Journal of Artificial Intelligence Research 66 (2019) 279-296 Submitted 11/2018; published 09/2019

Formulas Free From Inconsistency: An Atom-Centric Characterization in Priest's Minimally Inconsistent LP

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Abstract

As one of fundamental properties to characterize inconsistency measures for knowledge bases, the property of free formula independence well captures the intuition that free formulas are independent of the amount of inconsistency in a knowledge base for cases where inconsistency is characterized in terms of minimal inconsistent subsets. But it has been argued that not all the free formulas are independent of inconsistency in some other contexts of inconsistency characterization. In this paper, we propose a characterization of formulas independent of inconsistency in the framework of Priest's minimally inconsistent LP. Based on an atom-based counterpart of the notion of free formula, we propose a notion of Bi-free formula to describe formulas that are free from inconsistency in both syntax and paraconsistent models in this logic. Then we propose the property of Bi-free formula independence, which is more suitable for characterizing the role of formulas free from inconsistency in measuring inconsistency from both syntactic and semantic perspectives.

1. Introduction

Inconsistency handling is one of the important issues in knowledge representation and reasoning. In particular, it has been increasingly recognized that measuring inconsistency is a useful way to facilitate the process of inconsistency handling in a variety of applications such as requirements engineering (Mu, Hong, Jin, & Liu, 2013a; Mu, Jin, Liu, Zowghi, & Wei, 2013b), belief change (Hunter & Konieczny, 2010), news reports (Hunter, 2006), network security and intrusion detection (McAreavey, Liu, Miller, & Mu, 2011), and medical experts systems (Muiño, 2011). Techniques for measuring inconsistency for knowledge bases have been paid much attention recently (Liu & Mu, 2017). Here a knowledge base refers to a finite set of propositional formulas. Instead of considering any two inconsistent knowledge bases equally bad, the approach to measuring inconsistency gives a quantitative assessment for the inconsistency arising in a knowledge base in order to make a fine-grained distinction among inconsistent knowledge bases. Without loss of generality, we assume that an inconsistency measure is a function from the set of all knowledge bases to $[0, +\infty)$ such that the higher the value assigned to a knowledge base, the more inconsistent that knowledge base is.

A growing number of inconsistency measures have been proposed so far (Knight, 2002, 2003; Grant, 1978; Grant & Hunter, 2006; Hunter & Konieczny, 2010; Xiao, Lin, Ma, & Qi, 2010; Konieczny, Lang, & Marquis, 2003; Hunter, 2004; Hunter & Konieczny, 2010; Jabbour, Ma, & Raddaoui, 2014; Jabbour, Ma, Raddaoui, Sais, & Salhi, 2016; Mu, Liu, & Jin, 2011a; Mu, Liu, Jin, & Bell, 2011b; Mu, Wang, & Wen, 2014; Mu, 2015, 2018; Thimm, 2017). A more detailed survey of inconsistency measures has been given by Thimm (2018). Although each of these measures was proven to exhibit some good behaviors tailed to some certain (often restricted) perspectives, it has been reported that different inconsistency measures may bring different (possibly incompatible) characterizations of inconsistent knowledge bases (Grant & Hunter, 2011a, 2011b). Such a dilemma makes properties for characterizing desirable inconsistency measures more necessary.

The set of basic properties presented by Hunter and Konieczny (2006, 2008) provides a good starting point for characterizing inconsistency measures. A number of variants of Hunter and Konieczny's properties have been proposed by adapting or revising some ones of this set (Mu et al., 2011b; Jabbour et al., 2014, 2016; Besnard, 2014, 2017). Hunter and Konieczny's set of basic properties consists of *Normalization*, *Consistency*, *Monotony*, *Dominance*, and *Free Formula Independence*. Roughly speaking, the property of *Consistency* requires that any inconsistency measure should assign 0 to exactly all the consistent knowledge bases. It gives a very primitive requirement on nonnegative inconsistency measures, i.e., the ability to distinguish an inconsistent knowledge base from all consistent ones. It aims to guarantee a measure is indeed an inconsistency measure (Thimm, 2009). As an optional property, *Normalization* is used to adjust values of an inconsistency measure into $[0, 1]$. The property of *Monotony* requires that the inconsistency value cannot decrease when a knowledge base is enlarged, while the property of *Dominance* characterizes the intuition that logically stronger formulas bring more conflicts.

The property of *Free Formula Independence* aims to capture an intuition that formulas independent of inconsistency have no impact on the assessment of inconsistency in a knowledge base. Here free formulas of a knowledge base refer to ones that are not involved in any minimal inconsistent subset (an inconsistent subset without a proper inconsistent subset) of the knowledge base. Then *Free Formula Independence* grasps the intuition well in the context of inconsistency characterized by minimal inconsistent subsets.

Characterizing inconsistency in terms of minimal inconsistent subsets provides a syntactic perspective to analyze the inconsistency in a knowledge base. Besides this, atoms assigned to non-classical truth values by some paraconsistent models such as Belnap's four-valued semantics (Belnap, 1977; Arieli & Avron, 1998) and Priest's $\mathrm{LP_m}$ semantics (1991) have been also used to characterize inconsistency (Hunter & Konieczny, 2010; Ma, Qi, & Hitzler, 2011). However, the independence of free formulas from minimal inconsistent subsets cannot ensure that the property of *Free Formula Independence* is also appropriate for characterizing atom-centric inconsistency measures stemming from the inconsistency characterization based on paraconsistent models (Hunter & Konieczny, 2010). Here atom-centric inconsistency measures refer to ones that take into account the proportion of the language involved in inconsistency (Hunter & Konieczny, 2010). Then the property of *Free Formula Independence* has been weakened to characterize such situations by replacing free formulas with safe formulas (Thimm, 2009; Hunter & Konieczny, 2010). Here, a safe formula of a knowledge base refers to a consistent formula that does not contain any atom appearing in other formulas of the base.

On the other hand, it has been argued that the notion of safe formula cannot cover the *tautology*, which is free from inconsistency in intuition (Besnard, 2017). Then an alternative set of Hunter and Konieczny's properties has been presented by Besnard, in which two postulates *Tautology Independence* and *Conjunct Independence* together entail that adding a formula *safely consistent* for a knowledge base to that base has no impact on the inconsistency assessment of that base (Besnard, 2017). Here we say that a formula is safely consistent for a knowledge base if some substitution of the formula obtained by replacing all the occurrences of some of its atoms not appearing in formulas of that base with one or other of the two constants $\top$ (true) and $\perp$ (false) is a tautology (Besnard, 2017). We call such a formula *safely consistent formula* for the simplicity of discussion from now on. It has been shown that a new safe formula for a knowledge base is also a safely consistent formula for that base (Besnard, 2017). However, if we consider only the formulas built upon atoms involved in a knowledge base, then such a safely consistent formula for that base must be a tautology, as we will show later. This implies that the term of safely consistent formula is not general enough to cover formulas free from inconsistency characterization.

In this paper, we propose an atom-centric characterization of formulas that are independent of inconsistency in Priest's minimally inconsistent LP (Logic of Paradox) $\mathrm{LP_m}$ (1991), which is one of the simple paraconsistent logics often used to exemplify the inconsistency characterization in terms of paraconsistent semantics (Konieczny et al., 2003; Hunter & Konieczny, 2010). At first, we characterize free formulas from the perspective that the removal of such a formula from a knowledge base cannot change the minimal inconsistent subsets of that base. Inspired by this invariance of minimal inconsistent subsets, we then propose a new counterpart of the notion of free formula, which is used to capture formulas that are independent of atom-centric inconsistency characterization of a knowledge base in minimally inconsistent LP $\mathrm{LP_m}$. We show that the atom-centric counterpart can cover the tautology, the safe formula and the safely consistent formula. But the counterpart cannot rule out some formulas not really free from minimal inconsistent subsets. That is, such a semantic characterization of independence may lead to some undesired result in syntactic aspect. Then we enhance the counterpart and propose a notion of Bi-free formula, which is used to capture formulas that are independent of both syntactic and atom-centric inconsistency characterization in the framework of Priest's minimally inconsistent LP. Moreover, we show that our counterparts are more general than the other alternatives of free formulas. Finally, the corresponding counterparts of the property of Free Formula Independence are also given.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notions about inconsistency characterization for knowledge bases. In Section 3, we introduce alternatives of the property of *Free Formula Independence*. In Section 4, we propose the notion of B-atom-free formula, which is a counterpart of free formula in the context of atom-centric inconsistency characterization in Priest's minimally inconsistent LP $\mathrm{LP_m}$. In Section 5, we propose the notion of Bi-free formula, which is appropriate for characterizing formulas free from both syntactic and semantic inconsistency characterization in the framework of Priest's $\mathrm{LP_m}$. In Section 6, we discuss a new alternative of Hunter and Konieczny's properties based on the notion of Bi-free formula. Finally, we conclude this paper in Section 7.

2. Preliminaries

Throughout this paper we use a finite propositional language. Let $\mathcal{P}$ be a finite set of propositional atoms (or variables) and $\mathcal{L}$ a propositional language built from $\mathcal{P}$ and two propositional constants $\top$ (true) and $\bot$ (false) under connectives $\{\neg, \wedge, \vee\}$. We use a, b, c, $\cdots$ to denote propositional atoms, and α, β, γ, $\cdots$ to denote propositional formulas. In addition, we use $\alpha \equiv \top$ to denote that α is a *tautology*.

A *knowledge base* K is a finite set of propositional formulas. For any two disjoint knowledge bases K_1 and K_2, we use $K_1 + K_2$ instead of $K_1 \cup K_2$ to denote the union of K_1 and K_2.

Given a knowledge base K, we use $At(K)$ to denote the set of atoms appearing in formulas of K. For example, $At(\{a \vee b, \neg b \vee c\}) = \{a, b, c\}$.

K is *inconsistent* if there is a formula α such that $K \vdash \alpha$ and $K \vdash \neg\alpha$, where $\vdash$ is the classical consequence relation. We use $K \vdash \bot$ (resp. $K \nvdash \bot$) to denote that a knowledge base K is inconsistent (resp. consistent).

An inconsistent subset K' of K is called a *minimal inconsistent subset* of K if no proper subset of K' is inconsistent. Minimal inconsistent subsets of a knowledge base can be regarded as a characterization of inconsistency of that base from a syntactic perspective, since one needs to remove only one formula from each minimal inconsistent subset in order to resolve the inconsistency (Reiter, 1987). In this paper, we use $\mathcal{MI}(K)$ to denote the set of all the minimal inconsistent subsets of K, i.e.,

$$\mathcal{MI}(K) = \{K' \subseteq K | K' \vdash \bot \text{ and } \forall K'' \subset K', K'' \nvdash \bot\}.$$

A formula in K is called a *free formula* if this formula does not belong to any minimal inconsistent subset of K (Hunter & Konieczny, 2006). We use $\mathcal{FF}(K)$ to denote the set of free formulas of K. Then $K = \bigcup \mathcal{MI}(K) + \mathcal{FF}(K)$, where $\bigcup \mathcal{MI}(K) = \cup_{M \in \mathcal{MI}(K)} M$.

Note that an inconsistent knowledge base K has no classical model. Some paraconsistent models have been proposed to characterize inconsistent knowledge bases in semantics. It has been stated that Priest's minimally inconsistent LP (Logic of Paradox) $\mathrm{LP_m}$ (1991) is appropriate for exemplifying such a characterization since it is simple enough but agrees with classical logic whenever the knowledge base is consistent (Konieczny et al., 2003).

The $\mathrm{LP_m}$ model (Priest, 1991) of knowledge bases is given in the framework of Priest's Logic of Paradox (Priest's LP for short) (1979). Roughly speaking, Priest's LP provides three-valued models for classically inconsistent knowledge bases by expanding the classical truth values $\{\mathsf{T}, \mathsf{F}\}$ to the set $\{\mathsf{T}, \mathsf{F}, \{\mathsf{T}, \mathsf{F}\}\}$, in which the third truth value $\{\mathsf{T}, \mathsf{F}\}$ (also abbreviated as B in Hunter & Konieczny, 2010; Konieczny et al., 2003) is considered intuitively as both true and false (Priest, 1991). Here we use the following notations and the concepts about the $\mathrm{LP_m}$ model used by Hunter and Konieczny (2010). An interpretation ω for $\mathrm{LP_m}$ models maps each propositional variable to one of the three truth values T, F, B such that

- $\omega(\top) = \mathsf{T}, \omega(\bot) = \mathsf{F}$,
- $\omega(\neg\alpha) = \mathsf{B}$ if and only if $\omega(\alpha) = \mathsf{B}$,
- $\omega(\neg\alpha) = \mathsf{T}$ if and only if $\omega(\alpha) = \mathsf{F}$,
- $\omega(\alpha \wedge \beta) = \min_{\leq t}\{\omega(\alpha), \omega(\beta)\}$,
- $\omega(\alpha \vee \beta) = \max_{\leq t}\{\omega(\alpha), \omega(\beta)\}$,

where $\mathsf{F} <_t \mathsf{B} <_t \mathsf{T}$. Then *the set of models of a formula* α is defined as $\mathsf{Mod}_{\mathsf{LP}}(\alpha) = \{\omega | \omega(\alpha) \in \{\mathsf{T}, \mathsf{B}\}\}$. Further, *the set of models of a knowledge base* K is defined as

$$\mathsf{Mod}_{\mathsf{LP}}(K) = \{\omega | \omega \in \mathsf{Mod}_{\mathsf{LP}}(\alpha) \text{ for all } \alpha \in K\}.$$

Let K be a knowledge base and ω be a model of K, then we use $\omega!(K)$ to denote the set of propositional variables of K assigned to B by ω, i.e., $\omega!(K) = \{x \in At(K) | \omega(x) = \mathsf{B}\}$.

Based on $\omega!(K)$, we call a model ω of K a *minimal (inconsistent) model* of K if there is no $\omega' \in \mathsf{Mod}_{\mathsf{LP}}(K)$ such that $\omega'!(K) \subset \omega!(K)$. We use $\mathsf{MinMod}_{\mathsf{LP}}(K)$ to denote the set of minimal models of K. Essentially, each minimal model $\omega \in \mathsf{MinMod}_{\mathsf{LP}}(K)$ is one of the "most classical" models of K, and $\omega!(K)$ describes a minimal set of atoms that have to be assigned to B by ω. We call the elements of $\omega!(K)$ the *B-atoms* of ω w.r.t. K if ω is a minimal model of K. From now on, we use $B(\omega)$ to denote the set of B-atoms of the minimal model ω.

Based on minimal models, the $\mathrm{LP_m}$ consequence relation is defined by

$$K \models_{\mathrm{LP_m}} \alpha \text{ iff } \mathsf{MinMod}_{\mathsf{LP}}(K) \subseteq \mathsf{Mod}_{\mathsf{LP}}(\alpha).$$

Then α is a $\mathrm{LP_m}$ consequence of K if all the most classical models of K are models of α.

Inconsistency measures aim to evaluate the inconsistency in each knowledge base based on some inconsistency characterizations such as minimal inconsistent subsets and $\mathrm{LP_m}$ models, and then to distinguish knowledge bases from some perspectives. To characterize desirable inconsistency measures, some properties or postulates have been proposed so far. In particular, Hunter and Konieczny's set of properties (2006, 2008) has been considered as a good starting point for developing postulates. Without loss of generality, we assume that an *inconsistency measure* I assigns every knowledge base K a nonnegative number $I(K)$ such that the higher the value of a non-negative inconsistency measure, the more inconsistent a knowledge base is. Under this assumption, the set of basic properties proposed by Hunter and Konieczny (2006, 2008) can be given as follows:

- *Consistency* : $I(K) = 0$ if and only if K is consistent.
- *Monotony*: $I(K \cup K') \geq I(K)$.
- *Free Formula Independence*: If $\alpha \in \mathcal{FF}(K \cup \{\alpha\})$, then $I(K \cup \{\alpha\}) = I(K)$.
- *Dominance*: If $\alpha \vdash \beta$ and $\alpha \nvdash \bot$, then $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$.
- *Normalization*: $0 \leq I(K) \leq 1$.

The property of *Consistency* says that 0 is the designated value of any inconsistency measure for all consistent knowledge bases. The property of *Normalization* says that the range of an inconsistency measure can be $[0, 1]$. In this paper we ignore this optional property. The property of *Monotony* requires that the inconsistency measure is monotonic w.r.t. the extension of a knowledge base, whilst the property of *Dominance* requires that the inconsistency measure needs to capture the intuition that replacing a formula with another logically stronger one may bring more conflicts. The property of *Free Formula Independence* requires that the inconsistency measure needs to be independent of free formulas of a knowledge base. In this paper, we focus on the property of *Free Formula Independence* and its variants.

3. Alternatives of Free Formulas Independence

The intent of *Free Formula Independence* is to capture the intuition that formulas free from inconsistency of a knowledge base have no impact on the evaluation of the inconsistency of that base. In the expression of this property, the term *free formula* is used to characterize formulas free from inconsistency. Note that *free formulas* are free from minimal inconsistent subsets in essence. Then the expression of *Free Formula Independence* captures its intent well in the context of inconsistency being characterized by minimal inconsistent subsets.

It has been reported that not all the free formulas are free from the inconsistency of a knowledge bases from other contexts of inconsistency characterization (Hunter & Konieczny, 2010). To illustrate this, consider $K = \{a \wedge \neg a \wedge b, \neg b, c\}$, where both $\neg b$ and c are free formulas of K. However, $\neg b$ is different from c in the sense that K conveys information about both b and $\neg b$. Actually, the atom b is assigned to the truth value B by a unique minimal model of K, whilst c is assigned to T by the minimal model. Then a weaker alternative of *Free Formula Independence*, called *Safe Formula Independence*, has been proposed to accommodate such cases (Hunter & Konieczny, 2010):

- *Safe Formula Independence*: If α is a safe formula of $K \cup \{\alpha\}$, then $\mathsf{I}(K \cup \{\alpha\}) = \mathsf{I}(K)$.

Here a formula $\alpha \in K$ is called a *safe formula* if $\alpha \not\vdash \perp$ and $At(\{\alpha\}) \cap At(K \setminus \{\alpha\}) = \emptyset$ (Hunter & Konieczny, 2010; Thimm, 2017). For example, c is a unique safe formula of $\{a \wedge \neg a \wedge b, \neg b, c\}$.

A safe formula is a special kind of free formula. The independence of the safe formula of the inconsistency in a knowledge base stems from the split of atoms of that knowledge base. On the other hand, such a dependence on the split of atoms makes the term safe formula restricted in coverage. For example, the free formula $d \wedge e$ is independent of the inconsistency in $\{c, \neg c \wedge d, d \wedge e\}$ because there is only the conflict between c and $\neg c$ in both syntax and $\mathrm{LP_m}$ semantics, but it isn't a safe formula. However, Besnard has argued that the term safe formula cannot cover tautologies, and then this weaker version does not entail the following property of *Tautology Independence* (Besnard, 2017), which is a straightforward consequence of *Free Formula Independence*:

- *Tautology Independence*: If $\alpha \equiv \top$, then $I(K \cup \{\alpha\}) = I(K)$.

Then the property of *Free Formula Independence* is replaced by *Tautology Independence*, together with the following property of *Conjunct Independence* (Besnard, 2017):

- *Conjunct Independence*: If $\alpha \wedge \beta \notin K$, $\beta \notin K$, and α is safely consistent for $K \cup \{\beta\}$, then $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta\})$.

Moreover, it has been shown that *Tautology Independence* and *Conjunct Independence* together entails that $I(K \cup \{\alpha\}) = I(K)$ if α is safely consistent for K (Besnard, 2017).

Here a formula α is *safely consistent* for K (*a safely consistent formula* of $K \cup \{\alpha\}$ for short in this paper) if there exists a substitution σ such that $\sigma\alpha$ [1] is a tautology and $\sigma a = a$ for all $a \in At(K)$, but either $\sigma b = b$ or $\sigma b = \perp$ or $\sigma b = \top$ for all $b \in At(\{\alpha\}) \setminus At(K)$ (Besnard, 2017). Roughly speaking, such a formula has at least one classical model such that any valuation obtained from the model by changing truth assignments to atoms appearing in formulas of K is also a model of the formula. That is, models of such a formula can assign any truth values to atoms appearing in formulas of K. For example, consider $K = \{a, \neg a\}$ and $c \vee a$. Obviously, any valuation ω such that $\omega(c) = \mathsf{T}$ is a model of $c \vee a$, regardless of the truth value assigned to a by ω. Then the substitution $\sigma(c \vee a) = \top \vee a$ of $c \vee a$ by replacing c with the constant $\top$ is a tautology. So, $c \vee a$ is safely consistent for K. It has been shown that the term safely consistent formula can cover safe formulas, that is, if α is a safe formula for $K \cup \{\alpha\}$, then α is safely consistent for K (Besnard, 2017).

4. The B-Atom-Free Formula

The safely consistent formula is more general than the safe formula. It covers the tautology as well. In particular, the following proposition shows that if a safely consistent formula of a knowledge base has no atom different from ones of other formulas, then it must be a tautology.

Proposition 4.1 *Let K be a knowledge base and α a formula safely consistent for K such that $At(\{\alpha\}) \subseteq At(K)$, then α is a tautology.*

1. $\sigma\alpha$ is obtained from α by replacing each occurrence of b by σb throughout for each $b \in At(\{\alpha\})$.

Proof Let K be a knowledge base and α a safely consistent formula of $K \cup \{\alpha\}$, then there exists a substitution σ such that $\sigma\alpha$ is a tautology and $\sigma a = a$ for all $a \in At(K)$. If $At(\{\alpha\}) \subseteq At(K)$, then $\sigma\alpha = \alpha$. So, α is a tautology. $\square$

Evidently, we can get the following result from this proposition.

Corollary 4.1 *Let K be a knowledge base and $\alpha \in K$. If α is safely consistent for K, then α is a tautology.*

On the other hand, this proposition implies that the notion of safely consistent formula is not general enough to capture the formula free from inconsistency in the case where the formula is built upon atoms of other formulas of a base. To illustrate this, consider $K_0 = \{a \wedge c, \neg a \wedge b, b \wedge c\}$. Note that $At(\{b \wedge c\}) \subset At(K_0 \setminus \{b \wedge c\})$, and $b \wedge c$ is not a tautology. Then $b \wedge c$ is neither a safe formula nor a safely consistent formula of K_0. However, the free formula $b \wedge c$ of K_0 is free from inconsistency in both syntax and paraconsistent semantics because neither b nor c is assigned to the inconsistent truth value B by the minimal model of K_0.

The following example also shows that the notion of safely consistent formula is not general enough to capture the formula free from inconsistency in Priest's $\mathrm{LP_m}$.

Example 4.1 *Let $K_1 = \{a, \neg a \wedge b, b \wedge c\}$. Then ω_1 is the unique minimal model of K_1, where $\omega_1(a) = \mathsf{B}$ and $\omega_1(b) = \omega_1(c) = \mathsf{T}$.*

Note that $b \wedge c$ is a free formula of K such that neither b nor c is assigned to B by the minimal model ω_1 of K_1. This means that $b \wedge c$ is free from inconsistency characterization in Priest's $\mathrm{LP_m}$. But $b \wedge c$ is neither a safe formula of K nor a safely consistent formula for K_1.

In this section, we focus on characterizing formulas free from inconsistency in the framework of Priest's $\mathrm{LP_m}$. We start with the following characterization of the role of free formulas in terms of *invariance of minimal inconsistent subsets*. Let K be a knowledge base and $\alpha \in K$. Then

- α is a free formula of K if and only if $\mathcal{M}I(K) = \mathcal{M}I(K \setminus \{\alpha\})$.

This characterization inspires us to characterize the formulas free from inconsistency in the framework of Priest's $\mathrm{LP_m}$ using the invariance of inconsistency characterization. Intuitively, for each minimal model of K, the B-atoms $B(\omega)$ of ω w.r.t. K are exactly ones that have to be considered contradictory when we give a definitely true or false value to other atoms. We use $\mathsf{BA}(K)$ to denote the set of all B-atoms of minimal models of K, i.e., $\mathsf{BA}(K) = \{B(\omega) | \omega \in \mathsf{MinMod_{LP}}(K)\}$. Then $\mathsf{BA}(K)$ can be considered as a characterization of inconsistency in K from an atom-centric perspective.

Note that in the framework of Priest's $\mathrm{LP_m}$, neither the notion of safely consistent formulas nor that of safe formulas for a knowledge base can cover the formulas associated with the invariance of B-atoms. To illustrate this, consider the following example.

Example 4.2 *Consider $K_1 = \{a, \neg a \wedge b, b \wedge c\}$ again. Note that ω_1 is the unique minimal model of K_1, where $\omega_1(a) = \mathsf{B}$, $\omega_1(b) = \mathsf{T}$, $\omega_1(c) = \mathsf{T}$. So, $B(\omega_1) = \{a\}$ and $\mathsf{BA}(K_1) = \{\{a\}\}$. Obviously, $\mathsf{BA}(K_1 \setminus \{b \wedge c\}) = \mathsf{BA}(K_1) = \{\{a\}\}$. However, $b \wedge c$ is neither a safe formula of K_1 nor a safely consistent formula for K_1.*

To capture the invariance of inconsistency characterization in the framework of Priest's $\mathrm{LP_m}$, we give the following counterpart of the notion of free formula.

Definition 4.1 *Let K be a knowledge base and $\alpha \in K$. Then we call α a B-atom-free formula of K if* $\mathsf{BA}(K) = \mathsf{BA}(K \setminus \{\alpha\})$.

Essentially, the independence of the B-atom-free formula of inconsistency characterization stems from the invariance of the set of B-atoms in Priest's $\mathrm{LP_m}$. In this sense, the notion of B-atom-free formula captures the underlying idea of free formula.

Example 4.3 *Consider $K_1 = \{a, \neg a \wedge b, b \wedge c\}$ again. Then $b \wedge c$ is the unique B-atom-free formula of K_1.*

The following proposition shows that the notion of B-atom-free formula can cover both the tautology and the safe formula.

Proposition 4.2 *Let K be a knowledge base and $\alpha \in K$.*

(1) If $\alpha \equiv \top$, then α is a B-atom-free formula of K.

(2) If α is a safe formula, then α is a B-atom-free formula of K.

Proof Let K be a knowledge base and $\alpha \in K$.

(1) If $\alpha \equiv \top$, then α must be a tautology in Priest's LP (1979). So, any 3-valued interpretation ω is a model of α in LP. Therefore, $\mathsf{BA}(K) = \mathsf{BA}(K \setminus \{\alpha\})$.

(2) If α is a safe formula of K, then $\alpha \not\vdash \perp$ and $At(\{\alpha\}) \cap At(K \setminus \{\alpha\}) = \emptyset$. So, for every minimal model ω of K, $\omega(a) \neq \mathsf{B}$ for all $a \in At(\{\alpha\})$. Then ω is also a minimal model of $K \setminus \{\alpha\}$. On the other hand, if ω is a minimal model of $K \setminus \{\alpha\}$, then there exists at least one minimal model ω' of K such that $\omega!(K \setminus \{\alpha\}) = \omega'!(K)$. Therefore, $\mathsf{BA}(K) = \mathsf{BA}(K \setminus \{\alpha\})$.$\square$

It has been stated that if α is safely consistent for K, then α is free for K (Besnard, 2017). The following proposition shows that the notion of B-atom-free formula can cover safely consistent formulas.

Proposition 4.3 *Let K be a knowledge base and α a formula. If α is safely consistent for K, then α is a B-atom-free formula of $K \cup \{\alpha\}$.*

Proof Let α be safely consistent for K. If $\alpha \in K$, then $K \cup \{\alpha\} = K$. By Corollary 4.1 and Proposition 4.2, we get that α is a B-atom-free formula of K.

If $\alpha \notin K$, then there exists a substitution σ such that $\sigma\alpha$ is a tautology and

(1) $\sigma a = a$ for all $a \in At(K)$,

(2) either $\sigma b = b$ or $\sigma b = \perp$ or $\sigma b = \top$ for all $b \in At(\{\alpha\}) \setminus At(K)$.

Further, the set of atoms of α can be divided into the following four subsets:

- $At_K(\{\alpha\}) = At(\{\alpha\}) \cap At(K)$,
- $At_\top(\{\alpha\}) = \{b | b \in At(\{\alpha\}), \sigma b = \top\}$,

- $At_{\bot}(\{\alpha\}) = \{b | b \in At(\{\alpha\}), \sigma b = \bot\}$, and
- $At_{=}(\{\alpha\}) = \{b | b \in At(\{\alpha\}), \sigma b = b\} \setminus At(K)$.

Let ω be a minimal model of $K \cup \{\alpha\}$, then for all $b \in At(\{\alpha\}) \setminus At(K)$, $\omega(b) \neq \mathsf{B}$. Otherwise, we adapt ω to ω' in view of the substitution σ as follows:

$$\omega'(a) = \begin{cases} \mathsf{T}, & \text{if } a \in At_{\top}(\{\alpha\}) \cup At_{=}(\{\alpha\}), \\ \mathsf{F}, & \text{if } a \in At_{\bot}(\{\alpha\}), \\ \omega(a), & \text{otherwise.} \end{cases}$$

Note that $\sigma\alpha$ is also a tautology in LP (Priest, 1979), then ω' is a model of $K \cup \{\alpha\}$ such that $\omega'!(K \cup \{\alpha\}) \subset \omega!(K \cup \{\alpha\})$. This contradicts the minimality of ω.

Let ω be a minimal model of $K \cup \{\alpha\}$, then the previous part of the proof shows that ω is also a minimal model of K, moreover, $\omega!(K \cup \{\alpha\}) = \omega!(K)$. So, $\mathsf{BA}(K \cup \{\alpha\}) \subseteq \mathsf{BA}(K)$.

On the other hand, for any minimal model ω of K, we can adapt ω to ω' in view of the substitution σ as follows:

$$\omega'(a) = \begin{cases} \mathsf{T}, & \text{if } a \in At_{\top}(\{\alpha\}) \cup At_{=}(\{\alpha\}) \\ \mathsf{F}, & \text{if } a \in At_{\bot}(\{\alpha\}), \\ \omega(a), & \text{otherwise.} \end{cases}$$

Obviously, $\omega'!(K \cup \{\alpha\}) = \omega!(K)$. Then ω' is a minimal model of $K \cup \{\alpha\}$. So, $\mathsf{BA}(K \cup \{\alpha\}) \supseteq \mathsf{BA}(K)$. Then $\mathsf{BA}(K \cup \{\alpha\}) = \mathsf{BA}(K)$. Therefore, α is a B-atom-free formula of $K \cup \{\alpha\}$. □

Now we are ready to replace the property of *Free Formula Independence* with the following property:

- *B-atom-free Formula Independence*: If α is a B-atom-free formula of K, then $I(K \setminus \{\alpha\}) = I(K)$.

Then we can get the following results from Proposition 4.2 and 4.3, respectively.

Corollary 4.2 *Assuming* B-atom-free Formula Independence *entails* Tautology Independence *and* Safe Formula Independence .

Corollary 4.3 *Assuming* B-atom-free Formula Independence *entails that* $I(K \cup \{\alpha\}) = I(K)$ *if* α *is safely consistent for* K.

In addition, we have the following relation between *B-atom-free Formula Independence* and *Conjunct Independence*.

Proposition 4.4 *Assuming* B-atom-free Formula Independence *and* $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$ *entail* Conjunct Independence.

Proof. If $\alpha \wedge \beta \notin K$, $\beta \notin K$, and α is safely consistent for $K \cup \{\beta\}$, then α is B-atom-free formula of $K \cup \{\alpha, \beta\}$, by applying Proposition 4.3. So, *B-atom-free Formula Independence* entails that $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\beta\})$. Further, if $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$, then $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta\})$. □

However, ω is a model of $K \cup \{\alpha, \beta\}$ if and only if ω is a model of $K \cup \{\alpha \wedge \beta\}$. This implies that for any inconsistency measure I based on $\mathrm{LP_m}$ models, it holds that $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\alpha, \beta\})$.

5. Bi-Free Formulas

As a counterpart of the notion of free formula, the B-atom-free formula is defined on the invariance of inconsistency characterization. In the framework of Priest's $\mathrm{LP_m}$, the inconsistency of a knowledge base is characterized by B-atoms of minimal models of that base. Similar to the case of free formula, the minimality involved in inconsistency characterization makes the notion of B-atom-free formula cover some formulas that are not really free from minimal inconsistent subsets. That is, the invariance of inconsistency characterization in terms of minimal models cannot ensure that a B-atom-free formula must be a free formula. To illustrate this, consider the following example.

Example 5.1 *Consider $K_2 = \{a, \neg a, \neg a \vee b, \neg b\}$. Evidently, ω is a unique minimal model of K_2, where $\omega(a) = \mathsf{B}, \omega(b) = \mathsf{F}$. Then $\mathsf{BA}(K_2) = \{\{a\}\}$. So, both $\neg a \vee b$ and $\neg b$ are B-atom-free formulas of K_2. But, neither $\neg a \vee b$ nor $\neg b$ is a free formula of K_2.*

On the other hand, as illustrated by the following example, not all the free formulas of a knowledge base are really independent of B-atoms in the framework of Priest's $\mathrm{LP_m}$. We also need to exclude such a type of free formulas from free formulas.

Example 5.2 *Consider $K_3 = \{a \wedge \neg a \wedge b, \neg b\}$. Note that $\neg b$ is a unique free formula of K_3. Evidently, $\mathsf{BA}(K_3) = \{\{a, b\}\}$, but $\mathsf{BA}(K_3 \setminus \{\neg b\}) = \{\{a\}\}$. So, $\neg b$ is not free from B-atoms of K_3 in the framework of Priest's $\mathrm{LP_m}$.*

However, we can characterize the free formulas that are not really independent of inconsistency characterization by the notion of *false B-independent formula* in the framework of Priest's $\mathrm{LP_m}$.

Definition 5.1 *Let K be a knowledge base and α a free formula of K. We call α a false B-independent formula of K if $\exists S \subset K$ s.t. $S \vdash \perp$ and $\mathsf{BA}(S) \neq \mathsf{BA}(S \cup \{\alpha\})$.*

Note that the B-atom-free formulas describe ones that are free from contradictory atoms in minimal paraconsistent models, while the free formulas characterize ones that are from minimal inconsistent subsets in syntax. To address this, we propose a notion of Bi-free formula to describe formulas that are free from inconsistency characterization in both syntax and paraconsistent semantics.

Definition 5.2 *Let K be a knowledge base and $\alpha \in K$. Then we call α a Bi-free formula of K if for all $S \subseteq K$ s.t. $\alpha \in S$, it holds that $\mathsf{BA}(S) = \mathsf{BA}(S \setminus \{\alpha\})$.*

Example 5.3 *Consider $K_4 = \{a \wedge c, \neg a, \neg a \vee b, \neg b, c \wedge d\}$. Evidently, $\mathsf{BA}(K_4) = \{\{a\}\}$, and all of $\neg a \vee b$, $\neg b$, and $c \wedge d$ are B-atom-free formulas. Note that $\mathsf{BA}(\{a \wedge c, \neg a \vee b, \neg b\}) \neq \mathsf{BA}(\{a \wedge c, \neg b\})$ and $\mathsf{BA}(\{a \wedge c, \neg a \vee b, \neg b\}) \neq \mathsf{BA}(\{a \wedge c, \neg a \vee b\})$. Then neither $\neg a \vee b$ nor $\neg b$ is a Bi-free formula of K_4.*

But for all $S \subseteq K_4$ s.t. $c \wedge d \in S$, $\mathsf{BA}(S) = \mathsf{BA}(S \setminus \{c \wedge d\})$. So, $c \wedge d$ is the unique Bi-free formula of K_4.

Note that for any $M \in \mathcal{MI}(K)$, $\mathsf{BA}(M) \neq \mathsf{BA}(M \setminus \{\alpha\}) = \emptyset$ for all $\alpha \in M$. Then all the Bi-free formulas are free formulas. However, the following proposition shows that the notion of Bi-free formula can exclude the false B-independent formula.

Proposition 5.1 *Let K be a knowledge base and α a Bi-free formula of K. Then α is not a false B-independent formula of K.*

Proof If α is a Bi-free formula of K, then $\forall S \subseteq K$ s.t. $\alpha \in S$, it holds that $\mathsf{BA}(S) = \mathsf{BA}(S\setminus\{\alpha\})$. So, α is not a false B-independent formula of K. $\square$

Interestingly, we show that if α is a Bi-free formula, then α is a both B-atom-free and free formula.

Proposition 5.2 *Let K be a knowledge base and $\alpha \in K$. If α is a Bi-free formula of K, then $\mathsf{BA}(K) = \mathsf{BA}(K \setminus \{\alpha\})$ and $\mathcal{M}I(K) = \mathcal{M}I(K \setminus \{\alpha\})$.*

Proof Let K be a knowledge base and $\alpha \in K$. If α is a Bi-free formula of K, then $\forall S \subseteq K$ s.t. $\alpha \in S$, it holds that $\mathsf{BA}(S) = \mathsf{BA}(S \setminus \{\alpha\})$. Obviously, it holds that $\mathsf{BA}(K) = \mathsf{BA}(K \setminus \{\alpha\})$. Now we show that $\mathcal{MI}(K) = \mathcal{MI}(K \setminus \{\alpha\})$. Otherwise, $\exists M \in \mathcal{MI}(K)$ s.t. $\alpha \in M$, and $\mathsf{BA}(M) = \mathsf{BA}(M \setminus \{\alpha\})$. Obviously, $\emptyset = \mathsf{BA}(M \setminus \{\alpha\})$ and $\mathsf{BA}(M) \neq \emptyset$, and a contradiction arises. $\square$

Note that α being a false B-independent formula of K does not necessarily imply that $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{\alpha\})$. To illustrate this, consider $K = \{a \wedge \neg a \wedge \neg b, b \wedge c, b \wedge d\}$. Obviously, both $b \wedge c$ and $b \wedge d$ are false B-independent formula of K, but neither $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{b \wedge c\})$ nor $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{b \wedge d\})$ holds. However, we have a more interesting characterization of Bi-free formulas in the case that for any false B-independent formula α of K, it holds that $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{\alpha\})$.

Proposition 5.3 *Let K be a knowledge base such that $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{\beta\})$ for any false B-independent formula β. Then $\alpha \in K$ is a Bi-free formula of K if and only if $\mathsf{BA}(K) = \mathsf{BA}(K\setminus\{\alpha\})$ and $\mathcal{M}I(K) = \mathcal{M}I(K \setminus \{\alpha\})$.*

Proof Let K be a knowledge base and $\alpha \in K$. By Proposition 5.2, if α is a Bi-free formula of K, then $\mathsf{BA}(K) = \mathsf{BA}(K \setminus \{\alpha\})$ and $\mathcal{M}I(K) = \mathcal{M}I(K \setminus \{\alpha\})$.

On the other hand, let $\alpha \in K$ such that $\mathsf{BA}(K) = \mathsf{BA}(K \setminus \{\alpha\})$ and $\mathcal{M}I(K) = \mathcal{M}I(K \setminus \{\alpha\})$. Suppose that there exists $\exists S \subset K$ such that $\alpha \in S$ and $\mathsf{BA}(S) \neq \mathsf{BA}(S \setminus \{\alpha\})$. If $S \setminus \{\alpha\}$ is consistent then $S \in \mathcal{MI}(K)$ and $\alpha \in S$, it contradicts that $\mathcal{M}I(K) = \mathcal{M}I(K \setminus \{\alpha\})$. If $S \setminus \{\alpha\}$ is inconsistent then α is a false B-independent formula of K. So, $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{\alpha\})$. It contradicts $\mathsf{BA}(K) = \mathsf{BA}(K \setminus \{\alpha\})$.

Therefore, $\forall S \subseteq K$ s.t. $\alpha \in S$, it holds that $\mathsf{BA}(S) = \mathsf{BA}(S \setminus \{\alpha\})$. That is, α is a Bi-free formula of K. $\square$

Corollary 5.1 *Let K be a knowledge base such that $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{\beta\})$ for any false independent formula β. Then $\alpha \in K$ is a false B-independent formula if and only if $\mathcal{M}I(K) = \mathcal{M}I(K \setminus \{\alpha\})$ but $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{\alpha\})$.*

Corollary 5.2 *Let K be a knowledge base and $\alpha \in K$.*

(1) If α is a tautology, then α is a Bi-free formula.

(2) If α is a safe formula, then α is a Bi-free formula.

Proof Let K be a knowledge base and $\alpha \in K$.

(1) If α is a tautology, then for all $S \subseteq K \setminus \{\alpha\}$, ω is a minimal model of S if and only if ω is a minimal model of $S \cup \{\alpha\}$. Then $\mathsf{BA}(S) = \mathsf{BA}(S \cup \{\alpha\})$. So, α is a Bi-free formula.

(2) If α is a safe formula, then for all $S \subseteq K \setminus \{\alpha\}$, then any minimal model ω of $S \cup \{\alpha\}$ is also a minimal model of S. On the other hand, for any minimal model ω of S, there exists a minimal model ω' of $S \cup \{\alpha\}$ such that $\omega'!(S \cup \{\alpha\}) = \omega!(S)$. Then $\mathsf{BA}(S) = \mathsf{BA}(S \cup \{\alpha\})$. So, α is a Bi-free formula. $\Box$

Proposition 5.4 *Let K be a knowledge base and α a formula. If α is safely consistent for K, then α is a Bi-free formula of $K \cup \{\alpha\}$.*

Proof Let α be safely consistent for K. If $\alpha \in K$, then $K \cup \{\alpha\} = K$. By Corollary 4.1 and Corollary 5.2, we get that α is a Bi-free formula of K. If $\alpha \notin K$, then α is also safely consistent for S for all $S \subseteq K$. So, α is a B-atom-free formula of $S \cup \{\alpha\}$. Then it holds that $\mathsf{BA}(S) = \mathsf{BA}(S \cup \{\alpha\})$. Therefore, α is a Bi-free formula of $K \cup \{\alpha\}$. $\Box$

On the other hand, the following example shows that a Bi-free formula α of K is not necessarily safely consistent for $K \setminus \{\alpha\}$.

Example 5.4 *Consider $K_1 = \{a, \neg a \wedge b, b \wedge c\}$ again. Then $b \wedge c$ is a unique Bi-free formula of K_1. But $b \wedge c$ is not safely consistent for $\{a, \neg a \wedge b\}$.*

We use $\mathcal{BIFF}(K)$, $\mathcal{BAFF}(K)$, $\mathcal{FBF}(K)$, $\mathcal{SCF}(K)$, $\mathcal{SF}(K)$, and $\mathcal{T}(K)$ to denote the sets of Bi-free formulas, B-atom-free formulas, false B-independent formulas, safely consistent formulas, safe formulas, and tautologies of K, respectively. In summary, we obtain the following results illustrated by Fig. 1 (a), where $\mathcal{BAFF}(K)$ and $\mathcal{FF}(K)$ are demonstrated by blue circle and red circle, respectively:

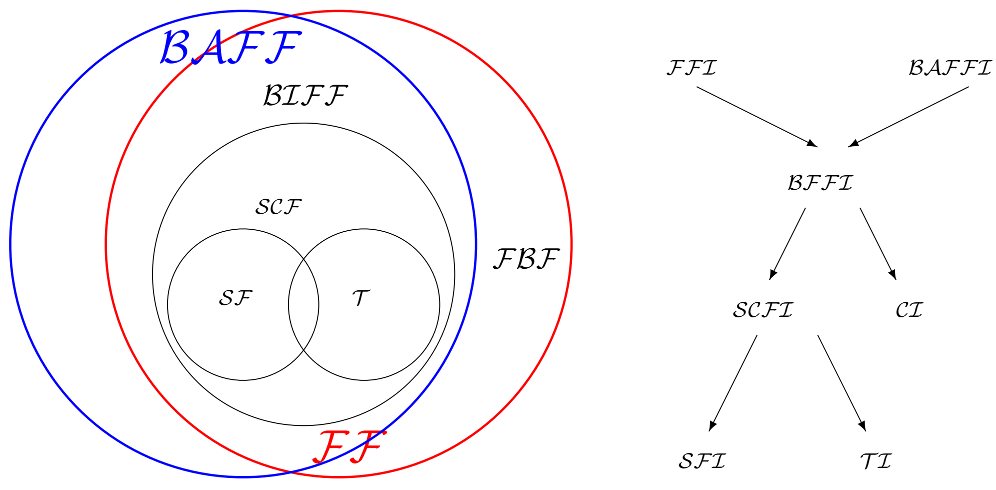
1. $\mathcal{BIFF}(K) \subseteq \mathcal{BAFF}(K) \cap \mathcal{FF}(K)$, in particular, $\mathcal{BIFF}(K) = \mathcal{BAFF}(K) \cap \mathcal{FF}(K)$ in the case that $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{\beta\})$ for any false B-independent formula β.
2. $\mathcal{FBF}(K) \subseteq \mathcal{FF}(K) \setminus \mathcal{BAFF}(K)$, in particular, $\mathcal{FBF}(K) = \mathcal{FF}(K) \setminus \mathcal{BAFF}(K)$ in the case that $\mathsf{BA}(K) \neq \mathsf{BA}(K \setminus \{\beta\})$ for any false B-independent formula β.
3. $\mathcal{SCF}(K) \subseteq \mathcal{BIFF}(K)$, and it holds that $\mathcal{SCF}(K) \subset \mathcal{BIFF}(K)$ for some K.
4. $\mathcal{SF}(K) \cup \mathcal{T}(K) \subseteq \mathcal{SCF}(K)$.

Now we provide another alternative of the property of *Free Formula Independence*, which is appropriate for describing measures from both syntax and paraconsistent semantics.

- *Bi-free Formula Independence*: If α is a Bi-free formula of K, then $I(K \cup \{\alpha\}) = I(K)$.

Evidently, we can get the following results.

Corollary 5.3 *Assuming* Bi-free Formula Independence *entails* Tautology Independence *and* Safe Formula Independence.



(a) (b)

Figure 1: The relations among these types of independent formulas

Corollary 5.4 *Assuming* Bi-free Formula Independence *entails that* $I(K \cup \{\alpha\}) = I(K)$ *if* α *is safely consistent for* K.

Corollary 5.5 *Assuming* Bi-free Formula Independence *and* $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$ *entail* Conjunct Independence.

Note that in the framework of Priest's $\mathrm{LP_m}$, if we only consider the measures based on the minimal models, then for such a measure I, it always hold that $I(K \cup \{\alpha, \beta\}) = I(K \cup \{\alpha \wedge \beta\})$. Then assuming *Bi-free Formula Independence* entails *Conjunct Independence* in this case.

In summary, the entailment of the other properties of independence from the *Bi-free Formula Independence* in the framework of Priest's $\mathrm{LP_m}$ is given in Fig. 1(b), where $\mathcal{FFI}$, $\mathcal{BAFFI}$, $\mathcal{BFFI}$, $\mathcal{SCFI}$, $\mathcal{SFI}$, $\mathcal{TI}$, and $\mathcal{CI}$ are used to denote *Free Formula Independence*, *B-atom-free Formula Independence*, *Bi-free Formula Independence*, *Safely Consistent Formula Independence*, *Safe Formula Independence*, *Tautology Independence*, and *Conjunct Independence*, respectively.

6. A New Alternative of Hunter and Konieczny's Properties

Informally speaking, the property of *B-atom-free Formula Independence* may be considered as a counterpart of *Free Formula Independence* for characterizing inconsistency measures based on minimal models, whilst the property of *Bi-free Formula Independence* is meaningful for both syntax-based and (paraconsistent) model-based measures in the framework of Priest's $\mathrm{LP_m}$. By replacing the property of *Free Formula Independence* with the properties of *B-atom-free Formula Independence* and *Bi-free Formula Independence*, respectively, we get two alternative sets of Hunter and Konieczny's properties.

In addition, Besnard argued that it is necessary to impose a condition $\alpha \notin K$ on the formula α involved in *Dominance* (Besnard, 2017). Here we take this new form of *Dominance*.

Let us consider the measure I_{LP_m} (Hunter & Konieczny, 2006, 2010), one of measures often used to exemplify atom-centric inconsistency measuring, which satisfies all the Hunter and

Konieczny's properties except *Free Formula Independence* (Hunter & Konieczny, 2010). Given a knowledge base K, $I_{LP_m}(K)$ is defined as $\frac{\min_{\omega \in \mathsf{Mod}_{\mathsf{LP}}(K)} \{|\omega!(K)|\}}{|\mathcal{P}|}$. Then we can get the following result.

Proposition 6.1 *The inconsistency measure I_{LP_m} satisfies B-atom-free Formula Independence and Bi-free Formula Independence.*

Proof Note that $I_{LP_m}(K) = \frac{|\mathsf{B}(\omega)|}{|\mathcal{P}|}$, where $\omega \in \mathsf{MinMod}_{\mathsf{LP}}(K)$. So, $I_{LP_m}(K) = I_{LP_m}(K \setminus \{\alpha\})$ if $\mathsf{BA}(K) = \mathsf{BA}(K \setminus \{\alpha\})$. $\square$

Besides such atom-centric measures, there are some formula-centric measures built upon minimal inconsistent subsets (Hunter & Konieczny, 2010; Mu, 2015). Here formula-centric measures refer to ones that take into account the number of formulas required for inconsistency (Hunter & Konieczny, 2010). We use the measure I_{dr} (Mu, 2015) (a slightly restricted version also called d-hit inconsistency measure in Grant & Hunter, 2013) to exemplify the formula-centric measure. Here $I_{dr}(K)$ is exactly the minimum number of formulas that have to be removed from K to break all the minimal inconsistent subsets of K. However, given an inconsistent knowledge base, neither the formula-centric nor the atom-centric gives a full characterization of inconsistency in that base. They describe the inconsistency from their own respective perspectives. Roughly speaking, atom-centric measures are not necessarily able to capture the effect of syntactic changes of a knowledge base on the characterization of inconsistency, whilst formula-centric measures cannot capture the proportion of language affected by inconsistency explicitly. To illustrate this, here we consider three knowledge bases K_5, K_6, and K_7 built upon variables $\{a, b, c\}$, where $K_5 = \{a, \neg a, b, \neg b \vee c, \neg c\}$, $K_6 = \{a \wedge b, \neg a, \neg b \vee c, \neg c\}$, and $K_7 = \{a \wedge b, \neg a, \neg c\}$. Evidently, $I_{LP_m}(K_5) = I_{LP_m}(K_6) = \frac{2}{3} > \frac{1}{3} = I_{LP_m}(K_7)$, and $I_{dr}(K_5) = 2 > 1 = I_{dr}(K_6) = I_{dr}(K_7)$. Note that we need remove at least two formulas to restore the consistency for K_5, whilst we need only remove one formula $a \wedge b$ to restore the consistency for K_6. Such a syntax-related difference in characterizing inconsistency between K_5 and K_6 captured by I_{dr} cannot be captured by I_{LP_m}. On the other hand, the atom-related difference between K_6 and K_7 captured by I_{LP_m} cannot be captured by I_{dr}. Then it is advisable to integrate the two kinds of measures for cases where both perspectives are considered important to characterize inconsistency. Here we construct a bi-measure $I_B^f(K) = \sqrt{(I_f(K))^2 + (I_{LP_m}(K))^2}$ to capture the inconsistency from both perspectives, where I_f is a normalized formula-centric measure. Obviously, $I_B^f(K) \leq I_B^f(K')$ if $I_f(K) \leq I_f(K')$ and $I_{LP_m}(K) \leq I_{LP_m}(K')$. Moreover, the inconsistency measure I_B^f satisfies the properties of *Consistency*, *Monotony*, *Dominance*, and *Bi-free Formula Independence*, if I_f satisfies the properties of *Consistency*, *Monotony*, *Dominance*, and *Free Formula Independence*. For example, it has been shown that the formula-centric inconsistency measure I_{dr} satisfies the properties of *Consistency*, *Monotony*, *Dominance*, and *Free Formula Independence* (Mu, 2018). Then we can construct such a bi-measure $I_B^{dr}(K) = \sqrt{(1 - e^{-I_{dr}(K)})^2 + (I_{LP_m}(K))^2}$ based on the normalization $1 - e^{-I_{dr}(K)}$ of $I_{dr}(K)$. Moreover, under this bi-measure, we have that $I_B^{dr}(K_5) = \sqrt{(1 - \frac{1}{e^2})^2 + \frac{4}{9}} > \sqrt{(1 - \frac{1}{e})^2 + \frac{4}{9}} = I_B^{dr}(K_6) > \sqrt{(1 - \frac{1}{e})^2 + \frac{1}{9}} = I_B^{dr}(K_7)$.

On the other hand, concepts of $\star$-innocuous formulas, $\star$-free formulas, and $\star$-conflicts parameterised by an arbitrary consequence operation $\mathrm{Cn}^\star$ presented by Bona and Hunter (2017) generalize the notions of free formula and minimal inconsistent subset to capture uncontroversial and controversial formulas with regard to consistency restore procedures formalized by the consequence operation $\mathrm{Cn}^\star$, respectively. These provide an interesting perspective to reveal hidden conflicts behind minimal inconsistent subsets, and then to find really uncontroversial formulas. Roughly speaking, $\star$-innocuous formulas of a knowledge base refer to those that might be consistently added after $\star$-consolidating the rest of the base, whilst $\star$-free formulas refer to those not involved in the derivation of a formula in a minimal inconsistent subset of the $\mathrm{Cn}^{\star}$-closure (Bona & Hunter, 2017). For example, when we use the identity function-based consequence operation Cn^{Id} such that $\mathrm{Cn}^{Id}(\{\varphi\}) = \{\varphi\}$ for any formula φ, then the corresponding consolidation is exactly AGM-consolidation. Moreover, $\star$-conflicts of a knowledge base are exactly minimal inconsistent subsets of that base, and both $\star$-innocuous formulas and $\star$-free formulas are exactly free formulas of a knowledge base (Bona & Hunter, 2017). But if we consider classical consequence operation Cn, then both $\star$-innocuous formulas and $\star$-free formulas are exactly tautologies (Bona & Hunter, 2017).

Note that these concepts assume that $\mathrm{Cn}^{\star}$ is a Tarskian consequence operation. This requires that $\mathrm{Cn}^{\star}$ is monotonic. However, it has been shown by Priest (1991) that $\mathrm{LP_m}$ consequence relation is non-monotonic. This implies that we may not instantiate such concepts based on $\mathrm{LP_m}$ consequence relation to analyze independence of formulas of inconsistency in Priest's minimally inconsistent LP directly.

7. Conclusion

Formulas Independent of inconsistency are of interest to analyzing and measuring inconsistency. The free formula has been considered as a kind of formulas independent of inconsistency when the inconsistency is characterized by minimal inconsistent subsets. However, not all the free formulas are independent of the inconsistency when the inconsistency is characterized by some paraconsistent models.

In this paper, we have identified formulas independent of inconsistency in the framework of Priest's minimally inconsistent LP $\mathrm{LP_m}$. The B-atom-free formula, as a counterpart of free formula, has been proposed based on the invariance of inconsistency characterization. Just as the case of free formula, the B-atom-free formula covers some formulas not independent of inconsistency in syntactic characterization. Then we proposed the notion of Bi-free formula, which exactly covers formulas independent of inconsistency in both formula-centric and atom-centric characterizations. Several corresponding alternatives of the property of Free Formula Independence have been also proposed.

Acknowledgements

The author is grateful to anonymous reviewers for their valuable comments. This work was partly supported by the National Natural Science Foundation of China under Grant No.61572002 and No.61170300.

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