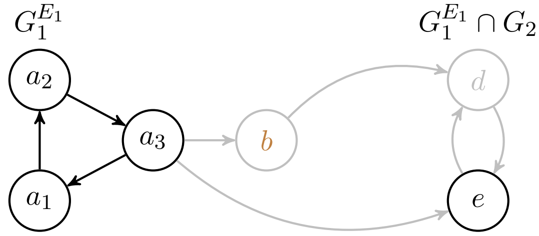
  Thus there is no w-admissible argument in $F^{E_1 \cup E_2} \cap F_2$ and since $E_1 \in pr^w(F_1)$, there is no w-admissible argument in $F^{E_1 \cup E_2} \cap F_1$, either. Thus $E_1 \cup E_2$ is w-preferred.

- By directionality, $E_1$ is w-admissible in $F_1$ and by Theorem 2.15, $E_2$ must be w-admissible in $F^{E_1}$. Finally $E_1 \cup E_2$ being preferred means $F_1^{E_1}$ does not tolerate w-admissible arguments and neither does $F^{E_1 \cup E_2} \cap F_2 = \left(F^{E_1} \cap F_2\right)^{E_2}$. By the former, $E_1 \in pr^w(F_1)$ and by the latter, $E_2 \in pr^w(F^{E_1} \cap F_2)$. □

**Example 5.18.** Consider $G_1$ and $G_2$ as well as $R_3 = \{(b, d), (a_3, e)\}$ as depicted below. The triple $(G_1, G_2, R_3)$ is a splitting of the following AF $G$.



We obtain $E_1 = \{b\}$ as the unique w-preferred extension of $G_1$. Moreover, $E_2 = \{e\}$ is the unique w-preferred extension of $G_1^{E_1} \cap G_2$. Applying the splitting theorem yields $E = \{b, e\}$ as the only w-preferred extension of $G$. Indeed, $E$ is conflict-free in $G$ and moreover, the attackers of $E$, namely $d$ and $a_3$ are either not contained in the three-cycle $G^E$ or not part of any w-admissible extension since $\bigcup adm^w\left(G^E\right) = \emptyset$.

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## 6. Conclusion

In this paper, we investigated the computational complexity of the standard reasoning problems for weakly admissibility-based semantics. More specifically we examined the verification problem, the problem of deciding whether or not a given AF possesses a non-empty extension, as well as credulous and skeptical acceptance of a given argument. It turns out that all of them, except the trivial skeptical acceptance for $adm^w$, are PSPACE-complete in general. The lower bound was proved by a suitable adjustment of the well-known standard translation from propositional formulas to AFs, with some noteworthy novel features: i) the argument $\phi$ representing whether or not the formula evaluates to true is not attacked by the arguments $C_i$ representing the clauses, but only by two of the variables, ii) the arguments representing the variables occurring in the given formula attack each other forming several layers in order to implement the quantifier alternation, iii) auxiliary arguments $p_j$ are required to guide the simulation of the aforementioned alternation, and iv) none of the arguments corresponding to variables in the QBF at hand are contained in a w-admissible extension of the constructed AF; the important part of the construction is the interaction of arguments which are *not* accepted.

The PSPACE-completeness of weakly admissibility-based semantics is in contrast to the complexity of standard admissibility-based semantics which are located in the first two levels of the polynomial hierarchy (Dvořák & Dunne, 2018). Moreover, this also holds for semantics like cf2 (Baroni et al., 2005) or stage2 (Dvořák & Gaggl, 2016), that (a) follow a similar endeavor of dealing with the acceptance w.r.t. odd-length cycles and (b) follow a recursive approach. Notice that when verifying an extension, the SCC-recursive approach can be easily resolved in an iterative fashion, which is in polynomial time and requires only a linear number of calls to a base function, applied to verify parts of the extension on SCCs. For cf2 or stage2 these base functions are of comparably low computational complexity and thus these semantics remain in the first two levels of the polynomial hierarchy (Gaggl & Woltran, 2013; Dvořák & Gaggl, 2016). On the other hand, in order to verify a weakly admissible extension on a strongly connected graph, we have to make several recursive calls on sub-graphs, which even makes evaluating the base function on a single SCC hard.

In the light of this high computational complexity we investigated graph classes that are known to yield computationally easier fragments for standard argumentation semantics, i.e., symmetric, acyclic, odd-cycle free, noeven, and bipartite AFs. Our results show that for some of these graph classes, there is a significant drop in the computational complexity. We also show that weakly preferred semantics can be computed along the strongly connected components. On the one hand this implies a fixed-parameter tractability result w.r.t. the size of the biggest SCC of the AF and on the other hand it allows to use the results on tractable fragments for SCCs that fall in such a tractable graph class even when the whole AF does not belong to such a graph class. An interesting future work in this direction is of course settling the exact complexity of weak-admissible based semantics for noeven AFs. Moreover, we initiated the investigation of fixed-parameter tractable algorithms, but these results need to be further developed.

The computational complexity of argumentation frameworks is not only well-understood in terms of the standard problems, but also more involved aspects have been studied especially in the field of dynamics like enforcing a desired set of arguments (Wallner, Niskanen, & Järvisalo, 2017; Niskanen, Wallner, & Järvisalo, 2018), incorporating new beliefs (Falappa, Kern-Isberner, & Simari, 2009; Haret, Wallner, & Woltran, 2018) or repairing a semantical collapse (Baumann & Ulbricht, 2019) and reasoning with incomplete argumentation frameworks (Baumeister, Järvisalo, Neugebauer, Niskanen, & Rothe, 2021; Fazzinga, Flesca, & Furfaro, 2020b). In light of the results obtained in this paper, reasoning problems like these are now also expected to be PSPACE-complete. Future work could also involve confirming these conjectures. Another possible direction for further investigation is a comparison of our insights with results concerning other variants of admissible semantics, e.g. in the probabilistic setting (Baier, Diller, Dubslaff, Gaggl, Hermanns, & Käfer, 2021).

Needless to say, an actual implementation utilizing our theoretical results would contribute to making weak admissibility ready for potential applications. A first step in that direction has been done in (Dvořák et al., 2021), where a worst-case complexity-adequate DATALOG encoding for weak-admissible semantics is presented. In particular due to the considerable worst case complexity, another conceivable technique might be utilizing graph convolutional networks as done in (Kuhlmann & Thimm, 2019; Malmqvist, Yuan, Nightingale, & Manandhar, 2020) for Dung's classical semantics, Although one has to keep in mind that these approaches calculate approximate solutions, the results reported in these papers are quite promising. Finally, we mention that the "weak" semantics have been further developed in (Dauphin, Rienstra, & van der Torre, 2021), which calls for an investigation on its own.

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